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FAMILIES OF AFFINE CUBIC SURFACES ARISING FROM GENERALIZED MONODROMY DATA

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Dedicated to Professor Sampei Usui on his 60th birthday

ABSTRACT. In this note, we will give a brief summary of geometric approach to understanding equations of Painlevé type ([O], [Sakai], [STT], [IIS1], [In]). Finally, we report the recent result on the moduli space of the generalized monodromy data associated to 10 families of isomonodromic problem related to the classical Painlevé equations in [PSa].

1. GEOMETRIC APPROACH TO EQUATIONS OF PAINLEVÉ TYPES

First of all, we would like to explain about differential equations of Painlevé type and known geometric approach to understand the equations.

1.1. Differential equations of Painlevé type.

A complex algebraic ordinary differential equation is said to have *the Painlevé property*, if its all solutions has no movable singularities other than poles. We call an algebraic ODE which satisfies Painlevé property an *ODE of Painlevé type*. Moreover, we can naturally extend the definition of Painlevé property for a partial differential equation, so we will also use the term "*an equation of Painlevé type*". After a result due to L. Fuchs and H. Poincaré for the first order case, P. Painlevé and his former student B. Gambier classified the rational differential equation of order two $q'' = R(t, q, q')$ which may satisfy Painlevé property into 6 types, $P_I, \cdots, P_{VI}$. We call them the (classical) Painlevé equations.

1.2. Okamoto–Painlevé pairs.

After the work of Okamoto [O], we understand the importance of study the families of spaces of initial conditions (or phase spaces) of classical Painlevé equations, and the relative compactifications of the families. In the works of Sakai [Sakai] and Saito-Takebe-Terajima [STT], the Okamoto compactifications of the initial spaces leads to the notion of Okamoto–Painlevé pair (S, Y), which is a pair of smooth projective rational surface S and an effective anti-canonical divisor $Y \in |-K_S|$ satisfying certain conditions. Then, one can understand the Painlevé equation from the view point of birational symmetries of families of Okamoto-Painlevé pairs or Kodaira-Spencer-Kawamata theory of deformation of Okamoto–Painlevé pairs (S, Y).

1.3. Isomonodromic deformations.

From these links between algebraic geometry and Painlevé equations, one may expect some kind of geometric origins of the equations of Painlevé type, which may clarify the meaning of the Painlevé property, and it is our main motivation in [IIS1] to seek them. Meanwhile, it is known that classical Painlevé equations can be derived from differential equations for isomonodromic deformations of certain linear connections over $\mathbf{P}^1$ with regular or irregular

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singularities (see e.g. [JMU]). This isomonodromic approach has been known to be useful in many fields of mathematics and physics. Moreover, one has proofs of Painlevé property of the differential equations derived from isomonodromic deformations of linear connections [Miw], [Mal2]. However, their proof is rather local which we mean that they only consider a Zariski open set of the phase spaces. On the other hand, Painlevé property is a global property of equations, so one has to consider the global moduli spaces of linear connections for all parameters, which are smooth algebraic varieties. In order to pursue this point of view, one needs very careful studies of the global moduli space of the linear connections with fixed type of regular or irregular singularities.

1.4. Moduli of stable parabolic connections and Riemann-Hilbert correspondence.

In [IIS1], in the regular singular case, we introduce the notion of a stable parabolic connection on a smooth projective curve (cf. [AL]). Then we can prove that there exists a smooth family

$$\pi : \mathcal{M} \longrightarrow T \times \mathcal{N}$$

of the moduli spaces of stable parabolic connections over the spaces $T \times \mathcal{N}$ which parametrizes natural time variables and ordered local exponents (cf. [IIS1], [In]). Note that the natural time variables in T come from the moduli of the curves and the locations of the regular singular points of connections. Moreover, there also exists the moduli space $\mathcal{R}$ of the monodromy representations related to the local system induced by the solutions of linear differential equations with regular singularities. For simplicity, it is natural to define $\mathcal{R}$ as the categorical quotient of a product of linear algebraic groups by the adjoint action, so that $\mathcal{R}$ becomes an affine algebraic variety.

Now it is easy to define the Riemann-Hilbert correspondence $\mathbf{RH} : \mathcal{M} \longrightarrow \mathcal{R}$ and also the correspondence of local exponents to the moduli of the local monodromies $\mathsf{rh} : \mathcal{N} \longrightarrow \mathcal{A}$, and this leads the following commutative diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathbf{RH}} & \tilde{\mathcal{R}} \\ \pi \downarrow & & \downarrow \phi \\ T \times \mathcal{N} & \xrightarrow{(1\times \mathsf{rh})} & T \times \mathcal{A}. \end{array}$$

Here $\phi : \tilde{\mathcal{R}} \longrightarrow T \times \mathcal{A}$ is a local trivial extension of $\mathcal{R} \longrightarrow \mathcal{A}$ over T.

Then in [IIS1] and [In], we prove that for any fixed element $(\mathbf{t}, \nu) \in T \times \mathcal{N}$, the induced map between the fibers

$$\mathbf{RH}_{(\mathbf{t},\nu)} : \mathcal{M}_{(\mathbf{t},\nu)} \longrightarrow \tilde{\mathcal{R}}_{(\mathbf{t},\mathsf{rh}(\nu))} \tag{1}$$

is a *proper subjective analytic bimeromorphic map*. For a general $\nu \in \mathcal{N}$, $\mathbf{RH}_{(\mathbf{t},\nu)}$ gives an analytic isomorphism of smooth fibers. On the other hand, for a special $\nu \in \mathcal{N}$, the fiber $\tilde{\mathcal{R}}_{(\mathbf{t},\mathsf{rh}(\nu))}$ has singularities, but the moduli space $\mathcal{M}_{(\mathbf{t},\nu)}$ is always smooth and ended with a natural holomorphic symplectic structure. So our Riemann-Hilbert correspondence $\mathbf{RH}_{(\mathbf{t},\nu)}$ gives an analytic resolution of singularities for such $\mathcal{R}_{(\mathbf{t},\mathsf{rh}(\nu))}$.

The differential equations with natural time variables in T coming from monodromy preserving deformations of connections is just given by the flatness conditions of some extended connections. So basically it is given by the zero-curvature equation, and so the differential equation becomes non-linear. More geometrically, we can explain as follows. In T direction, the family $\phi : \tilde{\mathcal{R}} \longrightarrow T \times \mathcal{A}$ is locally trivial. Fixing $\nu \in \mathcal{N}$, the pulling back the local trivial sections of $\mathcal{R}_{\mathsf{rh}(\nu)} \longrightarrow T \times \{\mathsf{rh}(\nu)\}$ by $\mathbf{RH}$, we obtain analytic horizontal sections for $\mathcal{M}_{\nu} \longrightarrow T \times \{\nu\}$.

These section gives analytic foliation on $\mathcal{M}_{\nu} \longrightarrow T \times \{\nu\}$, which is nothing but the differential equations for isomonodromic deformations of linear connections.

2. Painlevé equations and Families of Okamoto-Painlevé pairs

2.1. Classification of Painlevé equations (8 types).

Classically, Painlevé equations are classified into 6 types by Painlevé and Gambier. However, from the view point of geometry of phase spaces or Okamoto Painlevé pairs, it is natural to classify them into 8 types as in Table 1 according to Dynkin diagram of affine root systems ($D_r^{(1)}, 4 \leq r \leq 8$, $E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$) (Okamoto [O], Sakai [Sakai], Saito-Takebe-Terajima [STT]).

