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THE METHOD OF NEHARI MANIFOLD REVISITED

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1. Introduction and abstract setting

The method of Nehari manifold goes back to Nehari's work [5, 6], where he considered a boundary value problem for a certain nonlinear second order ordinary differential equation in an interval (a, b) and showed that it has a nontrivial solution which may be found by constrained minimization of the Euler-Lagrange functional corresponding to the problem.

In this paper we give a short account of the Nehari method in an abstract setting and give examples of its applications to nonlinear elliptic equations. An important assumption here is that the associated functional has a local minimum at 0. We also consider a recent extension of this method to problems where 0 is a saddle point of the functional. The approach we present here differs a little from the usual one and is taken from [11], where the details and a more extensive list of references may be found. We assume the reader is familiar with basic critical point theory and its applications to nonlinear boundary value problems for elliptic equations, see e.g. [1, 2, 9, 12].

Let E be real Banach space and $\Phi \in C^1(E, \mathbb{R})$ a functional. If $\Phi'(u) = 0$ and $u \neq 0$, then

$$u \in \mathcal{N} := \{u \in E \setminus \{0\} : \Phi'(u)u = 0\}.$$

So $\mathcal{N}$ is a natural constraint for the set of nontrivial solutions. It is called the *Nehari manifold* though it is not a manifold in general. Assume without loss of generality that $\Phi(0) = 0$ and let

$$c := \inf_{u \in \mathcal{N}} \Phi(u).$$

Put $S := S_1(0) = \{u \in E : \|u\| = 1\}$. Suppose:

(A_1) E is a Hilbert space,

(A_2) For each $w \in E \backslash \{0\}$ there exists s_w such that if $\alpha_w(s) := \Phi(sw)$, then $\alpha_w'(s) > 0$ for $0 < s < s_w$ and $\alpha_w'(s) < 0$ for $s > s_w$,

(A_3) There exists $\delta > 0$ such that $s_w \geq \delta$ for all $w \in S$ and for each compact subset $\mathcal{W} \subset S$ there exists a constant $C_{\mathcal{W}}$ such that $s_w \leq C_{\mathcal{W}}$ for all $w \in \mathcal{W}$.

It is easy to see that (A_2)-(A_3) imply:

- 0 is a local minimum for Φ,
- $\Phi(s_w w) = \max_{s>0} \Phi(sw)$,
- $0 = \alpha'_w(s_w) = \Phi'(s_w w)w = 0$, $s_w w \in \mathcal{N}$ (and $s_w w \notin \mathcal{N}$ for any other $s > 0$),
- $\mathcal{N}$ is bounded away from 0 and is radially homeomorphic with S,
- c if attained is positive.

Assuming $\Phi \in C^2(E, \mathbb{R})$, one also has

$$\alpha''_w(s_w) = \Phi''(s_w w)(w, w) = s_w^{-2}\Phi''(u)(u, u) \leq 0, \text{ where } u = s_w w \in \mathcal{N}.$$

If $\Phi''(u)(u, u) < 0$ for all $u \in \mathcal{N}$, then it follows from the implicit function theorem that $\mathcal{N}$ is a C^1-manifold of codimension 1 and $E = T_u(\mathcal{N}) \oplus \mathbb{R}u$ for each $u \in \mathcal{N}$, where $T_u(\mathcal{N})$ denotes the tangent space of $\mathcal{N}$ at u. Hence in this case if c is attained, then any $u \in \mathcal{N}$ with $\Phi(u) = c$ (i.e., any minimizer of $\Phi|_{\mathcal{N}}$) satisfies $\Phi'(u) = 0$. Such u is called a *ground state* (because it has minimal "energy" Φ in the set of all nontrivial solutions). As we shall see below, the assumptions $\Phi \in C^1(E, \mathbb{R})$ and (A_2)-(A_3) suffice in order to assert that the minimizers are critical points. Let

$$\widehat{m} : E \setminus \{0\} \to \mathcal{N}, \quad \widehat{m}(w) := s_w w$$

and

$$m : S \to \mathcal{N}, \quad m := \widehat{m}|_S.$$

Proposition 1.1 ([11], Proposition 8). *Suppose Φ satisfies (A_2)-(A_3). Then:*

(a) The mapping $\widehat{m}$ is continuous.

(b) The mapping m is a homeomorphism between S and $\mathcal{N}$. The inverse of m is given by $m^{-1}(u) = u/\|u\|$.

