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A period map for cubic surfaces

Atsushi Ikeda

Graduate School of Science, Osaka University

1 Construction of the period map

In this report, we construct a period map for cubic surfaces, and we prove the injectivity of the period map. Let $X = X_F \subset \mathbf{P}^3$ be a nonsingular cubic surface defined by $F \in \mathbf{C}[x_0, \dots, x_3]$. We remark that the cubic surface X has no holomorphic 2-form, therefore we cannot have a nontrivial Hodge structure by the period integral on X itself. We will consider another variety associated with X. Let $B = B_F$ be the zeros of the Hessian of the cubic polynomial F;

$$B_F = \{p \in X \mid \det \Big(\frac{\partial^2 F}{\partial x_i \partial x_j}(p)\Big)_{0 \leq i,j \leq 3} = 0\},$$

and let $\phi : Y = Y_F \to X$ be the double cover branched along B. We remark that B has at most node as its singularity, and Y is the canonical resolution of the finite double cover. Here we want to classify the isomorphism class of X by using period integral on Y.

The double cover Y is a minimal surface of general type with the geometric genus $p_g(Y) = 4$, the irregularity $q(Y) = 0$ and the square of the canonical divisor $K_Y^2 = 6$. Then the second Betti number is $h^2(Y, \mathbf{Z}) = 52$. The Néron-Severi group $\mathrm{NS}\,(Y)$ is contained in $H^2(Y, \mathbf{Z})$, and the Picard number depends on the equation F. We can prove that the Picard number of Y for the generic equation is 28. We denote by $H^2_{\mathrm{inv}}(Y, \mathbf{Z})$ the subgroup of rank 28 in $H^2(Y, \mathbf{Z})$ which corresponds to the Néron-Severi group of the generic equation, and we denote by $H^2_{\mathrm{var}}(Y, \mathbf{Z})$ the subgroup of rank 24 in $H^2(Y, \mathbf{Z})$ orthogonal to $H^2_{\mathrm{inv}}(Y, \mathbf{Z})$ by the symmetric form

$$\langle\ ,\ \rangle_Y : H^2(Y, \mathbf{Z}) \times H^2(Y, \mathbf{Z}) \longrightarrow H^4(Y, \mathbf{Z}) \simeq \mathbf{Z}.$$

We study the Hodge structure defined on $H^2_{\mathrm{var}}(Y, \mathbf{Z})$.

Let $(H, \langle\ ,\ \rangle)$ be a lattice which is isomorphic to $(H^2_{\mathrm{var}}(Y, \mathbf{Z}), \langle\ ,\ \rangle_Y)$. We have an Hermitian form on $H_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{Z}} H$ by

$$\langle\ ,\bar{\ }\rangle :\ H_{\mathbf{C}} \times H_{\mathbf{C}} \longrightarrow \mathbf{C};\ (\alpha, \beta) \longmapsto \langle \alpha, \bar{\beta} \rangle,$$

where $\bar{\beta}$ denotes the complex conjugate of $\beta \in H_{\mathbf{C}}$. We define the classifying space of the polarized Hodge structure by

$$D = \{W \in \mathrm{Grass}\,(4, H_{\mathbf{C}}) \mid W \subset W^{\perp},\ \langle\ ,\bar{\ }\rangle|_W > 0\},$$

and we call elements of D polarized Hodge structure on H. We define the polarized Hodge structure on $H^2_{\mathrm{var}}(Y, \mathbf{Z})$ by the image of the injective homomorphism

$$H^0(Y, \Omega^2_Y) \longrightarrow \mathrm{Hom}\left(\frac{H^2(Y,\mathbf{Z})}{H^2_{\mathrm{inv}}(Y,\mathbf{Z})}, \mathbf{C}\right) \simeq H^2_{\mathrm{var}}(Y, \mathbf{C});\ \eta \longmapsto \left[\alpha \mapsto \int_\alpha \eta\right].$$

Let $\mathcal{C} \subset \mathrm{Grass}\,(1, H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(3)))$ be the space of smooth cubic surfaces. We fix a base point $[F_0] \in \mathcal{C}$ and an isomorphism $(H^2_{\mathrm{var}}(Y_{F_0}, \mathbf{Z}), \langle\ ,\ \rangle_{Y_{F_0}}) \simeq (H, \langle\ ,\ \rangle)$. Then the monodromy group Γ is defined as the image of the monodromy representation

$$\pi_1(\mathcal{C}, [F_0]) \longrightarrow \mathrm{Aut}(H, \langle\ ,\ \rangle),$$

and we have a period map

$$\mathcal{C} \longrightarrow \Gamma\backslash D;\ [F] \longmapsto [H^0(Y_F, \Omega^2_{Y_F}) \subset H^2_{\mathrm{var}}(Y_F, \mathbf{C}) \simeq H^2_{\mathrm{var}}(Y_{F_0}, \mathbf{C}) \simeq H_{\mathbf{C}}],$$

where the isomorphism $H^2_{\mathrm{var}}(Y_F, \mathbf{C}) \simeq H^2_{\mathrm{var}}(Y_{F_0}, \mathbf{C})$ is defined by a path from $[F_0]$ to $[F]$ in $\mathcal{C}$. This map gives the period map $\Psi : \mathcal{M} \to \Gamma\backslash D$, where $\mathcal{M} = \mathcal{C}/\mathrm{PGL}(4)$ is the moduli space of nonsingular cubic surfaces.

