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A SEMILINEAR SCHRÖDINGER EQUATION WITH AHARONOV-BOHM MAGNETIC POTENTIAL

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1. INTRODUCTION AND PRELIMINARIES

In quantum mechanics the Hamiltonian for a nonrelativistic charged particle in an electromagnetic field is given by $(-i\nabla+A)^2+V$, where $V:\Omega\to\mathbb{R}$ is the electric (or scalar) potential, $A:\Omega\to\mathbb{R}^N$ is the magnetic (or vector) potential and Ω is the region to which the particle is confined. The vector potential $A=(A_1,A_2,\ldots,A_N)$ is a source for the magnetic field $B=\operatorname{curl}A$, where $\operatorname{curl}A$ is the $N\times N$ skew-symmetric matrix with entries $B_{jk}=\partial_jA_k-\partial_kA_j$. A can also be seen as a differential 1-form and then B is the 2-form given by $B=dA$. More precisely,

$$A=\sum_{j=1}^{N}A_j\,dx^j \quad\text{and}\quad B=dA=\sum_{j<k}B_{jk}\,dx^j\wedge dx^k,$$

where B_{jk} are as above. If $N=3$, then $\operatorname{curl}A$ also has the usual representation as a vector in $\mathbb{R}^3$.

Assume for simplicity that A and V are smooth, set $\Delta_A:=(-i\nabla+A)^2$ and consider the equation

$$(1)\qquad -\Delta_Au+V(x)u=|u|^{p-2}u,\quad x\in\Omega\subset\mathbb{R}^N,\ N\geq 2.$$

Here $u:\Omega\to\mathbb{C}$, $2<p<\infty$ if $N=2$, and $2<p\leq 2^*$ if $N\geq 3$, where $2^*:=2N/(N-2)$ is the critical Sobolev exponent. Formally, solutions to (1) are critical points of the functional

$$J(u):=\frac{1}{2}\int_\Omega(|\nabla_Au|^2+V(x)|u|^2)-\frac{1}{p}\int_\Omega|u|^p,$$

where $\nabla_Au:=(\nabla+iA(x))u$. The first paper in which problem (1) has been considered seems to be [3].

Suppose $V\geq 0$ and let E_A be the closure of $C_0^\infty(\Omega,\mathbb{C})$ with respect to the norm given by

$$\|u\|^2:=\int_\Omega(|\nabla_Au|^2+V|u|^2)\,dx.$$

Then $J \in C^1(E, \mathbb{R})$ and $J'(u) = 0$ if and only if $u \in E_A$ and u is a solution of (1). It is not difficult to verify that the following holds:

(i) $J(e^{i\theta}u) = J(u)$ for all $\theta \in \mathbb{R}$ (i.e., J is S^1-invariant).
(ii) If Ω is simply connected and $\operatorname{curl} \widetilde{A} = B = \operatorname{curl} A$, then there is φ such that $\widetilde{A}(x) = A(x) + \nabla\varphi(x)$.
(iii) $\widetilde{u} = e^{-i\varphi}u$ if and only if $\nabla_{\widetilde{A}}\widetilde{u} = e^{-i\varphi}\nabla_A u$.
(iv) $u \in E_A$ if and only if $\widetilde{u} \in E_{\widetilde{A}}$.
(v) u satisfies $(1)_A$ if and only if $\widetilde{u}$ satisfies $(1)_{\widetilde{A}}$.

Here and in what follows $(1)_A$ and $(1)_{\widetilde{A}}$ respectively denote equation (1) with vector potential A and $\widetilde{A}$. Property (i) is obvious, (iii)-(v) follow by direct computation (see also [1, 3]) and (ii) can be obtained either by integration or by using the fact that Ω has trivial de Rham cohomology and hence $d(\widetilde{A} - A) = 0$ implies that $\widetilde{A} - A = d\varphi$ for some φ (here φ is considered as a 0-form).

In conclusion, if Ω is simply connected, there is a one-to-one correspondence between the solutions u of $(1)_A$ and the solutions $\widetilde{u}$ of $(1)_{\widetilde{A}}$. Moreover, $|u|^2 = |\widetilde{u}|^2$, i.e., the (unnormalized) probability density of finding a particle at x is independent of the choice of A. Hence the magnetic field B and not the particular choice of the magnetic potential A is important. This is called *gauge invariance* and the transformation $u \mapsto \widetilde{u}$ is called *change of gauge.*

In this note we report on some results contained in [2]. We consider equation (1) with the Aharonov-Bohm potential (to be defined in the next section) in

$$\Omega = \Omega_{a,b} := \{x = (x_1, x_2, x_3) : a < x_1^2 + x_2^2 < b\},$$

where $0 \leq a < b \leq \infty$. So Ω is not simply connected here and as we shall see, A and not only B turns out to be important.