Proof. (a) Suppose $w_n \to w \neq 0$. Since $\widehat{m}(tw) = \widehat{m}(w)$ for each $t > 0$, we may assume without loss of generality that $w_n \in S$. It suffices to show that $\widehat{m}(w_n) \to \widehat{m}(w)$ after passing to a subsequence. Put $\widehat{m}(w_n) = s_n w_n$. By (A_2) and (A_3), (s_n) is bounded and bounded away from 0, hence, taking a subsequence, $s_n \to \bar{s} > 0$. Since $\mathcal{N}$ is closed and $\widehat{m}(w_n) \to \bar{s}w$, $\bar{s}w \in \mathcal{N}$. It follows that $\bar{s}w = s_w w = \widehat{m}(w)$.

(b) follows directly from (a). □

Let

$$\widehat{\Psi} : E \setminus \{0\} \to \mathbb{R}, \quad \widehat{\Psi}(w) := \Phi(\widehat{m}(w))$$

and

$$\Psi : S \to \mathbb{R}, \quad \Psi := \widehat{\Psi}|_S.$$

Proposition 1.2 ([11], Proposition 9). *Suppose Φ satisfies (A_2)-(A_3). Then $\widehat{\Psi} \in C^1(E \setminus \{0\}, \mathbb{R})$ and*

$$\widehat{\Psi}'(w)z = \frac{\|\widehat{m}(w)\|}{\|w\|}\Phi'(\widehat{m}(w))z \quad \text{for all } w, z \in E, \ w \neq 0.$$

Proof. Let $w \in E \setminus \{0\}$ and $z \in E$. Since $\Phi(s_w w) \geq \Phi(sw)$ for all $s > 0$, we see using the mean value theorem that if $|t|$ is small enough, then

$$\begin{aligned}\widehat{\Psi}(w+tz) - \widehat{\Psi}(w) &= \Phi(s_{w+tz}(w+tz)) - \Phi(s_w w) \\ &\leq \Phi(s_{w+tz}(w+tz)) - \Phi(s_{w+tz} w) \\ &= \Phi'(s_{w+tz}(w+\tau tz))s_{w+tz}tz\end{aligned}$$

for some $\tau = \tau(t) \in (0,1)$. Similarly,

$$\begin{aligned}\widehat{\Psi}(w+tz) - \widehat{\Psi}(w) &\geq \Phi(s_w(w+tz)) - \Phi(s_w w) \\ &= \Phi'(s_w(w+\eta tz))s_w tz\end{aligned}$$

for some $\eta = \eta(t) \in (0,1)$. Since the mapping $w \mapsto s_w$ is continuous according to Proposition 1.1, it follows from the inequalities above that

$$\lim_{t\to 0} \frac{\widehat{\Psi}(w+tz) - \widehat{\Psi}(w)}{t} = s_w\Phi'(s_w w)z = \frac{\|\widehat{m}(w)\|}{\|w\|}\Phi'(\widehat{m}(w))z.$$

Hence the Gâteaux derivative of $\widehat{\Psi}$ is bounded linear in z and continuous in w. So $\widehat{\Psi} \in C^1(E \setminus \{0\}, \mathbb{R})$, see e.g. [2, 12]. □

Corollary 1.3 ([11], Corollary 10). *Suppose Φ satisfies (A_2)-(A_3). Then:*

(a) $\Psi \in C^1(S, \mathbb{R})$ and

$$\Psi'(w)z = \|m(w)\|\Phi'(m(w))z \quad \text{for all } z \in T_w(S).$$

(b) If (w_n) is a Palais-Smale sequence for Ψ, then $(m(w_n))$ is a Palais-Smale sequence for Φ. If $(u_n) \subset \mathcal{N}$ is a bounded Palais-Smale sequence for Φ, then $(m^{-1}(u_n))$ is a Palais-Smale sequence for Ψ.

(c) w is a critical point of Ψ if and only if $m(w)$ is a nontrivial critical point of Φ. Moreover, the corresponding values of Ψ and Φ coincide

and $\inf_S \Psi = \inf_{\mathcal{N}} \Phi$.
(d) If Φ *is even, then so is* Ψ.

Proof. (a) follows from the preceding proposition. Note only that $m(w) = \widehat{m}(w)$ because $w \in S$ here.

(b) Let $u = m(w)$. Since $u \in \mathcal{N}$, $\Phi'(u)w = \Phi'(u)\frac{u}{\|u\|} = 0$. Hence it follows using (a) that

$$(1) \qquad \|\Psi'(w)\| = \sup_{\substack{z \in T_w(S) \\ \|z\|=1}} \Psi'(w)z = \|u\| \sup_{\substack{z \in T_w(S) \\ \|z\|=1}} \Phi'(u)z = \|u\|\|\Phi'(u)\|.$$

Since $\mathcal{N}$ is bounded away from 0 and $\Phi(u) = \Psi(w)$, we obtain the conclusion.