Theorem 1.1. *The period map Ψ is injective.*

Indeed, this theorem depends on the injectivity of another period map constructed by Allcock-Carlson-Toledo [1]. In the next section, we review the work of Allcock-Carlson-Toledo.

2 The period map by Allcock-Carlson-Toledo

Let $X = X_F \subset \mathbf{P}^3$ be a nonsingular cubic surface defined by $F(x_0, \ldots, x_3)$, and let $V = V_F \subset \mathbf{P}^4$ be the cubic 3-fold defined by the equation $F(x_0, \ldots, x_3) = x_4^3$. Then the natural projection

$$\rho : V \longrightarrow \mathbf{P}^3;\ [x_0 : \cdots : x_3 : x_4] \longmapsto [x_0 : \cdots : x_3]$$

is the triple Galois cover branched along X. Let σ be a generator of the Galois group. Since $H^3(V, \mathbf{Z})$ has no invariant vector by the Galois action, we consider $H^3(V, \mathbf{Z})$ as a $\mathbf{Z}[\omega]$-module of rank 5 by $\omega\alpha = \sigma^*(\alpha)$ for $\alpha \in H^3(V, \mathbf{Z})$, where $\omega \in \mathbf{C}$ denotes a primitive 3-rd root of unity. By using the alternating form

$$\langle\ ,\ \rangle_V : H^3(V, \mathbf{Z}) \times H^3(V, \mathbf{Z}) \longrightarrow H^6(V, \mathbf{Z}) \simeq \mathbf{Z},$$

we define a Hermitian form on $H^3(V, \mathbf{Z})$ by

$$h_V : H^3(V, \mathbf{Z}) \times H^3(V, \mathbf{Z}) \longrightarrow \mathbf{Z}[\omega];\ (\alpha, \beta) \longmapsto \langle \alpha, \omega\beta \rangle_V - \omega \langle \alpha, \beta \rangle_V.$$

Then we have a natural isomorphism of Hermitian space $H^3(V, \mathbf{C})_\omega \simeq \mathbf{C} \otimes_{\mathbf{Z}[\omega]} H^3(V, \mathbf{Z})$, where

$$H^3(V, \mathbf{C})_\omega = \{\alpha \in H^3(V, \mathbf{C}) \mid \sigma^*(\alpha) = \omega\alpha\}$$

is the eigenspace of ω in $H^3(V, \mathbf{C})$ by the action σ^*, and the Hermitian form on $H^3(V, \mathbf{C})_\omega$ is defined by

$$H^3(V, \mathbf{C})_\omega \times H^3(V, \mathbf{C})_\omega \longrightarrow \mathbf{C};\ (\alpha, \beta) \longmapsto (\omega^2 - \omega)\langle \alpha, \bar{\beta} \rangle_V.$$

Let (H', h) be a Hermitian $\mathbf{Z}[\omega]$-lattice which is isomorphic to $(H^3(V, \mathbf{Z}), h_V)$. The period domain of Allcock-Carlson-Toledo is defined by

$$D' = \{E \in \mathrm{Grass}\,(4, \mathbf{C} \otimes_{\mathbf{Z}[\omega]} H') \mid h|_E > 0\},$$

which is isomorphic to the 4-dimensional complex ball

$$\Delta = \{(a_1, \ldots, a_4) \in \mathbf{C}^4 \mid |a_1| + \cdots + |a_4| < 1\}.$$

We fix an isomorphism $(H^3(V_{F_0}, \mathbf{Z}), h_{V_{F_0}}) \simeq (H', h)$. Then an element of D' is defined by

$$H^{2,1}(V_F)_\omega \subset H^3(V_F, \mathbf{C})_\omega \simeq \mathbf{C} \otimes_{\mathbf{Z}[\omega]} H^3(V_F, \mathbf{Z}) \simeq \mathbf{C} \otimes_{\mathbf{Z}[\omega]} H^3(V_{F_0}, \mathbf{Z}) \simeq \mathbf{C} \otimes_{\mathbf{Z}[\omega]} H',$$

where the isomorphism $H^3(V_F, \mathbf{Z}) \simeq H^3(V_{F_0}, \mathbf{Z})$ is defined by a path from $[F_0]$ to $[F]$ in $\mathcal{C}$. This gives a period map

$$\Psi' : \mathcal{M} \longrightarrow \Gamma' \backslash D';\ [F] \longmapsto [H^{2,1}(V)_\omega \subset \mathbf{C} \otimes_{\mathbf{Z}[\omega]} H'],$$

where Γ' is the monodromy group.