2. The Aharonov-Bohm potential

In what follows we always assume $N = 3$ and we denote by A the Aharonov-Bohm magnetic potential given by

$$A(x) := \left(-\frac{x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2}, 0\right). \tag{2}$$

We note that $\operatorname{curl} A = 0$ in $\Omega_{0,\infty} = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 > 0\}$ and

$$\int_\gamma A = 2\pi, \tag{3}$$

where γ is a properly oriented closed curve enclosing the x_3-axis. This is in sharp contrast to the case of simply connected domain Ω where $\int_\gamma A = 0$ for any smooth A and any closed curve $\gamma \subset \Omega$ (by the Stokes theorem). The integral in (3) is called the magnetic flux and describes the influence of a magnetic potential on a charged quantum-mechanical particle moving in a region where the magnetic field is 0 (this is called the *Aharonov-Bohm effect*, see e.g. [5]). In our setting the particle is confined to a region $\Omega = \Omega_{a,b}$ outside a thin solenoid extending along the x_3-axis.

If A is given by (2), then $A = \nabla\theta$, where $\theta = \theta(x)$ is the polar angle of (x_1, x_2). Obviously, θ is not well defined in $\Omega_{0,\infty}$, however, $e^{i\theta}$ is. Suppose $u \neq 0$ is a real-valued solution of one of the equations

$$-\Delta u + V(x)u = \lambda u \quad \text{and} \quad -\Delta u + V(x)u = |u|^{p-2}u.$$

Then $\widetilde{u} = e^{-i\theta}u$ satisfies the corresponding equation, with $-\Delta$ replaced by $-\Delta_A$. If a particle travels in the x_1x_2-plane from a point P to a point Q lying on the opposite side of the x_3-axis (say $P = (0,1,0)$ and $Q = (0,-1,0)$), then the phase θ of $\widetilde{u}$ will differ by 2π depending on from which side the particle passes this axis. This difference in phases has been experimentally observed, see [5].

Consider now the vector potentials sA and tA, where A is given by (2) and $s, t \in \mathbb{R}$. Then there is no gauge equivalence, not even up to 2π, between $(1)_{sA}$ and $(1)_{tA}$ if $s - t \notin \mathbb{Z}$. Indeed, we have $sA - tA = (s-t)\nabla\theta$, so if $sA - tA = \nabla\varphi$, then $\varphi = (s-t)\theta + C$, hence neither φ nor $e^{i\varphi}$ is well defined.

3. Existence of solutions - the subcritical case

Consider the equation

$$(4) \qquad -\Delta_{sA}u + u = |u|^{p-2}u, \quad u \in H^1_{sA,0}(\Omega_{a,b}, \mathbb{C}),$$

where A is the Aharonov-Bohm potential (2), $s \in \mathbb{R}$, $\Omega \equiv \Omega_{a,b} = \{x \in \mathbb{R}^3 : a < x_1^2 + x_2^2 < b\}$, $0 \leq a < b \leq \infty$, $2 < p < 2^* = 6$ and the space $H^1_{sA,0}(\Omega_{a,b}, \mathbb{C})$ is the closure of $C_0^\infty(\Omega_{a,b}, \mathbb{C})$ with respect to the norm given by

$$\|u\|^2_{sA} := \int_\Omega (|\nabla_{sA}u|^2 + |u|^2)\, dx.$$

Let $G = SO(2)$ denote the group of rotations of the plane $\mathbb{R}^2 \equiv \mathbb{C}$, or equivalently, the group of unit complex numbers acting by multiplication on $\mathbb{C}$.