(c) By (1), $\Psi'(w) = 0$ if and only if $\Phi'(m(w)) = 0$. The other part is clear.

(d) If Φ is even, then $s_w = s_{-w}$. Hence $\widehat{m}(-w) = -\widehat{m}(w)$ and $\Psi(-w) = \Psi(w)$ by the definition of Ψ. □

Remark 1.4. (i) It is easy to see from the definitions that the value c has the following minimax characterization:

$$c = \inf_{u \in \mathcal{N}} \Phi(u) = \inf_{w \in E \setminus \{0\}} \max_{s>0} \Phi(sw) = \inf_{w \in S} \max_{s>0} \Phi(sw).$$

(ii) Note that the conditions (A_2)-(A_3) do *not* imply that $\widehat{m} \in C^1$ and $\mathcal{N} \in C^1$, yet we have $\Psi \in C^1$.

(iii) Propositions 1.1, 1.2 and Corollary 1.3 remain valid for a large class of Banach spaces - see Section 3.1 of [11]. In particular, they remain valid in the Sobolev spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$.

2. Elliptic equations in a bounded domain

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and consider the boundary value problem

$$(2) \qquad \begin{cases} -\Delta u - \lambda u = f(x,u), & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $\lambda < \lambda_1$ (λ_1 is the first Dirichlet eigenvalue for $-\Delta$ in Ω), f is continuous and

$$(3) \qquad |f(x,u)| \le a(1 + |u|^{q-1}) \quad \text{for some } a > 0 \text{ and } 2 < q < 2^*.$$

Here 2^* is the critical exponent with respect to the embedding of the Sobolev space $H^1(\Omega)$ into $L^q(\Omega)$, i.e., $2^* := 2N/(N-2)$ whenever $N \ge 3$. If $N = 1$ or 2, we put $2^* := \infty$. Since $\lambda < \lambda_1$, the quadratic form

$u \mapsto \int_\Omega (|\nabla u|^2 - \lambda u^2)\,dx$ is positive definite and defines an equivalent norm in $H_0^1(\Omega)$.

Let

$$F(x,u) := \int_0^u f(x,s)\,ds$$

and

$$\Phi(u) := \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda u^2)\,dx - \int_\Omega F(x,u)\,dx \equiv \frac{1}{2}\|u\|^2 - I(u).$$

Then $\Phi \in C^1(E,\mathbb{R})$, where $E := H_0^1(\Omega)$ and critical points of Φ coincide with (weak) solutions of (2).

Theorem 2.1 ([11], Theorem 16). *In addition to the above assumptions, suppose that*
(i) $f(x,u) = o(u)$ *uniformly in* x *as* $u \to 0$,
(ii) $u \mapsto f(x,u)/|u|$ *is strictly increasing on* $(-\infty,0)$ *and* $(0,\infty)$,
(iii) $F(x,u)/u^2 \to \infty$ *uniformly in* x *as* $|u| \to \infty$.
Then equation (2) has a ground state solution. Moreover, if f *is odd in* u, *then (2) has infinitely many pairs of solutions.*

Proof (outline). Since

$$\alpha_w(s) = \Phi(sw) = \frac{1}{2}s^2\|w\|^2 - \int_\Omega F(x,sw)\,dx,$$

it is easy to see from (i) and (iii) that $\alpha_w(s) > 0$ for $s > 0$ small and $\alpha_w(s) < 0$ for s large. Moreover,

$$\alpha_w'(s) = \frac{d}{ds}\Phi(sw) = s\left(\|w\|^2 - \int_\Omega \frac{f(x,sw)}{sw}w^2\,dx\right).$$

It follows from (ii) that the term in brackets above is strictly decreasing in s if $w \neq 0$ and $s > 0$. Hence there exists a unique s_w with $\alpha_w'(s_w) = 0$. So (A_2) holds and one sees that so does (A_3). Consequently, according to Corollary 1.3, $\Psi \in C^1(S,\mathbb{R})$ and critical points of Ψ coincide with solutions of (2).