Theorem 2.1 (Hartling [4], Allcock-Carlson-Toledo [1]). *The period map Ψ' is injective.*

This theorem depends on the Torelli theorem for cubic 3-folds by Clemens-Griffiths [2] and Tjurin [6].

3 Relation between the period maps Ψ and Ψ'

In this section, we will see that the polarized Hodge structure $(H^2_{\mathrm{var}}(Y, \mathbf{Z}), \langle\ ,\ \rangle_Y)$ is obtained from the Hodge structure of Allcock-Carlson-Toledo $(H^3(V, \mathbf{Z}), h_V)$. Let

(H', h) be a Hermitian $\mathbf{Z}[\omega]$-lattice which is isomorphic to $(H^3(V, \mathbf{Z}), h_V)$. The cyclic group $\mathbf{Z}/3\mathbf{Z}$ acts on H' by

$$\mathbf{Z}/3\mathbf{Z} \times H' \longrightarrow H'; \ ([m], u) \longmapsto \omega^m u,$$

and we have a alternating form on H' by

$$\bigwedge\nolimits^2_{\mathbf{Z}} H' \longrightarrow \mathbf{Z}; \ u \wedge v \longmapsto \frac{1}{\omega^2 - \omega}(h(u, v) - \overline{h(u, v)}).$$

Let $\alpha_0, \ldots, \alpha_4, \beta_0, \ldots, \beta_4$ be a symplectic basis of H'. We set $\theta = \sum_{i=0}^4 \alpha_i \wedge \beta_i \in \bigwedge^2_{\mathbf{Z}} H'$, which does not depend on the choice of the symplectic basis. We define a symmetric form on $\bigwedge^2_{\mathbf{Z}} H'$ by

$$\begin{array}{rccc} \langle \ , \ \rangle_h : & \bigwedge^2_{\mathbf{Z}} H' \times \bigwedge^2_{\mathbf{Z}} H' & \longrightarrow & \bigwedge^{10}_{\mathbf{Z}} H' \simeq \mathbf{Z}; \\ & (u_1 \wedge u_2, v_1 \wedge v_2) & \longmapsto & \frac{1}{6}\theta^{\wedge 3} \wedge u_1 \wedge u_2 \wedge v_1 \wedge v_2. \end{array}$$

We denote by $\left(\bigwedge^2_{\mathbf{Z}} H'\right)_0$ the kernel of the alternating form $\bigwedge^2_{\mathbf{Z}} H' \to \mathbf{Z}$. Then we have a lattice $\left(\left(\bigwedge^2_{\mathbf{Z}} H'\right)_0^{\mathbf{Z}/3\mathbf{Z}}, \langle \ , \ \rangle_h\right)$. We set a $\mathbf{C}$-linear map j by

$$j : \mathbf{C} \otimes_{\mathbf{Z}[\omega]} H' \simeq H'_{\mathbf{C},\omega} \subset H'_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{Z}} H'; \ 1 \otimes v \longmapsto \frac{1}{\omega^2 - \omega}(\omega^2 \otimes v - 1 \otimes \omega v).$$

Then a Hodge structure of Allcock-Carlson-Toledo $E \subset \mathbf{C} \otimes_{\mathbf{Z}[\omega]} H'$ gives a polarized Hodge structure on $\left(\left(\bigwedge^2_{\mathbf{Z}} H'\right)_0^{\mathbf{Z}/3\mathbf{Z}}, \langle \ , \ \rangle_h\right)$ by

$$j(E) \wedge \overline{j(E^{\perp})} \subset \left(\bigwedge\nolimits^2_{\mathbf{C}} H'_{\mathbf{C}}\right)_0^{\mathbf{Z}/3\mathbf{Z}}.$$

Theorem 3.1 ([5]). *There is a natural isomorphism of polarized Hodge structures*

$$\left(\left(\bigwedge\nolimits^2_{\mathbf{Q}} H^3(V, \mathbf{Q})(1)\right)_0^{\mathrm{Gal}(\rho)}, \frac{1}{3}\langle \ , \ \rangle_{h_V}\right) \simeq (H^2_{\mathrm{var}}(Y, \mathbf{Q}), \langle \ , \ \rangle_Y).$$