Theorem 3.1 ([2], Theorem 1.1). *Let $0 \leq a < b \leq \infty$. Then for every $n \in \mathbb{Z}$ there exists a nontrivial solution u_n of $(4)_{sA}$ with the following properties:*
(a) $u_0 = |u_0|$ and $|u_n| > 0$ in $\Omega_{a,b}$,
(b) $|u_n| \neq |u_m|$ if $|s+m| \neq |s+n|$,
(c) $u_n(g(x_1,x_2),x_3) = g^n u_n(x_1,x_2,x_3)$ for every $g \in SO(2)$, $x \in \Omega_{a,b}$,
(d) $u_n(x_1,x_2,x_3) = u_n(x_1,x_2,-x_3)$ for every $x \in \Omega_{a,b}$,
(e) $\|u_m\|_{sA} < \|u_n\|_{sA}$ if $|s+m| < |s+n|$,
(f) $\lim_{|n|\to\infty} \|u_n\|_{sA} = \infty$.

Below we shall outline the proof of some parts of this theorem. For notational convenience we let Ω be the shorthand for $\Omega_{a,b}$. Put

$$S_{\Omega,s,p} := \inf_{\substack{u \in H^1_{sA,0}(\Omega,\mathbb{C}) \\ u \neq 0}} \frac{\|u\|^2_{sA}}{\|u\|^2_p}. \tag{5}$$

If $S_{\Omega,s,p}$ is attained at u such that $\|u\|_p = 1$ (where $\|\,.\,\|_p$ denotes the L^p-norm), then $S^{1/(p-2)}_{\Omega,s,p} u$ is a solution of $(4)_{sA}$ and it has minimal energy among all nontrivial solutions. It is therefore called a *ground state.*

It is well known that equation $-\Delta u + u = |u|^{p-2}u$ in $H^1(\mathbb{R}^3,\mathbb{R})$ has a unique radially symmetric positive solution w and

$$S_p := \inf_{\substack{u \in H^1(\mathbb{R}^3,\mathbb{R}) \\ u \neq 0}} \frac{\|u\|^2}{\|u\|^2_p} = \frac{\|w\|^2}{\|w\|^2_p},$$

where $\|u\|$ denotes the $H^1(\mathbb{R}^3,\mathbb{R})$-norm. The following relation between $\nabla_A u$ and $\nabla|u|$ is called the *diamagnetic inequality,* see [4]:

$$|\nabla(|u(x)|)| \leq |\nabla_{sA} u(x)| \text{ a.e. in } \Omega.$$

Proposition 3.2 ([2], Proposition 2.1). *If $\Omega = \Omega_{a,\infty}$, then $S_{\Omega,s,p} = S_p$.*

Proof (outline). Let $v_n(x) := \psi(x_1,x_2)w(x_1-n,x_2,x_3)$, where ψ a cut-off function such that $\psi(x) = 0$ if $x_1^2 + x_2^2 \leq a^2$ and $\psi(x) = 1$ if $x_1^2 + x_2^2 \geq 2a^2$. Then an easy computation using the diamagnetic inequality and the fact that $|A(x)| \to 0$ as $x_1^2 + x_2^2 \to \infty$ shows that

$$S_p \leq \frac{\|\,|v_n|\,\|^2}{\|v_n\|^2_p} \leq \frac{\|v_n\|^2_{sA}}{\|v_n\|^2_p} = S_p + o(1).$$

□

By property (iii) in Section 1 (or a straightforward computation), we have the following relation induced by a change of gauge:

$$\nabla_{sA}(e^{-im\theta}u) = e^{-im\theta}\nabla_{(s-m)A}u, \quad m \in \mathbb{Z}. \tag{6}$$

Proposition 3.3 ([2], Proposition 2.2). *Suppose $S_{\Omega,s,p} = S_p$. Then $S_{\Omega,s,p}$ is attained if and only if $\Omega = \Omega_{0,\infty}$ and $s \in \mathbb{Z}$.*

Proof. Suppose $\Omega = \Omega_{0,\infty}$ and $s = m \in \mathbb{Z}$. Since a line in $\mathbb{R}^3$ has capacity zero, $H_0^1(\Omega, \mathbb{C}) = H^1(\mathbb{R}^3, \mathbb{C})$ and hence $w \in H_0^1(\Omega, \mathbb{C})$. Let $u_{-m} := e^{-im\theta}w$. Then $u_{-m} \in H^1_{mA,0}(\Omega, \mathbb{C})$ and

$$\frac{\|u_{-m}\|^2_{mA}}{\|u_{-m}\|^2_p} = \frac{\|w\|^2}{\|w\|^2_p} = S_p = S_{\Omega,m,p}.$$