Notation	Dynkin	Equation
P_I	$E_8^{(1)}$	$\frac{d^2x}{dt^2} = 6x^2 + t$
P_{II}	$E_7^{(1)}$	$\frac{d^2x}{dt^2} = 2x^3 + tx + \alpha$
$P_{III}^{D_6^{(1)}}$	$D_6^{(1)}$	$\frac{d^2x}{dt^2} = \frac{1}{x}\left(\frac{dx}{dt}\right)^2 - \frac{1}{t}\frac{dx}{dt} + \frac{1}{t}(\alpha x^2 + \beta) + 4x^3 - \frac{4}{x}$
$P_{III}^{D_7^{(1)}}$	$D_7^{(1)}$	$\frac{d^2x}{dt^2} = \frac{1}{x}\left(\frac{dx}{dt}\right)^2 - \frac{1}{t}\frac{dx}{dt} + \frac{1}{t}(2x^2 + \beta) - \frac{4}{x}$
$P_{III}^{D_8^{(1)}}$	$D_8^{(1)}$	$\frac{d^2x}{dt^2} = \frac{1}{x}\left(\frac{dx}{dt}\right)^2 - \frac{1}{t}\frac{dx}{dt} + \frac{1}{t}(4x^2 + 4),$
P_{IV}	$E_6^{(1)}$	$\frac{d^2x}{dt^2} = \frac{1}{2x}\left(\frac{dx}{dt}\right)^2 + \frac{3}{2}x^3 + 4tx^2 + 2(t^2 - \alpha)x + \frac{\beta}{x}$
P_V	$D_5^{(1)}$	$\frac{d^2x}{dt^2} = \left(\frac{1}{2x} + \frac{1}{x-1}\right)\left(\frac{dx}{dt}\right)^2 - \frac{1}{t}\frac{dx}{dt} + \frac{(x-1)^2}{t^2}\left(\alpha x + \frac{\beta}{x}\right)$ $+\gamma\frac{x}{t} + \frac{1}{2}\frac{x(x+1)}{x-1},$
P_{VI}	$D_4^{(1)}$	$\frac{d^2x}{dt^2} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-t}\right)\left(\frac{dx}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{x-t}\right)\left(\frac{dx}{dt}\right)$ $\frac{x(x-1)(x-t)}{t^2(t-1)^2}\left(\alpha - \beta\frac{t}{x^2} + \gamma\frac{t-1}{(x-1)^2} + \left(\frac{1}{2} - \delta\right)\frac{t(t-1)}{(x-t)^2}\right)$

Table 1. Classification of Painlevé equations

2.2. Geometry of Painlevé equations and Okamoto–Painlevé pairs.

It is known that the equations P_J are equivalent to a Hamiltonian system for some Hamiltonian function $H_J(x, y, t, \nu)$. Here $\nu$is the set of constants which corresponds to $\alpha, \beta, \cdots$ in Table 1.

(2)
$$(H_J) : \begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H_J}{\partial y}, \\ \dfrac{dy}{dt} = -\dfrac{\partial H_J}{\partial x}, \end{cases}$$

The differential equation (H_J) is then equivalent to the following rational vector field.

$$\tilde{v} = \frac{\partial}{\partial t} + \frac{\partial H_J}{\partial y}\frac{\partial}{\partial x} - \frac{\partial H_J}{\partial x}\frac{\partial}{\partial y} \in H^0(\mathbf{C}^2 \times B_J, \Theta) \subset H^0(\mathbf{P}^2 \times B_J, \Theta(*H)).$$

Moreover, in (H_J), one can take the Hamiltonian $H_J(x,y,t,\boldsymbol{\nu})$ as a polynomial function in x, y and a rational in t. Then fixing a parameter $\boldsymbol{\nu}$, we obtain the following commutative diagram:

$$\begin{array}{ccccc} \mathbf{C}^2 \times B_J & \hookrightarrow & \mathbf{P}^2 \times B_J & \hookleftarrow & H \\ \downarrow & & \downarrow & & \\ B_J & = & B_J & & \end{array} \tag{3}$$

where $B_J = \mathbf{C}, \mathbf{C} \setminus \{0\}$, or $\mathbf{C} \setminus \{0,1\}$. In this notation, it is known that

$$\tilde{v} \in H^0(\mathbf{P}^2 \times B_J, \Theta_{\mathbf{P}^2 \times B_J}(-\log H) \otimes \mathcal{O}(H)). \tag{4}$$

Moreover, at the boundary divisor H, the rational vector field $\tilde{v}$ has *accessible singularities*. Then we can resolve **the accessible singularities** of $\tilde{v}$ on H by the blowings-up

$$\pi : \mathcal{S} \longrightarrow \mathbf{P}^2 \times B_J$$

and obtain

$$\begin{array}{ccc} \mathcal{S} & \hookleftarrow & \mathcal{D} \\ \downarrow & \swarrow & \\ B_J & & \end{array} \tag{5}$$

Finally, we can obtain a rational vector field (=Painlevé equaions)

$$\boxed{\tilde{v} \in H^0(\mathcal{S}, \Theta_{\mathcal{S}}(-\log \mathcal{D}) \otimes \mathcal{O}(\mathcal{D}))} \tag{6}$$

which has no accessible singularities and the foliation of $\tilde{v}$ can be separated at least locally (see Figure 1). This procedures for all classical Painlevé equations were done by Okamoto [O], so one can say that the space $\mathcal{S}$ is the family of spaces of initial conditions of Okamoto or relative compactification of spaces of initial conditions.

2.3. Okamoto–Painlevé pairs.

Recall that we have a family of smooth projective surfaces

$$\begin{array}{ccc} \mathcal{S} & \hookleftarrow & \mathcal{D} \\ \downarrow & \swarrow & \\ B_J & & \end{array} \tag{7}$$

One can see that each fiber $\mathcal{S}_t$ has an anti-canonical divisor $\mathcal{Y}_t \in |-K_{\mathcal{S}_t}|$ with $(\mathcal{Y}_t)_{red} = \mathcal{D}_t$ such that $(\mathcal{S}_t, \mathcal{Y}_t)$ is an Okamoto–Painlevé pairs in the following sense.

Definition 2.1. Let S be a complex projective smooth surface, and $Y \in |-K_S|$ an anticanonical divisor. Let $Y = \sum_{i=1}^r m_i Y_i$ be the irreducible decomposition of Y. Then (S, Y) is said to be an *Okamoto-Painlevé Pair* it and only if the following condition is satisfied.

$$Y \cdot Y_i = \deg Y_{|Y_i} = \deg -K_{S|Y_i} = 0 \quad \text{for all } i, 1 \le i \le r \tag{8}$$