Let $(H, \langle \ , \ \rangle)$ be a lattice which is isomorphic to $(H^2_{\mathrm{var}}(Y, \mathbf{Z}), \langle \ , \ \rangle_Y)$, and let

$$\iota : \left(\left(\bigwedge\nolimits^2 H'_{\mathbf{Q}}\right)_0^{\mathbf{Z}/3\mathbf{Z}}, \frac{1}{3}\langle \ , \ \rangle_h\right) \simeq (H_{\mathbf{Q}}, \langle \ , \ \rangle).$$

be the isomorphism of lattices given by Theorem 3.1 for a base point $[F_0] \in \mathcal{C}$. Then Theorem 3.1 means that the diagram

$$\begin{array}{ccccc} & & \mathcal{M} & & \\ & {\scriptstyle \Psi'} \swarrow & & \searrow {\scriptstyle \Psi} & \\ \Gamma' \backslash D' & & \xrightarrow[\Phi]{} & & \Gamma \backslash D \end{array}$$

is commutative, where the morphism Φ is defined by

$$\Phi : D' \longrightarrow D; \ E \longmapsto \iota(j(E) \wedge \overline{j(E^{\perp})}).$$

And we can prove that $\Phi : D' \to D$ is injective. These imply Theorem 1.1.

4 Geometry of lines

In this section, we explain the isomorphism in Theorem 3.1. Let $\Lambda(\mathbf{P}^n)$ be the Grassmannian variety of lines in $\mathbf{P}^n$. Let X be a nonsingular cubic surface. We define a subvariety of $\mathbf{P}^3 \times \Lambda(\mathbf{P}^3)$ by

$$Y = \{(p, L) \in \mathbf{P}^3 \times \Lambda(\mathbf{P}^3) \mid \mathrm{mult}_p(L.X) \geq 3\},$$

which is the double cover branched along B by the first projection

$$\phi : Y \longrightarrow X; \ (p, L) \longmapsto p.$$

We define a divisor on Y by

$$Y_\infty = \{(p, L) \in \mathbf{P}^3 \times \Lambda(\mathbf{P}^3) \mid p \in L \subset X\} = L_1^+ \amalg \cdots \amalg L_{27}^+.$$

We remark that X contains 27 lines $L_1, \ldots, L_{27}$ in $\mathbf{P}^3$, and L_i^+ denotes the component of Y_∞ which corresponds to L_i. Then we can prove that

$$H^2_{\mathrm{inv}}(Y, \mathbf{Z}) = \phi^* H^2(X, \mathbf{Z}) + \sum_{i=1}^{27} \mathbf{Z}[L_i^+].$$

Let

$$\pi : Y \longrightarrow Z \subset \Lambda(\mathbf{P}^3); \ (p, L) \longmapsto L$$

be the second projection, where Z denotes its image. Then π is the birational morphism which contracts curves L_i^+, and we have an isomorphism of Hodge structures

$$H^2_{\mathrm{prim}}(Z, \mathbf{Q}) \simeq H^2_{\mathrm{var}}(Y, \mathbf{Q}), \tag{1}$$

where $H^2_{\mathrm{prim}}(Z, \mathbf{Q})$ is the subspace in $H^2(Z, \mathbf{Q})$ orthogonal to the class of the hyperplane section by the Plücker embedding of $\Lambda(\mathbf{P}^3)$.

Next we review some results on Fano surface of lines on cubic 3-folds by Clemens-Griffiths [2] and Tjurin [6]. Let $V \subset \mathbf{P}^4$ be a nonsingular cubic 3-fold, and let

$$S = \{L \in \Lambda(\mathbf{P}^4) \mid L \subset V\}$$

be the Fano surface of lines on V. Then there are isomorphisms of Hodge structures

$$\bigwedge\nolimits^2_{\mathbf{Q}} H^1(S, \mathbf{Q}) \simeq H^2(S, \mathbf{Q}), \tag{2}$$

$$H^3(V, \mathbf{Q})(1) \simeq H^1(S, \mathbf{Q}) \tag{3}$$

by [2], [3] or [6].

Let $X \subset \mathbf{P}^3$ be a nonsingular cubic surface, and let $V \subset \mathbf{P}^4$ be the cubic 3-fold which is the triple Galois cover $\rho : V \to \mathbf{P}^3$ branched along X. If L is a line on V, then the image of the projection $\rho(V)$ is a line of $\mathbf{P}^3$ which is contained in X or intersects X with the multiplicity 3. Therefore we have the triple Galois cover

$$S \longrightarrow Z;\ L \longmapsto \rho(L),$$

and we have an isomorphism of Hodge structures

$$H^2(S, \mathbf{Q})^{\mathrm{Gal}(\rho)} \simeq H^2(Z, \mathbf{Q}). \tag{4}$$

By these isomorphisms (1) – (4), we have the isomorphism in Theorem 3.1.

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