Suppose now $S_{\Omega,s,p} = S_p$ is attained at some v. Then, by the diamagnetic inequality,

$$S_p\|v\|_p \leq \|\,|v|\,\|^2 \leq \|v\|^2_{sA} = S_p\|v\|_p.$$

So $|v| = w$, possibly after a translation of w and normalization of v, and hence $\Omega = \Omega_{0,\infty}$. Moreover, $\|\,|v|\,\| = \|v\|_{sA}$ and therefore (cf. [1])

$$sA = -\operatorname{Im}\frac{\nabla v}{v}.$$

Let $\gamma(t) := (\cos t, \sin t, 0)$ and $u(t) := v(\gamma(t))$. Then u is a curve in $\mathbb{R}^2 \equiv \mathbb{C}$ and

$$\frac{u'(t)}{u(t)} = \frac{\nabla v(\gamma(t))}{v(\gamma(t))} \cdot \gamma'(t).$$

So setting $z = x_1 + ix_2$,

$$\begin{aligned} 2\pi s &= \int_\gamma sA = -\operatorname{Im}\int_\gamma \frac{\nabla v}{v} = -\operatorname{Im}\int_0^{2\pi}\frac{u'(t)}{u(t)}\,dt \\ &= -\operatorname{Im}\int_\gamma \frac{dz}{z} = -2\pi m, \end{aligned}$$

where m is the winding number of u with respect to 0. It follows that $s = -m \in \mathbb{Z}$. □

As before, let $G = SO(2)$ denote the group of rotations of the plane $\mathbb{R}^2$ and put $(gu)(x) := u(g^{-1}(x_1, x_2), x_3)$ for all $g \in SO(2)$. This defines an action of $SO(2)$ on $H^1_{sA,0}(\Omega, \mathbb{C})$. Let $H^1_{sA,0}(\Omega, \mathbb{C})^G$ be the subspace of $H^1_{sA,0}(\Omega, \mathbb{C})$ consisting of functions which are invariant with respect to this action, i.e.,

$$\begin{aligned} H^1_{sA,0}(\Omega, \mathbb{C})^G &= \{u \in H^1_{sA,0}(\Omega, \mathbb{C}) : (gu)(x) = u(x) \text{ for all } g \in SO(2)\} \\ &\equiv \{u \in H^1_{sA,0}(\Omega, \mathbb{C}) : u = u(\sqrt{x_1^2 + x_2^2}, x_3)\}. \end{aligned}$$

Denote

$$S^G_{\Omega,s,p} := \inf_{\substack{u\in H^1_{sA,0}(\Omega,\mathbb{C})^G\\ u\neq 0}} \frac{\|u\|^2_{sA}}{\|u\|^2_p}. \tag{7}$$

The following result is a special case of Proposition 2.3 in [2].

Proposition 3.4. $S^G_{\Omega,s,p}$ *is attained at some* $v \in H^1_{sA,0}(\Omega,\mathbb{C})^G$. *After normalization, this* v *is a solution of* $(4)_{sA}$.

According to Proposition 3.3, a ground state solution to (4) in the full space $H^1_{sA,0}(\Omega,\mathbb{C})$ exists if and only if $\Omega = \Omega_{0,\infty}$ and $s \in \mathbb{Z}$. Proposition 3.4 shows that if one restricts the space of admissible functions to those which are $SO(2)$-invariant, then, in this restricted space, a ground state exists for any $\Omega = \Omega_{a,b}$ and $s \in \mathbb{R}$. So in particular, $S_{\Omega,s,p} < S^G_{\Omega,s,p}$ unless $\Omega = \Omega_{0,\infty}$ and $s \in \mathbb{Z}$.

Proof of Proposition 3.4 (outline). If the infimum is attained at v, then it follows from the principle of symmetric criticality (see e.g. [6]) that after normalization v is a solution of $(4)_{sA}$.