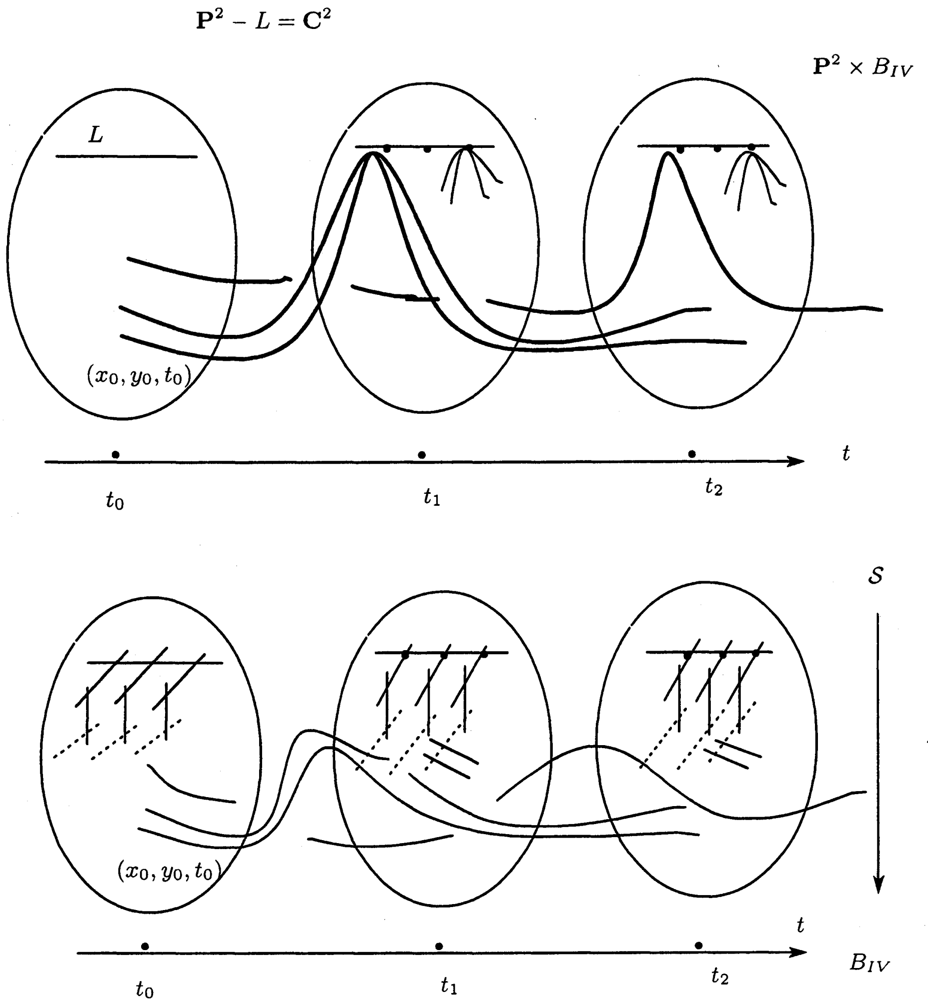

FIGURE 1

Proposition 2.2. *Let (S, Y) be a rational Okamoto-Painlevé pair. Then S can be obtained by 9 points blowings-up of $\mathbf{P}^2$. Moreover*

(1) $\dim |-nK_S| = \dim |nY| \leq 1$ *for all* $n \geq 1$.
(2) *If* $\dim |-nK_S| = \dim |nY| = 1$ *for some* $n \geq 1$, *there exists an elliptic fibraiton* $f : S \longrightarrow \mathbf{P}^1$ *with* $f^*(\infty) = nY$.

Definition 2.3. Let (S, Y) be a rational Okamoto–Painlevé pair. Then (S, Y) is said to be *of fiberd type* if there exists an elliptic fibration $f : S \longrightarrow \mathbf{P}^1$ such that $f^*(\infty) = nY$ for some $n \geq 1$. Otherwise (S, Y) is said to be *of non-fiberd type* . The later is equivalent to $\dim |-nK_S| = 0$ for all $n \geq 1$.

Proposition 2.4. *Let (S, Y) be a rational Okamoto-Painlevé pair such that Y_{red} is a divisor with only normal crossings. Then the type of Y is same as one in the list of Table 2.*

2.4. **Deformation of Okamoto–Painlevé pairs.** After we introduce the notion of Okamoto–Painlevé pairs, in [STT] we prove the following theorem which shows that Painlevé vector field

Y or $R(Y)$	$\tilde{E}_8$	$\tilde{D}_8$	$\tilde{E}_7$	$\tilde{D}_7$	$\tilde{D}_6$	$\tilde{E}_6$	$\tilde{D}_5$	$\tilde{D}_4$	$\tilde{A}_{r-1}$ $1 \leq r \leq 9$	$\tilde{A}_0{}^*$ $r = 1$
Kodaira's notation	II^*	I_4^*	III^*	I_3^*	I_2^*	IV^*	I_1^*	I_0^*	I_r	I_0
Painlevé equation	P_I	$P_{III}^{\tilde{D}_8}$	P_{II}	$P_{III}^{\tilde{D}_7}$	P_{III}	P_{IV}	P_V	P_{VI}	none Difference Eq.	none Difference Eq.
r	9	9	8	8	7	7	6	5	r	1

TABLE 2

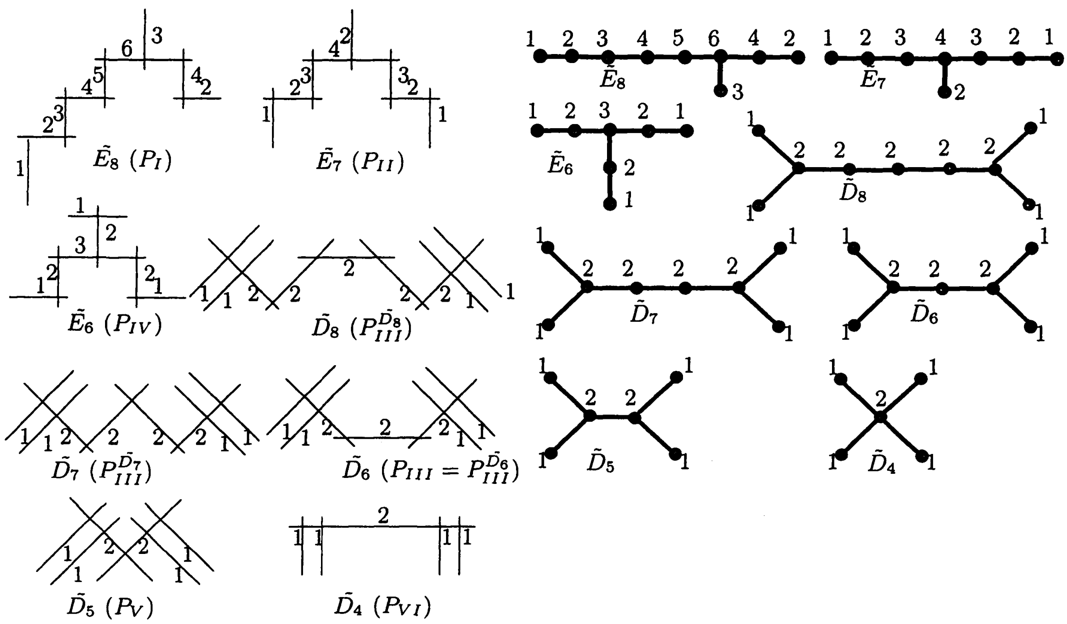

FIGURE 2. Configuration of $-K_S = Y$ for Okamoto-=Painlevé pair (S, Y)

$\tilde{v}$ is the unique rational vector fields on a global semi universal family of Okamoto–Painlevé pairs of each additive type. Moreover the Painlevé vector field can be derived from the special deformation of pairs (S, Y).

Theorem 2.5. *Let $R = R(Y)$ be one of types of the root systems $\tilde{D}_i, 4 \leq i \leq 8$ or $\tilde{E}_j, 6 \leq j \leq 8)$. Let r be the number of irreducible components of $D = Y_{red}$, and set $s = s(R) = 9 - r$. Then there exist Zariski open affine subsets $\mathcal{N}_R \subset \mathbf{C}^s = \operatorname{Spec} \mathbf{C}[\alpha_1, \cdots, \alpha_s]$, $\mathcal{B}_R \subset \mathbf{C} = \operatorname{Spec} \mathbf{C}[t]$, and a commutative diagram*

$$\begin{array}{ccc} \mathcal{S} & \hookleftarrow & \mathcal{D} \\ \pi \downarrow & \swarrow & \varphi \\ \mathcal{N}_R \times \mathcal{B}_R & & . \end{array}$$

which satisfies the following properties.

(1) • *$\mathcal{S}$: a smooth quasi-projective manifold*
• *$\mathcal{D}$: a divisor with normal crossing of $\mathcal{S}$.*

- *π: a smooth and projective morphism, φ is a flat morphism.*

(2)

$$\exists \quad \omega_{\mathcal{S}} \in \Gamma(\mathcal{S}, \Omega^2_{\mathcal{S}/\mathcal{N}_R \times \mathcal{B}_R}(*\mathcal{D}))$$

$\mathcal{Y}$:= the pole divisor of $\omega_{\mathcal{S}}$. $\forall(\boldsymbol{\alpha}, t) \in \mathcal{N}_R \times \mathcal{B}_R$,

$(\mathcal{S}_{\boldsymbol{\alpha},t}, \mathcal{Y}_{\boldsymbol{\alpha},t})$ *a rational O.P. pair of type $R = R(Y)$ (non-fibered type).*

(3)

$$\exists^1 \quad \tilde{v} \in \Gamma(\mathcal{S}, \Theta_{\mathcal{S}}(-\log \mathcal{D}) \otimes \mathcal{O}_{\mathcal{S}}(\mathcal{D}))$$

such that $\pi_*(\tilde{v}) = \frac{\partial}{\partial t}$. (Painlevé vector fields).