Below we shall show that Φ satisfies the Palais-Smale condition on $\mathcal{N}$, i.e., if $(u_n) \subset \mathcal{N}$, $\Phi(u_n)$ is bounded and $\Phi'(u_n) \to 0$, then (u_n) has a convergent subsequence. Assuming this, it follows from Corollary 1.3 that Ψ (as a mapping from S to $\mathbb{R}$) satisfies the Palais-Smale condition. But then a well known result in critical point theory implies that $\inf_S \Psi = c > 0$ is attained. Hence (2) has a ground state solution.

Suppose f is odd in u, then Φ is even. Let

$$\Gamma_j := \{A \subset S : A = -A,\ A \text{ is compact and } \gamma(A) \geq j\},$$

where γ is the Krasnoselskii genus [1, 2, 9], and

$$c_j := \inf_{A\in\Gamma_j}\sup_{u\in A}\Phi(u), \quad j = 1,2,\dots$$

Since $c = c_1 \le c_2 \le \cdots$, $c_j < \infty$ for all j and Ψ satisfies the Palais-Smale condition, we can invoke another well known result in critical point theory (see e.g. [1, 2, 9]) in order to conclude that Ψ has infinitely many pairs of critical points and hence (2) possesses infinitely many pairs of solutions. □

Proposition 2.2. *Φ satisfies the Palais-Smale condition on $\mathcal{N}$.*

Proof. Suppose $(u_n) \subset \mathcal{N}$, $\Phi(u_n)$ is bounded and $\Phi'(u_n) \to 0$. If (u_n) is bounded, we may assume passing to a subsequence that $u_n \rightharpoonup u$. Since I' is completely continuous (i.e., $I'(u_n) \to I'(u)$ if $u_n \rightharpoonup u$) and $\Phi'(u_n) = u_n - I'(u_n) \to 0$, $u_n \to I'(u) = u$.

We complete the proof by showing that $\Phi|_{\mathcal{N}}$ is coercive, i.e., $\Phi(u_n) \to \infty$ as $\|u_n\| \to \infty$, $(u_n) \subset \mathcal{N}$. Suppose $\Phi(u_n) \le d$ and $\|u_n\| \to \infty$. We may assume passing to a subsequence that $v_n := u_n/\|u_n\| \rightharpoonup v$, $v_n \to v$ in $L^q_{loc}(\mathbb{R}^N)$ and a.e. If $v = 0$, then weak continuity of I implies that

$$d \ge \Phi(u_n) = \Phi(s_{v_n}v_n) \ge \Phi(sv_n) = \frac{1}{2}s^2 - \int_\Omega F(x, sv_n)\,dx \to \frac{1}{2}s^2$$

for all $s > 0$, a contradiction. So $v \ne 0$. Since $|u_n(x)| \to \infty$ whenever $v(x) \ne 0$, we obtain passing to a subsequence, using Fatou's lemma and (iii) that

$$0 \le \frac{\Phi(u_n)}{\|u_n\|^2} = \frac{1}{2} - \int_\Omega \frac{F(x,u_n)}{u_n^2}v_n^2\,dx \to -\infty,$$

a contradiction again. □

Remark 2.3. (i) Each ground state solution u_0 is positive or negative. For suppose u is a sign-changing solution and let $u^+ := \max\{u, 0\}$, $u^- := \min\{u, 0\}$. Multiplying (2) by $u^\pm$ and integrating, we see that $u^\pm \in \mathcal{N}$. Hence $\Phi(u) = \Phi(u^+) + \Phi(u^-) \ge 2c$ and $u \ne u_0$. It follows that $u_0 \ge 0$ or $u_0 \le 0$ and by Harnack's inequality [3], $u_0 > 0$ or $u_0 < 0$ in Ω.

(ii) It is well known that if (ii) and (iii) in Theorem 2.1 are replaced by the following Ambrosetti-Rabinowitz superlinearity condition:

$$\text{There exist } \mu > 2 \text{ and } R > 0 \text{ such that } 0 < \mu F(x,u) \le f(x,u)u \text{ for all } |u| \ge R, \tag{4}$$

then (2) has a positive solution, and if in addition f is odd in u, then there are infinitely many pairs of solutions (see e.g. [1, 9]). It is easy to see that (4) implies $|F(x,u)| \geq a_1|u|^\mu - a_2$ for some $a_1, a_2 > 0$ while (ii), (iii) hold for certain functions which increase slower than that, e.g., $f(x,u) = u\log(1+|u|)$. On the other hand, there are functions satisfying (4) but not (ii).