Let $\|v_n\|_p = 1$, $\|v_n\|^2_{sA} \to S^G_{\Omega,s,p}$. Passing to a subsequence, $v_n \rightharpoonup v$. If

$$\sup_{x\in\mathbb{R}^3} \int_{B_1(x)} |v_n|^2 \to 0,$$

where $B_1(x)$ denotes the open ball with radius 1 and center at x, then $\|v_n\|_p \to 0$ by P.L. Lions' lemma [6]. This is impossible because $\|v_n\|_p = 1$. Hence there exist ξ_n such that

$$\int_{B_1(\xi_n)} |v_n|^2 \geq \delta > 0 \tag{8}$$

for almost all n. Since the norms in (7) are invariant with respect to translations along the x_3-axis, we may assume $\xi_n = (\xi_{n1}, \xi_{n2}, 0)$. Then it is clear that the sequence (ξ_n) is bounded if $\Omega = \Omega_{a,b}$, $b < \infty$. This is also true if $b = \infty$. Indeed, let $m = m(n)$ be the largest number of elements $g_1, \dots, g_m \in SO(2)$ such that

$$B_1(g_j(\xi_{n1}, \xi_{n2}), 0) \cap B_1(g_k(\xi_{n1}, \xi_{n2}), 0) = \emptyset \text{ whenever } j \neq k.$$

Since $v_n(g(x_1,x_2),x_3) = v_n(x_1,x_2,x_3)$ for all $g \in SO(2)$, we obtain

$$\|v_n\|^2_{sA} > \int_\Omega |v_n|^2 \geq m \int_{B_1(\xi_{n1},\xi_{n2},0)} |v_n|^2 \geq m\delta.$$

If $|\xi_n| \to \infty$, then $m = m(n) \to \infty$ contradicting the boundedness of $\|v_n\|_{sA}$. It follows that (ξ_n) is bounded and hence $v \neq 0$ according to (8) and the fact that $v_n \to v$ in $L^2_{loc}(\Omega)$.

One completes the proof by showing that $v_n \to v$ in $L^p(\Omega)$. □

Proof of Theorem 3.1 (outline). If u is $SO(2)$-invariant, then it is constant on each circle $x_1^2 + x_2^2 = r^2$, $x_3 = c$, hence $\nabla u(x) \cdot A(x) = 0$. So

$$\|u\|_{sA}^2 = \int_\Omega (|\nabla u + isAu|^2 + |u|^2) = \int_\Omega (|\nabla u|^2 + |sA|^2|u|^2 + |u|^2).$$

Using this, it is not difficult to show that if u is a minimizer for $S^G_{\Omega,s,p}$, then so is $|u|$, cf. [2], Lemma 2.4. Hence by the Harnack inequality there is a minimizer $v_s > 0$ which, after normalization, is a solution of $(4)_{sA}$.

If $v_s = v_t = v$, then

$$-\Delta v + \left(|sA|^2 + 1\right) v = v^{p-1} = -\Delta v + \left(|tA|^2 + 1\right) v,$$

which implies $s^2 = t^2$. So $v_s \neq v_t$ if $|s| \neq |t|$.

Let

$$u_n := e^{in\theta} v_{s+n}, \quad n \in \mathbb{Z}.$$

Then u_n is a solution of (4) according to (6) and property (v) in Section 1. Moreover, $u_0 > 0$, $|u_n| > 0$, $|u_n| \neq |u_m|$ if $|s+n| \neq |s+m|$ and it is easy to verify that $u_n(g(x_1, x_2), x_3) = g^n u_n(x_1, x_2, x_3)$. This completes the proof of (a)-(c). Using the moving plane method one sees that (d) holds, possibly after translating u_n along the x_3-axis. We omit the proof of (e) and (f). □

Below we state a result showing that the solutions u_n are minimizers for the quotient in (5) on a suitable subspace of $H^1_{sA,0}(\Omega, \mathbb{C})$.

Proposition 3.5 ([2], Proposition 2.6). *Let*

$$X_n := \{u \in H^1_{sA,0}(\Omega, \mathbb{C}) : u(g(x_1, x_2), x_3) = g^n u(x_1, x_2, x_3) \\ \forall g \in SO(2),\ x \in \Omega\}.$$

Then

$$\frac{\|u_n\|_{sA}^2}{\|u_n\|_p^2} = \inf_{\substack{u \in X_n \\ u \neq 0}} \frac{\|u\|_{sA}^2}{\|u\|_p^2}.$$

Hence u_n are ground states in the set of nontrivial solutions of $(4)_{sA}$ satisfying the symmetry property specified above.