(4) *The family is semi universal at a general point $(\boldsymbol{\alpha}, t) \in \mathcal{N}_R \times \mathcal{B}_R$, that is, the Kodaira-Spencer map*

$$\rho : T_{\boldsymbol{\alpha},t}(\mathcal{N}_R \times \mathcal{B}_R) \longrightarrow H^1(\mathcal{S}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t}))$$

is an isomorphism for a general point $(\boldsymbol{\alpha}, t)$.

The Kodaira–Spencer class $\rho(\frac{\partial}{\partial t})$ of t-direction ($= \mathcal{B}_R$) lies in the image of the natural map :

$$\delta : \mathbf{C} \simeq H^1_{\mathcal{D}_{\boldsymbol{\alpha},t}}(\mathcal{S}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t})) \hookrightarrow H^1(\mathcal{S}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t})).$$

(5)

$$\exists \quad \tilde{\omega}_{\mathcal{S}} \in \Gamma(\mathcal{S}, \Omega^2_{\mathcal{S}/\mathcal{N}_R}(\mathcal{Y}))$$

which is a lift of $\omega_{\mathcal{S}}$ and satisfy

$$\iota_{\tilde{v}}(\tilde{\omega}_{\mathcal{S}}) = 0.$$

(6) *There exists affine open subsets $\tilde{U}_i = \mathbf{C}^2 \times \mathcal{N}_R \times B_{\mathbf{R}}$ of $\mathcal{S}$ with canonical coordinates (x_i, y_i) and Hamiltonian functions $H_i(x_i, y_i, \boldsymbol{\alpha}, t)$*

$$\tilde{\omega}_{\mathcal{S}|\tilde{U}_i} = dx_i \wedge dy_i - dt \wedge dH_i(x_i, y_i, \boldsymbol{\alpha}, t)$$

In view of Theorem 2.5, we can expect the following geometric scheme to understand equations of Painlevé type.

Differential equations of Painlevé type	phase space $\Longrightarrow$ $\Longleftarrow$ Kodaira-Spencer	Logarithmic symplectic varieties with certain conditions and special deformations

However, we can also ask the following question.

Question 2.6. *What is more intrinsic meaning of semi universal family of Okamoto-Painlevé pairs $\pi : \mathcal{S} \longrightarrow \mathcal{N}_R \times \mathcal{B}_R$ in Theorem 2.5?*

3. Moduli space of stable parabolic connections

In order to answer Question 2.6 in the former section, we will introduce the notion of stable parabolic connection and explain about the results in [IIS1], [IIS2] and [In]. The answer of the question should be as in the following. *The semi universal family of Okamoto-Painlevé pairs can be constructed by the natural relative compactification of the family of moduli spaces of linear connections with singularities over a curve.* We can prove this statement for regular singular cases in [IIS1] and [In]. For connections with general singularities (regular or irregular singularities), we did not see any difficulty to extend the methods in [IIS1] and [In].

3.1. **Stable parabolic connection.** Let us fix integers $g \geq 0, n > 0, r > 0, d$. Let C be a nonsingular projective curve of genus g and $\mathbf{t} = \{t_1, \cdots, t_n\}$ a set of ordered n-distinct points on C. For a data $\mathbf{t} = \{t_1, \cdots, t_n\}$, we set $D(\mathbf{t}) = t_1 + \cdots + t_n$ the divisor associated to $\mathbf{t}$. In this section, according to [IIS1], [IIS2] and [In], we review known results on the moduli space of stable parabolic connections of rank r and degree d on C with at most regular singularities at $D(\mathbf{t})$.

A logarithmic connection on C with singularity at $D(\mathbf{t})$ is a pair (E, ∇) where E is an algebraic vector bundle on C of rank r and degree d and a morphism ∇ of sheaves

$$\nabla : E \longrightarrow E \otimes \Omega^1_C(D(\mathbf{t})) \tag{9}$$

which satisfies Leibniz rule, i.e., for any local section of $a \in \mathcal{O}_C, \sigma \in E$

$$\nabla(a\sigma) = \sigma \otimes da + a\nabla(\sigma). \tag{10}$$

Let (E, ∇) be a logarithmic connection with singularities at $D(\mathbf{t})$. For each $t_i \in \mathbf{t}$, we can define a residue homomorphism $\mathrm{res}_{t_i}(\nabla) \in \mathrm{End}(E_{|t_i})$ which is a $\mathbf{C}$-linear morphism on $E_{|t_i} \simeq \mathbf{C}^r$. We denote by $\{\nu_0^{(i)}, \nu_1^{(i)}, \cdots, \nu_{r-1}^{(i)}\}$ the set of (ordered) eigenvalues of $\mathrm{res}_{t_i}(\nabla)$ which are called local exponents of ∇ at t_i . Moreover we define the set of all local exponents $\boldsymbol{\nu}$ of (E, ∇) by

$$\boldsymbol{\nu} = (\nu_j^{(i)})^{1 \leq i \leq n}_{0 \leq j \leq r-1}. \tag{11}$$

The following lemma is a generalization of Fuchs relation when $C \simeq \mathbf{P}^1$.

Lemma 3.1. (Fuchs relation) *For a logarithmic connection (E, ∇) with singularity at $D(\mathbf{t})$ as above, let $\boldsymbol{\nu}$ be the set of local exponents as in (11). Then*

$$\sum_{i=1}^{n}(\sum_{j=0}^{r-1} \nu_j^{(i)}) = -\deg E = -\deg \wedge^r E = -d. \tag{12}$$

By this lemma, for each (n, r, d), it is natural to define the set of local exponents by

$$\mathcal{N}_r^n(d) := \left\{ \boldsymbol{\nu} = (\nu_j^{(i)})^{1 \leq i \leq n}_{0 \leq j \leq r-1} \in \mathbf{C}^{nr} \quad \middle| \quad d + \sum_{1 \leq i \leq n} \sum_{0 \leq j \leq r-1} \nu_j^{(i)} = 0 \right\} \simeq \mathbf{C}^{nr-1}. \tag{13}$$

Definition 3.2. For $(C, \mathbf{t})$ and $\boldsymbol{\nu} \in \mathcal{N}_r^n(d)$, a *$\boldsymbol{\nu}$-parabolic connection of rank r and degree d on C* with at most logarithmic singularity at $D(\mathbf{t})$ is a collection of data $(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})$ consisting of:

(1) a logarithmic connection (E, ∇) on C with a singularity at $D(\mathbf{t})$ such that $\mathrm{rank}\, E = r$ and $\deg E = \deg \wedge^r E = d$,
(2) and a filtration $l_*^{(i)} : E_{|t_i} = l_0^{(i)} \supset l_1^{(i)} \supset \cdots \supset l_{r-1}^{(i)} \supset l_r^{(i)} = 0$ for each $i, 1 \leq i \leq n$ such that $\dim(l_j^{(i)}/l_{j+1}^{(i)}) = 1$ and $(\mathrm{res}_{t_i}(\nabla) - \nu_j^{(i)})(l_j^{(i)}) \subset l_{j+1}^{(i)}$ for $j = 0, 1, \cdots, r-1$.

Note that for each fixed i, $1 \leq i \leq n$, $\{\nu_j^{(i)}\}_{0 \leq j \leq r-1}$ is the set of ordered eigenvalues of the residue matrix $\mathrm{res}_{t_i}(\nabla)$, so the parabolic structure $\{l_*^{(i)}\}$ gives the eigenspaces for $\mathrm{res}_{t_i}(\nabla)$.