Next we formulate a generalization of Theorem 2.1 to the p-Laplacian. Let $W_0^{1,p}(\Omega)$ be the usual Sobolev space and consider the boundary value problem

$$\begin{cases} -\Delta_p u - \lambda|u|^{p-2}u = f(x,u), & x \in \Omega \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{5}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian ($p > 1$),

$$\lambda < \lambda_1 := \inf_{\substack{u \in W_0^{1,p}(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^p\, dx}{\int_\Omega |u|^p\, dx},$$

f is continuous and

$$|f(x,u)| \leq a(1+|u|^{q-1}) \quad \text{for some } a > 0 \text{ and } p < q < p^*.$$

Here $p^* := Np/(N-p)$ if $N > p$ and $p^* := \infty$ otherwise. It is well known (see e.g. Section 7.5A in [2]) that λ_1 is attained and is the first Dirichlet eigenvalue of the p-Laplacian.

The functional

$$\Phi(u) := \frac{1}{p}\int_\Omega (|\nabla u|^p - \lambda|u|^p)\, dx - \int_\Omega F(x,u)\, dx,$$

is in $C^1(E,\mathbb{R})$, where $E := W_0^{1,p}(\Omega)$, and critical points of Φ are weak solutions of (5).

Theorem 2.4 ([11], Theorem 19). *In addition to the above assumptions, suppose that*
(i) $f(x,u) = o(|u|^{p-1})$ *uniformly in* x *as* $u \to 0$,
(ii) $u \mapsto f(x,u)/|u|^{p-1}$ *is strictly increasing on* $(-\infty, 0)$ *and* $(0,\infty)$,
(iii) $F(x,u)/|u|^p \to \infty$ *uniformly in* x *as* $|u| \to \infty$.
Then equation (5) has a ground state solution. Moreover, if f *is odd in* u*, then (5) has infinitely many pairs of solutions.*

The proof follows the same pattern as that of Theorem 2.1 but some work is needed in order to extend the abstract results of Section 1 to an appropriate class of Banach spaces.

3. An elliptic equation in $\mathbb{R}^N$

Consider now the problem

$$
(6) \qquad \begin{cases} -\Delta u + V(x)u = f(x,u), & x \in \mathbb{R}^N \\ u(x) \to 0, & |x| \to \infty, \end{cases}
$$

where V, f are continuous and

$$
(7) \qquad |f(x,u)| \le a(|u| + |u|^{q-1}) \quad \text{for some } a > 0 \text{ and } 2 < q < 2^*.
$$

Theorem 3.1 ([11], Theorem 20). *Suppose f satisfies the growth condition (7) and*
(i) V, f are 1-periodic in $x_1, \dots, x_N$ and $V(x) > 0$ for all x,
(ii) $f(x,u) = o(u)$ uniformly in x as $u \to 0$,
(iii) $u \mapsto f(x,u)/|u|$ is strictly increasing on $(-\infty, 0)$ and $(0, \infty)$,
(iv) $F(x,u)/u^2 \to \infty$ uniformly in x as $|u| \to \infty$.
Then equation (6) has a ground state solution.

This extends a result by Li, Wang and Zeng [4] where more regularity on f was assumed. The proof is similar to that of Theorem 2.1 with one important exception: Φ does not satisfy the Palais-Smale condition on $\mathcal{N}$ here. Yet one can show using a concentration-compactness type argument that Φ attains the infimum on $\mathcal{N}$. And also here it is possible to show that if u is a ground state solution, then either $u > 0$ or $u < 0$ for all x.

Note that it follows from the periodicity of V and f that if u is a solution of (6), then so is $u(\cdot - y)$ for any $y \in \mathbb{Z}^N$. Two solutions which are not translates of each other by an element of $\mathbb{Z}^N$ will be called *geometrically distinct.* One can show, using a rather lengthy and technical argument in [10] that if f is odd in u, then in fact (6) has infinitely many pairs of geometrically distinct solutions.

If V is a positive constant and $f = f(u)$, then $\Phi(u) = \Phi(u(\cdot - y))$ for all $y \in \mathbb{R}^N$ and any translate $u(\cdot - y)$ of a solution $u \neq 0$ is again a solution. Hence the proper notion of geometrically distinct solutions here would be the requirement that they are not translates of each other by any $y \in \mathbb{R}^N$. It follows that existence of a single nontrivial solution automatically leads to the existence of infinitely many geometrically distinct ones in the $\mathbb{Z}^N$-sense. However, as is well known, it is not necessarily true that the number of those which are $\mathbb{R}^N$-distinct is infinite, see e.g. [11].