4. Existence of solutions - the critical case

Now we take $p = 2^* = 6$ and consider the equation

$$-\Delta_{sA} u = |u|^4 u, \quad u \in \mathcal{D}^1_{sA,0}(\Omega, \mathbb{C}), \tag{9}$$

where $\mathcal{D}^1_{sA,0}(\Omega_{a,b}, \mathbb{C})$ is the closure of $C_0^\infty(\Omega_{a,b}, \mathbb{C})$ in the norm given by

$$\|u\|^2_{sA,*} := \int_\Omega |\nabla_{sA} u|^2$$

(as before, $\Omega = \Omega_{a,b}$).

Theorem 4.1 ([2], Theorem 1.3). *Let $a = 0$, $b = \infty$ or $0 < a < b < \infty$. Then for every $n \in \mathbb{Z}$ there exists a nontrivial solution u_n of $(9)_{sA}$ with the following properties:*
(a) $u_0 = |u_0|$ and $|u_n| > 0$ in $\Omega_{a,b}$,
(b) $|u_n| \neq |u_m|$ if $|s + m| \neq |s + n|$,
(c) $u_n(g(x_1, x_2), x_3) = g^n u_n(x_1, x_2, x_3)$ for every $g \in SO(2)$, $x \in \Omega_{a,b}$,
(d) $u_n(x_1, x_2, x_3) = u_n(x_1, x_2, -x_3)$ for every $x \in \Omega_{a,b}$,
(e) $\|u_m\|_{sA,*} < \|u_n\|_{sA,*}$ if $|s + m| < |s + n|$.

Note that we have no conclusion (f) of Theorem 3.1 here (we do not know whether this conclusion holds for (9)).

The proof of the above theorem follows the same pattern as that of Theorem 3.1 but there are some differences which we briefly describe. Let S be the usual Sobolev constant given by

$$S := \inf_{\substack{u \in \mathcal{D}^{1,2}(\mathbb{R}^3, \mathbb{R}) \\ u \neq 0}} \frac{\|u\|^2}{\|u\|^2_{2^*}}$$

(here $\|u\|$ denotes the $\mathcal{D}^{1,2}(\mathbb{R}^3, \mathbb{R})$-norm) and denote

$$S_{\Omega,s,*} := \inf_{\substack{u \in \mathcal{D}^{1,2}_{sA,0}(\Omega, \mathbb{C}) \\ u \neq 0}} \frac{\|u\|^2_{sA,*}}{\|u\|^2_{2^*}}. \tag{10}$$

Proposition 4.2 ([2], Proposition 3.4). *$S_{\Omega,s,*} = S$ for any $\Omega = \Omega_{a,b}$, $0 \le a < b \le \infty$. $S_{\Omega,s,*}$ is attained if and only if $\Omega = \Omega_{0,\infty}$ and $s \in \mathbb{Z}$.*

Proof (outline). The first conclusion is proved by taking the truncated instanton (see [6])

$$u_{\varepsilon,x_0}(x) := \psi(x) \frac{\varepsilon^{1/2}}{(\varepsilon^2 + |x - x_0|^2)^{1/2}},$$

where ψ is a cutoff function around $x_0 \in \Omega$. Letting $\varepsilon \to 0$ we see that the quotient in (10) approaches S. Since $S \leq S_{\Omega,s,*}$ by the diamagnetic inequality, these two constants must be equal.

The proof of the second conclusion is the same as that of Proposition 3.3. $\square$

Let $\mathcal{D}^1_{sA,0}(\Omega_{a,b},\mathbb{C})^G$ be the subspace of $\mathcal{D}^1_{sA,0}(\Omega_{a,b},\mathbb{C})$ consisting of $SO(2)$-invariant functions and let $S^G_{\Omega,s,*}$ be the corresponding constant as in (10).

Proposition 4.3 ([2], Propositions 3.2 and 3.3). *If $\Omega = \Omega_{0,\infty}$ or $\Omega = \Omega_{a,b}$, $0 < a < b < \infty$, then $S^G_{\Omega,s,*}$ is attained at some $v \in \mathcal{D}^1_{sA,0}(\Omega_{a,b},\mathbb{C})^G$. After normalization, this v is a solution of $(9)_{sA}$.*

The proof of this proposition relies on the concentration-compactness principle and is omitted.

Now Theorem 4.1 can be proved using the same arguments as in Theorem 3.1. Finally, let us mention that similarly as in Proposition 3.5, the solutions u_n are minimizers for the quotient in (10) on a suitable subspace of $\mathcal{D}^1_{sA,0}(\Omega_{a,b},\mathbb{C})$.

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