In order to construct the good moduli space of $\boldsymbol{\nu}$-connections, it is necessary to introduce the stability condition on the $\boldsymbol{\nu}$-parabolic connections $(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})$. Let $\boldsymbol{\alpha} = (\alpha_j^{(i)})^{1 \leq i \leq n}_{1 \leq j \leq r}$ be a sequence of rational numbers such that

$$0 < \alpha_1^{(i)} < \alpha_2^{(i)} < \cdots < \alpha_r^{(i)} < 1 \tag{14}$$

for $i = 1, \dots, n$ and $\alpha_j^{(i)} \neq \alpha_{j'}^{(i')}$ for $(i,j) \neq (i',j')$. We choose $\boldsymbol{\alpha} = (\alpha_j^{(i)})$ sufficiently generic. Let $(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})$ be a $\boldsymbol{\nu}$-parabolic connection, and $F \subset E$ a nonzero subbundle satisfying $\nabla(F) \subset F \otimes \Omega_C^1(D(\mathbf{t}))$. We define integers $\mathrm{length}(F)_j^{(i)}$ by

$$\mathrm{length}(F)_j^{(i)} = \dim(F|_{t_i} \cap l_{j-1}^{(i)})/(F|_{t_i} \cap l_j^{(i)}). \tag{15}$$

Note that $\mathrm{length}(E)_j^{(i)} = \dim(l_{j-1}^{(i)}/l_j^{(i)}) = 1$ for $1 \leq j \leq r$.

Definition 3.3. A $\boldsymbol{\nu}$-parabolic connection $(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})$ is $\boldsymbol{\alpha}$-stable if for any proper nonzero subbundle $F \subsetneq E$ satisfying $\nabla(F) \subset F \otimes \Omega_C^1(D(\mathbf{t}))$, the inequality

$$\frac{\deg F + \sum_{i=1}^m \sum_{j=1}^r \alpha_j^{(i)} \mathrm{length}(F)_j^{(i)}}{\mathrm{rank}\, F} < \frac{\deg E + \sum_{i=1}^n \sum_{j=1}^r \alpha_j^{(i)} \mathrm{length}(E)_j^{(i)}}{\mathrm{rank}\, E} \tag{16}$$

holds.

For a fixed $(C, \mathbf{t})$ and $\boldsymbol{\nu} \in \mathcal{N}_r^n(d)$, let us define the coarse moduli space by

$$\mathcal{M}_{(C,\mathbf{t})}^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, r, n, d) = \left\{(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n}) \;\middle|\; \begin{array}{l} \text{an } \boldsymbol{\alpha}\text{-stable } \boldsymbol{\nu}\text{-parabolic connection} \\ \text{of rank } r \text{ and degree } d \text{ over } C \end{array} \right\} / \simeq . \tag{17}$$

Let $M_{g,n}$ be the moduli space of n-pointed smooth projective curves $(C, \mathbf{t})$. Taking a finite covering $\tilde{M}_{g,n} \longrightarrow M_{g,n}$, we may assume that there exists a universal family $(\mathcal{C}, \tilde{\mathbf{t}}) = (\mathcal{C}, \tilde{t_1}, \cdots, \tilde{t_n})$ over $\tilde{M}_{g,n}$. We have the following fundamental result (cf. [IIS1], [In]).

Theorem 3.4. *For sufficiently generic weight $\boldsymbol{\alpha}$, there exists a relative fine moduli scheme*

$$\pi : \mathcal{M}_{(\mathcal{C},\tilde{\mathbf{t}})/\tilde{M}_{g,n} \times \mathcal{N}_r^n(d)}^{\boldsymbol{\alpha}}(r, d, n) \longrightarrow \tilde{M}_{g,n} \times \mathcal{N}_r^n(d) \tag{18}$$

of $\boldsymbol{\alpha}$-stable parabolic connection of rank r and degree d, which is smooth and quasi-projective. The fiber of π over $((C, \mathbf{t}), \boldsymbol{\nu}) \in \tilde{M}_{g,n} \times \mathcal{N}_r^n(d)$ is isomorphic to the moduli space $\mathcal{M}_{(C,\mathbf{t})}^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, r, n, d)$ in (17). The moduli space $\mathcal{M}_{(C,\mathbf{t})}^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, r, n, d)$ is a smooth quasi-projective algebraic scheme of dimension $2r^2(g-1) + nr(r-1) + 2$.

Remark 3.5. We can also introduce the notion of stable $\boldsymbol{\nu}$-parabolic ϕ-connections on $(C, \mathbf{t})$. The moduli space of the objects gives a compactification $\overline{\mathcal{M}^{\boldsymbol{\alpha}}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, n, d)$of $\mathcal{M}_{(C,\mathbf{t})}^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, r, n, d)$. (See [IIS1], [In]).

3.2. **Fixing the determinant-SL_r-case.** Under the same notation as above, let us consider the case $r = 1$. Let (L, ∇) be a line bundle on C with a logarithmic connection $\nabla : L \longrightarrow L \otimes \Omega_C^1(D(\mathbf{t}))$. At each singular point t_i, we have a trivial parabolic structure $1_{\mathbf{t}} = \{l_*^{(i)}\}$ by $L_{|t_i} = l_0^{(i)} \supset l_1^{(i)} = 0$. Moreover, for any weight $\boldsymbol{\alpha}$, a parabolic connection $(L, \nabla) = (L, \nabla, 1_{\mathbf{t}})$ with the trivial parabolic structure is α-stable, hence we do not specify the weight and stability conditions for the case of rank 1.

For $\boldsymbol{\nu}' = (\nu_1, \cdots, \nu_n) \in \mathcal{N}_1^n(d)$, the moduli space $\mathcal{M}_{(C,\mathbf{t})}(\boldsymbol{\nu}, 1, n, d)$ of the isomorphism class of $\boldsymbol{\nu}'$-parabolic connection $(L, \nabla, 1_{\mathbf{t}})$ is defined in the same way as above.

For an exponent $\boldsymbol{\nu} = (\nu_j^{(i)})_{0 \leq j \leq r-1}^{1 \leq i \leq n} \in \mathcal{N}_r^n(d)$, we define the trace of the exponent

$$\mathrm{tr}(\boldsymbol{\nu}) = (\sum_{j=0}^{r-1} \nu_j^{(1)}, \cdots, \sum_{j=0}^{r-1} \nu_j^{(n)}) \in \mathcal{N}_1^n(d),$$

which induces the morphism tr : $\mathcal{N}_r^n(d) \longrightarrow \mathcal{N}_1^n(d)$. We can obtain the following natural morphism between two moduli spaces

$$\begin{array}{rccc} \det & : \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, n, d) & \longrightarrow & \mathcal{M}_{(C,\mathbf{t})}(\mathrm{tr}(\boldsymbol{\nu}), 1, n, d) \\ & (E, \nabla, \{l_*^{(i)}\}_{1 \le i \le n}) & \mapsto & (\wedge^r E, \wedge^r \nabla) \end{array} \tag{19}$$

For $\boldsymbol{\nu}' \in \mathcal{N}_1^n(d)$, define

$$\mathcal{N}_r^n(d)(\boldsymbol{\nu}') = \mathrm{tr}^{-1}(\boldsymbol{\nu}') \subset \mathcal{N}_r^n(d)$$

For any $(L, \nabla_1) \in \mathcal{M}_{(C,\mathbf{t})}(\boldsymbol{\nu}', 1, n, d)$ and $\boldsymbol{\nu} \in \mathcal{N}_r^n(d)(\boldsymbol{\nu}')$, we define the submoduli space of $\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, n, d)$ by

(20)
$$\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, n, (L, \nabla_1)) = \det{}^{-1}((L, \nabla_1)) = \{(E, \nabla, \{l_*^{(i)}\}_{1 \le i \le n}) | (\wedge^r E, \wedge^r \nabla) \simeq (L, \nabla_1)\} / \simeq .$$

