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A mathematical approach to the computer simulations of Lorenz attractors

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1 Introduction

We shall study the double scroll solution behaviour of Lorenz equation (L) :

$$\frac{dx}{dt}=a(y-x),$$
$$\frac{dy}{dt}=cx-y-xz$$
$$\frac{dz}{dt}=-bz+xy$$

where a, b and c are positive constants. In 1960's, meteoroligist Lorenz carried out a computer simulation of (L) in the case where $a=10$, $b=\frac{8}{3}$ and $c=28$, and found a complicated solution behavior, which is now known in the 3-dimensional graphics as Figure [1, p.303] and called a double scroll. However this behavior has not seemed to be proved mathematically, and hence in this note we shall prove the existence of the double scroll for the Euler difference scheme of (1) :

$$\frac{\Delta x}{\Delta t}=a(y-x)$$
$$\frac{\Delta y}{\Delta t}=cx-y-xz$$
$$\frac{\Delta z}{\Delta t}=-bz+xy.$$

Setting $\Delta x=x'-x$, $\Delta y=y'-y$, $\Delta z=z'-z$ and $\Delta t=h$, we obtain the mapping T : $(x,y,z)\rightarrow(x',y',z')$ such that

$$\begin{aligned} x'&=(1-ah)x+ahy\\ y'&=hcx+(1-h)y-hxz\\ z'&=(1-bh)z+hxy. \end{aligned} \tag{1}$$

The equilibrium points of (L), $P_0(0,0,0)$, $P_1=(\sqrt{b(c-1)},\sqrt{b(c-1)},c-1)$ and $P_2=(-\sqrt{b(c-1)},-\sqrt{b(c-1)},c-1)$, are fixed points of T, and vice versa. First of all, we shall prove the existence of nontrivial periodic point $P = (x, y, z)$ such that $TP = (-x, -y, z)$, which implies that $T^2P = P$ by the symmetry of the right hand side of (1). Next we shall treat the case where P has a T^2-invariant unstable manifield around TP. In fact we shall treat the case where the Jacobian matrix of T^2 around P has as eigenvalues one real number λ, where $|\lambda| \neq 1$, and two complex conjugate number $\alpha \pm i\beta$, where $\alpha^2 + \beta^2 > 1$. In this case it follows from Hartman-Grobman's theorem [1, p.313] that there exists the manifold H as above and furthermore solutions of (1) rotates around P on H and around TP on PH, respectively, as repetition of T^2.

2 Periodic points

We shall find the nontrivial solution of the equation $T(x, y, z) = (-x, -y, z)$, which is equal to

$$\begin{aligned} (ah-2)x + ahy &= 0 \\ (2-h)y + hcx - hxz &= 0 \\ bz &= xy \end{aligned}$$

that is,

$$\begin{aligned} x^2 &= \frac{b\{a(c-1)h^2 + 2(a+1)h - 4\}}{h(ah-2)} \\ y &= \left(\frac{ah-2}{ah}\right)x \\ z &= \frac{1}{ah^2}\{a(c-1)h^2 + 2(a+1)h - 4\}. \end{aligned} \tag{2}$$

It is noted that (2) is meaningful in the case where either $0 < h < h'$ or $h > \frac{2}{a}$, where h' is a positive solution of the equation $a(c-1)h^2 + 2(a+1)h - 4 = 0$ and $h' < \frac{2}{a}$, and that $x^2 \to b(c-1)$, $\frac{y}{x} \to 1$ and $z \to c-1$ as $h \to \infty$. Namely 2-periodic points (x, y, z) approach P_1 and P_2, as $h \to \infty$, which is the case where we shall consider in the following. It is noted that the solutions (x, y, z) are different from P_1 and P_2, and moreover any point in a neighbourhood of (x, y, z) may be transfered into a neighbourhood of $(-x, -y, z)$ by T, which suggest the travelling of solutions of (L) between P_1 and P_2.

Next we shall consider a 2-dimensional, T^2-invariant, unstable manifold around (x, y, z) in the case where h is sufficiently large. The Jacobian matrix of T^2 around (x, y, z) is the following

$$A = \begin{pmatrix} 1-ah & ah & 0 \\ h(c-z) & 1-h & hx \\ -hy & -hx & 1-bh \end{pmatrix} \begin{pmatrix} 1-ah & ah & 0 \\ h(c-z) & 1-h & -hx \\ hy & hx & 1-bh \end{pmatrix}$$

When A has as eigenvalues a real number λ, where $|\lambda| \neq 1$, and complex conjugate numbers $\alpha \pm i\beta$, where $\alpha^2 + \beta^2 > 1$, it follows that (x, y, z) has a 2-dimensional, T^2- invariant, unstable manifold H around itself, on which solutions of (1) rotates around (x, y, z) as repetition of T^2. Since TH is the manifold corresponding to $(-x, -y, z)$, we may claim that this behavior of solutions is the double scroll of (1). Now setting $B = \lim_{h \to \infty} \frac{1}{h^2} A$, where B is a 3×3 matrix, we may verify

$$B = \begin{pmatrix} a^2 + a & -a^2 - a & -ax \\ -a - 1 + b(c-1) & a + 1 + b(c-1) & (1-b)x \\ (a - 1 - b)x & (-a - b + 1)x & b^2 + b(c-1) \end{pmatrix}$$

where $x^2 = b(c-1)$, and $|B| = 4a^2b^2(c-1)^2$. Since we assume that $c \neq 1$, any eigenvalue of B is not zero. If B has as eigenvalues complex conjugate number, then each eigenvalue of B is simple, and hence for large h, A has a real eigenvalue λ, where $|\lambda| > 1$, and complex conjugate number $\alpha \pm i\beta$ as eigenvalue, where $\alpha^2 + \beta^2 > 1$. Therefore in this case, we may claim that (1) shows the double scroll. As affirmative examples to this case we shall state the two examples such that $a = b = 1$ and $c = 2$ and that $a = 10$, $b = 3$ and $c = 28$; the latter one is close to the case treated by Lorenz, where $a = 10$, $b = \frac{8}{3}$ and $c = 28$.

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References

[1] T.Matsumoto, M.Komuro, H.Kokubu, R.Tokunaga, Bifurcations, sights, sounds and mathematics, Springer Verlag (1993)