4. Generalized Nehari manifold

Let E be a Hilbert space, $\Phi \in C^1(E, \mathbb{R})$, $\Phi(0) = 0$ and let

$$E = E^+ \oplus E^0 \oplus E^- \equiv E^+ \oplus F, \quad \text{where } \dim E^0 < \infty,$$

be an orthogonal decomposition. We shall write

$$u = u^+ + u^0 + u^- = u^+ + v, \quad u^\pm \in E^\pm, \ u^0 \in E^0, \ v \in F.$$

and

$$S^+ := S \cap E^+ = \{u \in E^+ : \|u\| = 1\}.$$

For $u \notin F$, let

$$E(u) := \mathbb{R}u \oplus F \equiv \mathbb{R}u^+ \oplus F$$

and

$$\widehat{E}(u) := \mathbb{R}^+u \oplus F \equiv \mathbb{R}^+u^+ \oplus F.$$

Assume that Φ satisfies the following conditions:

(B_1) $\Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - I(u)$, where I is weakly lower semicontinuous and $\frac{1}{2}I'(u)u > I(u) > 0$ for all $u \neq 0$.

(B_2) For each $w \in E \setminus F$ there exists a unique nontrivial (i.e., $\neq 0$) critical point $\widehat{m}(w)$ of $\Phi|_{\widehat{E}(w)}$. Moreover, $\widehat{m}(w)$ is the unique global maximum of $\Phi|_{\widehat{E}(w)}$.

(B_3) There exists $\delta > 0$ such that $\|\widehat{m}(w)^+\| \geq \delta$ for all $w \in E \setminus F$, and for each compact subset $\mathcal{W} \subset E \setminus F$ there exists a constant $C_{\mathcal{W}}$ such that $\|\widehat{m}(w)\| \leq C_{\mathcal{W}}$ for all $w \in \mathcal{W}$.

It is easy to see that the inequalities in (B_1) are satisfied if $I(u) = \int_\Omega F(x, u)\, dx$, where $f(x, u) = o(u)$ uniformly in x as $u \to 0$ and $u \mapsto f(x, u)/|u|$ is strictly increasing (i.e., if f satisfies (i), (ii) of Theorem 2.1).

Let

$$\mathcal{M} := \{u \in E \setminus F : \Phi'(u)u = 0 \text{ and } \Phi'(u)v = 0 \text{ for all } v \in F\}.$$

We shall call this set the *generalized Nehari manifold.* It has been introduced by Pankov in [7]. By (B_1), $\Phi \leq 0$ on F and if $u \neq 0$ and $\Phi'(u) = 0$, then $\Phi(u) = \Phi(u) - \frac{1}{2}\Phi'(u)u = \frac{1}{2}I'(u)u - I(u) > 0$. Hence $\mathcal{M}$ contains all nontrivial critical points of Φ and, as easily follows from (B_2), $\widehat{E}(w) \cap \mathcal{M} = \{\widehat{m}(w)\}$ whenever $w \in E \setminus F$. Note that if $F = \{0\}$, then (B_2), (B_3) are equivalent to (A_2), (A_3) and $\mathcal{M} = \mathcal{N}$. So indeed, $\mathcal{M}$ is a generalization of the Nehari manifold.

If, in addition, $\Phi \in C^2(E, \mathbb{R})$ and the restriction of $\Phi''(\widehat{m}(w))$ to $E(w)$ is negative definite for all $w \in E \setminus F$, then $\mathcal{M}$ is a manifold of class

C^1 by the implicit function theorem (cf. [7]). Under our assumptions $\mathcal{M}$ is a topological manifold as we shall see but in general it may not be C^1.

Let $\widehat{m}$ be as above and let m be the restriction of $\widehat{m}$ to S^+. Then we have

$$\widehat{m} : E \setminus F \to \mathcal{M}, \quad m := \widehat{m}|_{S^+} : S^+ \to \mathcal{M},$$

and similarly as in Section 1, we put

$$\widehat{\Psi} : E^+ \setminus \{0\} \to \mathbb{R}, \quad \widehat{\Psi}(w) := \Phi(\widehat{m}(w))$$

and

$$\Psi : S^+ \to \mathbb{R}, \quad \Psi := \widehat{\Psi}|_{S^+}.$$

We also set

$$c := \inf_{u \in \mathcal{M}} \Phi(u).$$

It follows from (B_2) that c, if attained, is positive. Below we state results which correspond to Propositions 1.1, 1.2 and Corollary 1.3. As a consequence of Corollary 4.3, minimizers for c are critical point of Φ. Hence also in the present situation, if c is attained, then there exist ground states (which have the explicit characterization given in Remark 4.4 below).