The moduli space $\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, n, (L, \nabla_1))$ can be considered as the moduli space of $\boldsymbol{\alpha}$-parabolic connection with the fixed determinant (L, ∇_1). From Theorem 3.4, one can easily see the following

Theorem 3.6. *The moduli space $\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, n, (L, \nabla_1))$ is a smooth quasiprojective scheme of dimension $2r^2(g-1) + nr(r-1) + 2 - 2g$.*

4. The Riemann-Hilbert correspondences and Painlevé property for Isomonodromic Flows

4.1. **Moduli space of monodromy representations.** For each n-pointed curve $(C, \mathbf{t}) = (C, t_1, \cdots, t_n) \in T = \mathcal{M}'_{g,n}$ $(g \ge 0, n \ge 1)$, set $D(\mathbf{t}) = t_1 + \cdots + t_n$. By abuse of notation, we denote by $\pi_1(C \setminus D(\mathbf{t}), *)$ the fundamental group of $C \setminus \{t_1, \cdots, t_n\}$ with a starting point $* \in C$. The set

$$\mathrm{Hom}(\pi_1(C \setminus D(\mathbf{t}), *), GL_r(\mathbf{C})) \tag{21}$$

of $GL_r(\mathbf{C})$-representations of $\pi_1(C \setminus D(\mathbf{t}), *)$ is an affine variety, on which $GL_r(\mathbf{C})$ naturally acts by the adjoint action. It is natural to define the moduli space by

$$\mathcal{RP}^r_{(C,\mathbf{t})} = \mathrm{Hom}(\pi_1(C \setminus D(\mathbf{t}), *), GL_r(\mathbf{C})) // Ad(GL_r(\mathbf{C})), \tag{22}$$

where the quotient // means the categorical quotient. More precisely, since $\pi_1(C \setminus D(\mathbf{t}), *)$ is generated by $(2g+n)$-elements $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_n$ with one relation

$$\prod_{i=1}^{g} [\alpha_i, \beta_i] \gamma_1 \cdots \gamma_n = 1,$$

the ring R of invariants of the simultaneous adjoint action of $GL_r(\mathbf{C})$ on the coordinate ring of $GL_r(\mathbf{C})^{2g+n-1}$, then we have an isomorphism

$$\mathcal{RP}^r_{(C,\mathbf{t})} \simeq \mathrm{Spec}(R). \tag{23}$$

Hence the moduli space $\mathcal{RP}^r_{(C,\mathbf{t})}$ becomes an affine algebraic scheme. Furthermore, each closed point of $\mathcal{RP}^r_{(C,\mathbf{t})}$ corresponds to a Jordan equivalence class of a representation (cf. [Section 4, [IIS1]], [Proposition 6.1, [Sim2]]).

Let us set

$$\mathcal{A}_r^{(n)} := \left\{ \mathbf{a} = (a_j^{(i)})^{1 \le i \le n}_{0 \le j \le r-1} \in \mathbf{C}^{nr} \,\middle|\, a_0^{(1)} a_0^{(2)} \cdots a_0^{(n)} = (-1)^{rn} \right\}. \tag{24}$$

For each $\mathbf{a} = (a_j^{(i)}) \in \mathcal{A}_r^{(n)}$ and $i, 1 \leq i \leq n$, we set $\mathbf{a}^{(i)} = (a_0^{(i)}, \cdots, a_{r-1}^{(i)})$ and define

(25) $$\chi_{\mathbf{a}^{(i)}}(s) = s^r + a_{r-1}^{(i)} s^{r-1} + \cdots + a_0^{(i)}.$$

Moreover we define a morphism

(26) $$\phi^r_{(C,\mathbf{t})} : \mathcal{RP}^r_{(C,\mathbf{t})} \longrightarrow \mathcal{A}_r^{(n)}$$

by the relation

(27) $$\det(sI_r - \rho(\gamma_i)) = \chi_{\mathbf{a}^{(i)}}(s)$$

where $[\rho] \in \mathcal{RP}^r_{(C,\mathbf{t})}$ and γ_i is a counterclockwise loop around t_i.

For $\mathbf{a} = (a_j^{(i)}) \in \mathcal{A}_r^{(n)}$, we denote by $\mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}$ the fiber of $\phi^r_{(C,\mathbf{t})}$ over $\mathbf{a}$, that is,

(28) $$\mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}} = \{[\rho] \in \mathcal{RP}^r_{(C,\mathbf{t})} | \det(sI_r - \rho(\gamma_i)) = \chi_{\mathbf{a}^{(i)}}(s), 1 \leq i \leq n\}.$$

For any covering $T' \to T$, we can define a relative moduli space $\mathcal{RP}^r_{n,T'} = \coprod_{(C,\mathbf{t}) \in T'} \mathcal{RP}^r_{(C,\mathbf{t})}$ of representations with the natural morphism

(29) $$\mathcal{RP}^r_{n,T'} \longrightarrow T'.$$

As in Section 4, [IIS1], there exists a finite covering $T' \longrightarrow T$ with the morphism

(30) $$\phi^r_n : \mathcal{RP}^r_{n,T'} \longrightarrow T' \times \mathcal{A}_r^{(n)},$$

such that

$$(\phi^r_n)^{-1}((C,\mathbf{t}),\mathbf{a}) = \mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}.$$

4.2. **Riemann-Hilbert correspondences.**

Next we define the Riemann-Hilbert correspondence from the moduli space of $\boldsymbol{\alpha}$-stable parabolic connections to the moduli space of the representations.

Let us fix positive integers r, d, $\boldsymbol{\alpha} = (\alpha_j^{(i)})$ as in (14), and $(C,\mathbf{t}) \in T' = \mathcal{M}'_{g,n}$. For simplicity, we set $\mathcal{M}^{\boldsymbol{\alpha}}_{((C,\mathbf{t}),\boldsymbol{\nu})} = \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, n, d)$ (cf. (17)).

We define a morphism

(31) $$\mathsf{rh} : \mathcal{N}_r^{(n)}(d) \longrightarrow \mathcal{A}_r^{(n)}, \quad \mathsf{rh}(\boldsymbol{\nu}) = \mathbf{a}$$

by the relation

(32) $$\prod_{j=0}^{r-1}(s - \exp(-2\pi\sqrt{-1}\nu_j^{(i)})) = s^r + a_{r-1}^{(i)} s^{r-1} + \cdots + a_0^{(i)}.$$

For each member $(E, \nabla, \{l_j^{(i)}\}) \in \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t}),\boldsymbol{\nu}}$, the solution subsheaf of E^{an}

(33) $$\ker(\nabla^{an}|_{C \setminus D(\mathbf{t})}) \subset E^{an}$$

becomes a local system on $C \setminus D(\mathbf{t})$ and corresponds to a representation

(34) $$\rho : \pi_1(C \setminus \{\mathbf{t}\}, *) \longrightarrow GL_r(\mathbf{C}).$$

Since the eigenvalues of the residue matrix of ∇^{an} at t_i are $\nu_j^{(i)}$, $0 \leq j \leq r-1$, considering the local fundamental solutions of $\nabla^{an} = 0$ near t_i, the monodromy matrix of $\rho(\gamma_i)$ has eigenvalues $\exp(-2\pi\sqrt{-1}\nu_j^{(i)})$, $0 \leq j \leq r-1$. Hence under the relation (32), or $\mathbf{a} = \mathsf{rh}(\boldsymbol{\nu})$, one can define a morphism

(35) $$\mathbf{RH}_{(C,\mathbf{t}),\boldsymbol{\nu}} : \mathcal{M}^{\boldsymbol{\alpha}}_{((C,\mathbf{t}),\boldsymbol{\nu})} \longrightarrow \mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}.$$

Replacing $T = \mathcal{M}'_{g,n}$ by a certain finite étale covering $u : T' \longrightarrow T$ and varying $((C,\mathbf{t}),\nu) \in T' \times \mathcal{N}_r^{(n)}(d)$ we can define a morphism

$$\mathbf{RH} : \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})/T'}(r,n,d) \longrightarrow \mathcal{RP}^r_{n,T'} \tag{36}$$

which makes the diagram

$$\begin{array}{ccc} \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\tilde{\mathbf{t}})/T'}(r,n,d) & \xrightarrow{\mathbf{RH}} & \mathcal{RP}^r_{n,T'} \\ {\scriptstyle \varphi_{r,n,d}}\downarrow & & \downarrow{\scriptstyle \phi^r_n} \\ T' \times \mathcal{N}_r^{(n)}(d) & \xrightarrow{Id \times \mathsf{rh}} & T' \times \mathcal{A}_r^{(n)} \end{array} \tag{37}$$

commute. The following result is proved in [In]. (See also [IIS1]).

Theorem 4.1. *Assume that* $\boldsymbol{\alpha}$ *is so generic that* $\boldsymbol{\alpha}$*-stable* $\Leftrightarrow$ $\boldsymbol{\alpha}$*-semistable. Moreover we assume that* $r \geq 2, rn - 2r - 2 > 0$ *if* $g = 0$, $n \geq 2$ *if* $g = 1$ *and* $n \geq 1$ *if* $g \geq 2$. *Then the morphism*

$$\mathbf{RH} : \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\tilde{\mathbf{t}})/T'}(r,n,d) \longrightarrow \mathcal{RP}^r_{n,T'} \times_{\mathcal{A}_r^{(n)}} \mathcal{N}_r^{(n)}(d) \tag{38}$$

induced by (36) is a proper surjective bimeromorphic analytic *morphism. In particular, for each* $((C,\mathbf{t}),\boldsymbol{\nu}) \in T' \times \mathcal{N}_r^{(n)}(d)$, *the restricted morphism*

$$\mathbf{RH}_{((C,\mathbf{t}),\boldsymbol{\nu})} : \mathcal{M}^{\boldsymbol{\alpha}}_{((C,\mathbf{t}),\boldsymbol{\nu})}(r,n,d) \longrightarrow \mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}} \tag{39}$$

gives an analytic resolution of singularities of $\mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}$ *where* $\mathbf{a} = rh(\boldsymbol{\nu})$.

Remark 4.2. Take $\boldsymbol{\nu} \in \mathcal{N}_r^{(n)}(d)$ such that $rh(\boldsymbol{\nu}) = \mathbf{a}$. A representation ρ such that $[\rho] \in \mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}$ is said to be *resonant* if

$$\dim(\ker(\rho(\gamma_i) - \exp(-2\pi\sqrt{-1}\nu_j^{(i)}))) \geq 2 \text{ for some } i,j. \tag{40}$$

The singular locus of $\mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}$ is given by the set

$$\left(\mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}\right)^{sing} := \left\{ [\rho] \in \mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}} \;\middle|\; \begin{array}{l} \rho \text{ is reducible or} \\ \text{resonant} \end{array} \right\}. \tag{41}$$

Moreover we denote the smooth part of $\mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}$ by

$$\left(\mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}\right)^{\sharp} = \mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}} \setminus \left(\mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}\right)^{sing}. \tag{42}$$

Theorem 4.1 implies that the restriction

$$\mathbf{RH}_{((C,\mathbf{t}),\boldsymbol{\nu})|\left(\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t}),\boldsymbol{\nu}}\right)^{\sharp}} : \left(\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t}),\boldsymbol{\nu}}\right)^{\sharp} \xrightarrow{\simeq} \left(\mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}\right)^{\sharp} \tag{43}$$

is an analytic isomorphism, where

$$\left(\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t}),\boldsymbol{\nu}}\right)^{\sharp} = \mathbf{RH}^{-1}_{((C,\mathbf{t}),\boldsymbol{\nu})}(\left(\mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}\right)^{\sharp}).$$

4.3. **Isomonodromic Flows and their Pailevé property.**

Let us fix $\boldsymbol{\nu} \in \mathcal{N}_r^{(n)}(d)$ and set $\mathbf{a} = \mathsf{rh}(\boldsymbol{\nu}) \in \mathcal{A}_r^{(n)}$. Then restricting $\mathbf{RH}$ to over the base $T' \times \{\boldsymbol{\nu}\}$ we obtain the following diagram.

$$\begin{array}{ccc} \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\tilde{\mathbf{t}})/T'}(r,n,d)_{\boldsymbol{\nu}} & \xrightarrow{\mathbf{RH}_{\boldsymbol{\nu}}} & \mathcal{RP}^r_{n,T',\mathbf{a}} \\ {\scriptstyle \varphi_{r,n,d}}\downarrow & & \downarrow{\scriptstyle \phi^r_n} \\ T' \times \{\boldsymbol{\nu}\} & \xrightarrow{Id \times \mathsf{rh}} & T' \times \{\mathbf{a}\} \end{array} \tag{44}$$

Isomonodromic flows on the phase space $\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\tilde{\mathbf{t}})/T'}(r,n,d)_{\boldsymbol{\nu}}$ is defined by the pullback of the locally constant flows on the right hand side of (44) by $\mathbf{RH}_{\boldsymbol{\nu}}$. The property of $\mathbf{RH}_{\boldsymbol{\nu}}$ proved in Theorem 4.1 gives the following theorem, which shows that for isomonodromic flows arising from linear connections on curves with at most logarithmic singularities have the Painlevé property ([IIS1], [In]).

Theorem 4.3. *The isomonodromic flows associated to the (linear) stable parabolic connections on $\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\tilde{\mathbf{t}})/T'}(r,n,d)_{\boldsymbol{\nu}}$ satisfies Painlevé property .*

Note that there are works on Painlevé property for isomonodromic flows by Miwa [Miw] and Malgrange [Mal1], [Mal2]. However their treatments of the phase spaces are not sufficient for rigorous proofs. We may point out that if one does not consider parabolic structures and stability condition, one cannot have a smooth family of moduli spaces, in particular for the case of resonant local exponents.

When $C = \mathbf{P}^1$, $T = \{(0,1,t,\infty)\} = \mathbf{P}^1 \setminus \{0,1,\infty\}$ (hence $n = 4$), $r = 2$, and SL-case, the family of moduli spaces of stable parabolic connections $\mathcal{M}^{\alpha}(2,4,-1) \longrightarrow T \times \mathcal{N}_2^{(4)}(-1)^{sl}$gives an semi universal family of the open parts of Okamoto–Painlevé pairs (cf. [IIS2], [IISA]). (In fact, a family of moduli spaces of associated stable parabolic ϕ-connection gives semi universal family of Okamoto–Painlevé pairs). Moreover the set of all isomonodromic flows are equivalent to Painlevé equations of type P_{VI} [IIS2]. Thus we obatin a rigorous proof of the following

Corollary 4.4. *Painlevé equations of types P_{VI} satisfy Painlevé property.*

5. 10 FAMILIES OF LINEAR RANK 2 CONNECTIONS ON $\mathbf{P}^1$ WITH SINGULARITIES

We have the sufficient geometric scheme to prove the Painlevé property of isomonodromic flows corresponding to linear connections with regular singularities like Painlevé equations of type P_{VI}. In order to include other classical Painlevé equations (order two differential equations) in Table 1 into this scheme, one needs to consider the linear connections with irregular and regular singularities.

We will restrict our consideration to the rank 2 sl-connections on $\mathbf{P}^1$. Then we will have 10 families of moduli spaces of linear connections on rank 2 bundles of degree 0 over $\mathbf{P}^1$ with singularities satisfying the following two conditions.

(1) The moduli space of connections with fixed formal types has dimension two.
(2) The moduli space has a natural one dimensional time parameter which generalized Riemann-Hilbert correspondence forgets like locations of regular singular points.