Proposition 4.1 ([11], Proposition 31). *Suppose Φ satises (B_1)-(B_3). Then:*
(a) The mapping $\widehat{m}$ is continuous.
(b) The mapping m is a homeomorphism between S^+ and $\mathcal{N}$, and the inverse of m is given by $m^{-1}(u) = u^+/\|u^+\|$.

Proposition 4.2 ([11], Proposition 32). *Suppose Φ satises (B_1)-(B_3). Then $\widehat{\Psi} \in C^1(E^+ \setminus \{0\}, \mathbb{R})$ and*

$$\widehat{\Psi}'(w)z = \frac{\|\widehat{m}(w)^+\|}{\|w\|}\Phi'(\widehat{m}(w))z \quad \text{for all } w, z \in E^+,\ w \neq 0.$$

Corollary 4.3 ([11], Corollary 33). *Suppose Φ satises (B_1)-(B_3). Then:*
(a) $\Psi \in C^1(S^+, \mathbb{R})$ and

$$\Psi'(w)z = \|m(w)^+\|\Phi'(m(w))z \quad \text{for all } z \in T_w(S^+).$$

(b) If (w_n) is a Palais-Smale sequence for Ψ, then $(m(w_n))$ is a Palais-Smale sequence for Φ. If $(u_n) \subset \mathcal{M}$ is a bounded Palais-Smale sequence for Φ, then $(m^{-1}(u_n))$ is a Palais-Smale sequence for Ψ.
(c) w is a critical point of Ψ if and only if $m(w)$ is a nontrivial critical

point of Φ. Moreover, the corresponding values of Ψ and Φ coincide and $\inf_{S^+}\Psi = \inf_{\mathcal{M}}\Phi$.
(d) If Φ is even, then so is Ψ.

The proofs of Proposition 4.2 and Corollary 4.3 are rather similar to those of Proposition 1.2 and Corollary 1.3 while the proof of (i) of Proposition 4.1 is different and more difficult than that of Proposition 1.1. See [10, 11] for the details.

Remark 4.4. Similarly as in Remark 1.4(i), we have the following minimax characterization of c:

$$c = \inf_{u\in\mathcal{M}}\Phi(u) = \inf_{w\in E\setminus F}\ \max_{u\in\widehat{E}(w)}\Phi(u) = \inf_{w\in S^+}\ \max_{u\in\widehat{E}(w)}\Phi(u).$$

5. Application to elliptic equations and systems

First we return to equation (2), where as before, Ω is bounded, f is continuous and satisfies the growth restriction (3), but instead of $\lambda<\lambda_1$, we now assume $\lambda\geq\lambda_1$.

Let $E = H_0^1(\Omega)$ and denote the Dirichlet eigenvalues of $-\Delta$ by $\lambda_1,\lambda_2,\ldots$, and a corresponding orthogonal (in E) set of eigenfunctions by $e_1,e_2,\ldots$. Suppose $\lambda_k<\lambda=\lambda_{k+1}=\cdots=\lambda_m<\lambda_{m+1}$, where $1\leq k<m$ and set

$$E^- = \operatorname{span}\{e_1,\ldots,e_k\},\quad E^0 = \operatorname{span}\{e_{k+1},\ldots,e_m\},$$
$$E^+ = \operatorname{cl}\operatorname{span}\{e_{m+1},e_{m+2},\ldots\}$$

(cl denotes the closure). Then $E=E^+\oplus E^0\oplus E^-$ is the orthogonal decomposition associated with the spectrum of $-\Delta-\lambda$ in E. We also include the cases $k=0$ and $k=m\geq 1$ which correspond to $E^-=\{0\}$ and $E^0=\{0\}$ respectively. Let $u=u^++u^0+u^-\in E^+\oplus E^0\oplus E^-$. In E we may introduce an equivalent norm such that

$$\int_\Omega(|\nabla u|^2-\lambda u^2)\,dx = \|u^+\|^2-\|u^-\|^2.$$

Then the functional Φ corresponding to (2) is given by

$$\Phi(u) = \frac{1}{2}\int_\Omega(|\nabla u|^2-\lambda u^2)\,dx - \int_\Omega F(x,u)\,dx$$
$$\equiv \frac{1}{2}\|u^+\|^2-\frac{1}{2}\|u^-\|^2 - I(u).$$

The conclusions are as before: there is a ground state and if f is odd in u, there are infinitely many pairs of solutions. More precisely, the following holds:

Theorem 5.1 ([10], Theorems 3.1, 3.2; [11], Theorem 37). *In addition to the above assumptions, suppose that*
(i) $f(x,u) = o(u)$ *uniformly in* x *as* $u \to 0$,
(ii) $u \mapsto f(x,u)/|u|$ *is strictly increasing on* $(-\infty, 0)$ *and* $(0,\infty)$,
(iii) $F(x,u)/u^2 \to \infty$ *uniformly in* x *as* $|u| \to \infty$.
Then equation (2) has a ground state solution. Moreover, if f *is odd in* u, *then (2) has infinitely many pairs of solutions.*

Note that since $\dim(E^0 \oplus E^-) > 0$, we are no longer in the situation encountered in Section 2, hence we must use the generalized Nehari manifold. The proof is similar to that of Theorem 2.1. However, there is one major difference: it is much more difficult to verify (B_2) than it was to verify (A_2). In particular, the following calculus lemma turns out to play an important role:

Lemma 5.2 ([10], Lemma 2.2; [11], Lemma 38). *Let* u, s, v *be real numbers such that* $s \geq -1$ *and let* $w := su + v \neq 0$. *Then*

$$f(x,u)\left[s\left(\frac{s}{2}+1\right)u + (1+s)v\right] + F(x,u) - F(x,u+w) < 0$$

for all $x \in \Omega$.

Also Theorem 3.1 has a counterpart here. Let $E = H^1(\mathbb{R}^N)$ and suppose V is continuous and 1-periodic in $x_1, \dots, x_N$. It is well known [8] that the spectrum $\sigma(-\Delta + V)$ in $L^2(\mathbb{R}^N)$ is completely continuous, bounded below but not above and consists of closed disjoint intervals. An interval (a,b) such that $a, b \in \sigma(-\Delta+V)$ and $(a,b) \cap \sigma(-\Delta+V) = \emptyset$ is called a *spectral gap*. The number of spectral gaps may be 0 (then $\sigma(-\Delta + V) = [a,\infty)$ for some a).

Suppose 0 is in a spectral gap of V. Then $E = E^+ \oplus E^-$ (i.e., $E^0 = \{0\}$) and $\dim E^{\pm} = \infty$. So again, we are in the situation of Section 4.

Theorem 5.3 ([10], Theorem 1.1; [11], Theorem 40). *Suppose* f *satisfies the growth condition (7) and*
(i) V, f *are 1-periodic in* $x_1, \dots, x_N$ *and 0 is in a spectral gap of* V,
(ii) $f(x,u) = o(u)$ *uniformly in* x *as* $u \to 0$,
(iii) $u \mapsto f(x,u)/|u|$ *is strictly increasing on* $(-\infty,0)$ *and* $(0,\infty)$,
(iv) $F(x,u)/u^2 \to \infty$ *uniformly in* x *as* $|u| \to \infty$.
Then equation (6) has a ground state solution.

Also here it can be shown that if in addition to the assumptions of Theorem 5.3 f is odd in u, then there exist infinitely many pairs of geometrically distinct solutions, see [10], Theorem 1.2.

Finally we mention a result for systems of equations. Let Ω be bounded and suppose that the functions g, h are continuous and satisfy the growth restriction (3). Consider the system

$$\begin{cases} -\Delta u_1 = h(x, u_2), & x \in \Omega \\ -\Delta u_2 = g(x, u_1), & x \in \Omega \\ u_1 = u_2 = 0, & x \in \partial\Omega. \end{cases} \tag{8}$$

Here we take $E := H_0^1(\Omega) \times H_0^1(\Omega)$, $u = (u_1, u_2) \in E$ and

$$\Phi(u) := \int_\Omega \nabla u_1 \cdot \nabla u_2 \, dx - \int_\Omega (G(x, u_1) + H(x, u_2)) \, dx,$$

where $G(x, u) := \int_0^u g(x, u) \, dx$ and $H(x, u) := \int_0^u h(x, u) \, dx$. Then we have

$$E = E^+ \oplus E^-, \quad E^\pm = \{u \in E : u_2 = \pm u_1\}, \quad \dim E^\pm = \infty.$$

Theorem 5.4 ([11], Theorem 41). *Suppose g, h satisfy (3) and (i)-(iii) of Theorem 5.1. Then system (8) has a ground state solution. Moreover, if g is odd in u_1 and h odd in u_2, then (8) has infinitely many pairs of solutions.*

A similar system in $\mathbb{R}^N$, with g, h 1-periodic in $x_1, \dots, x_N$, can also be treated by the same methods, cf. [11], Theorem 42.

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