The families satisfying these two conditions can be classified by the Katz invariant $r(s)$ at 4 points $s \in \{0,1,t,\infty\}$. We have 10 families of moduli spaces of connections over $\mathbf{P}^1$ as in Table 3. Note that the parameters in the space $\mathcal{N}$ are essentially given by Eigen values of formal monodromies at each singular point. For detail, see [PSa]. In [PSa], we also give an explicit family of connections which correspond to points in a Zariski open set of moduli space of connections, and corresponding isomonodromic equations.

Let us review former related works. In [JMU], Jimbo, Miwa and Ueno developed a theory of monodromy preserving deformation on linear connections on $\mathbf{P}^1$ with irregular singularities at most lever one. Then in [JM], Jimbo, Miwa treated six explicit isomonodromic families of connections of rank 2 with Katz invariants $(0,0,0,0)$, $(0,0,1)$, $(1,-,1)$, $(0,-,2)$, $(0,-,3)$, $(-,-,5/2)$ and derived Painlevé equations of six types P_J, $J = I, \cdots, VI$. Flaschka, Newell

Dynkin	Painlevé equation	$r(0)$	$r(1)$	$r(\infty)$	$r(t)$	$\dim \mathcal{N} = \dim \mathcal{A}$
$\tilde{D}_4$	P_{VI}	0	0	0	0	4
$\tilde{D}_5$	P_V	0	0	1	-	3
$\tilde{D}_6$	deg $P_V = P_{III}^{\tilde{D}_6}$	0	0	1/2	-	2
$\tilde{D}_6$	$P_{III}^{\tilde{D}_6}$	1	-	1	-	2
$\tilde{D}_7$	$P_{III}^{\tilde{D}_7}$	1/2	-	1	-	1
$\tilde{D}_8$	$P_{III}^{\tilde{D}_8}$	1/2	-	1/2	-	0
$\tilde{E}_6$	P_{IV}	0	-	2	-	2
$\tilde{E}_7$	$P_{IIFN} = P_{II}$	0	-	3/2	-	1
$\tilde{E}_7$	P_{II}	-	-	3	-	1
$\tilde{E}_8$	P_I	-	-	5/2	-	0

TABLE 3

[FN] treated the case $(0, -, 3/2)$, and derived the Painlevé equation P_{IIFN} which is equivalent to the original P_{II}. Ohyama and Okumura [OO] extend the result in [?] and then they also obtained 10 families with degeneration scheme from Painlevé VI. We note that three other cases $(0,0,1/2), (1/2,-,1), (1/2,-,1/2)$ correspond to each equations of type degenerate Painlevé P_V, $P_{III}^{\tilde{D}_7}$, $P_{III}^{\tilde{D}_8}$ (for detail see e.g., [OO], [OKSO], [PSa]).

5.1. **The moduli spaces of generalized monodromy and 10 families of affine cubic surfaces.** As in Table 3, we have 10 families of the moduli spaces of linear rank 2 connection with singularities with fixed type of Katz invariants. Locally at each singularity, the analytic isomorphism class of singularities are given by Stokes data and formal monodromy. Local topological monodromy around a singularity can be determined by Stokes data and formal monodromy, and with these data and data of links which connect the spaces of formal solutions at two different singular points determine the generalized monodromy data. The moduli space $\mathcal{R}$ of these generalized monodromy data can be constructed by a categorical quotients as in the case of regular singularities. Moreover we have the moduli space $\mathcal{A}$ of formal monodromy at each singular point, and then we have a natural morphism $\mathcal{R} \longrightarrow \mathcal{A}$ between two affine varieties. For the type of Katz invariant $(1,1,1,1)$, the moduli space $\mathcal{R}$ is calculated by Fricke-Klein [FK65], who gives a family of affine cubic surfaces (cf. [Iw1, Iw2]). In other cases, in [PSa], we calculate the moduli spaces $\mathcal{R}$ and obtain the following families of affine cubic surfaces in each of 10 types.

(0,0,0,0). P_{VI}. $\frac{d}{dz} + \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t}$, all $tr(A_*) = 0$.

$x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 - s_1x_1 - s_2x_2 - s_3x_3 + s_4 = 0$, (Fricke-Klein cubic [FK65])
$s_i = a_ia_4 + a_ja_k$, $(i,j,k) =$ a cyclic permutation of $(1,2,3)$,
$s_4 = a_1a_2a_3a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4$ with $a_1, a_2, a_3, a_4 \in \mathbb{C}$.

(0,0,1). P_V. $\frac{d}{dz} + \frac{A_0}{z} + \frac{A_1}{z-1} + t/2 \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, all $tr(A_*) = 0$.

$x_1x_2x_3 + x_1^2 + x_2^2 - (s_1 + s_2s_3)x_1 - (s_2 + s_1s_3)x_2 - s_3x_3 + s_3^2 + s_1s_2s_3 + 1 = 0$ with $s_1, s_2 \in \mathbb{C}$, $s_3 \in \mathbb{C}^*$.

(0,0,1/2). deg $P_V = P_{III}^{\tilde{D}_6}$. $\frac{d}{dz} + \frac{A_0}{z} + \frac{A_1}{z-1} + \begin{pmatrix} 0 & tq \\ 0 & 0 \end{pmatrix}$, all $tr(A_*) = 0$.

$x_1x_2x_3 + x_1^2 + x_2^2 + s_0x_1 + s_1x_2 + 1 = 0$ with $s_0, s_1 \in \mathbb{C}$.

(1,-,1). $P_{III}^{\tilde{D}_6}$. $z\frac{d}{dz} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z^{-1} + A_0 + A_1 z$, all $tr(A_*) = 0$.

$x_1x_2x_3 + x_1^2 + x_2^2 + (1+\alpha\beta)x_1 + (\alpha+\beta)x_2 + \alpha\beta = 0$ with $\alpha, \beta \in \mathbb{C}^*$.

(1/2,-,1). $P_{III}^{\tilde{D}_7}$. $z\frac{d}{dz} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z^{-1} + A_0 + A_1 z$, all $tr(A_*) = 0$.

$x_1x_2x_3 + x_1^2 + x_2^2 + \alpha x_1 + x_2 = 0$ with $\alpha \in \mathbb{C}^*$.

(1/2,-,1/2). $P_{III}^{\tilde{D}_8}$. $z\frac{d}{dz} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z^{-1} + \begin{pmatrix} a & -te \\ 1 & -a \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -e^{-1} & 0 \end{pmatrix} z$.

$x_1x_2x_3 + x_1^2 - x_2^2 - 1 = 0$.

(0,-,2). P_{IV} . $z\frac{d}{dz} + A_0 + A_1 z + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z^2$.

$x_1x_2x_3 + x_1^2 - (s_2^2 + s_1s_2)x_1 - s_2^2x_2 - s_2^2x_3 + s_2^2 + s_1s_2^3$ with $s_1 \in \mathbb{C}$, $s_2 \in \mathbb{C}^*$.

(0,-,3/2). $P_{IIFN} = P_{II}$. $z\frac{d}{dz} + \begin{pmatrix} a & c \\ -b & -a \end{pmatrix} + \begin{pmatrix} 0 & 2t+b \\ 1 & 0 \end{pmatrix} z + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z^2$

$x_1x_2x_3 + x_1 - x_2 + x_3 + s = 0$, with $s \in \mathbb{C}$.

(-,-,3). P_{II}. $\frac{d}{dz} + A_0 + A_1 z + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z^2$, all $tr(A_*) = 0$.

$x_1x_2x_3 - x_1 - \alpha x_2 - x_3 + \alpha + 1 = 0$ with $\alpha \in \mathbb{C}^*$.

(-,-,5/2). P_I. $\frac{d}{dz} + \begin{pmatrix} a & 2t+d^2 \\ -d & -a \end{pmatrix} + \begin{pmatrix} 0 & d \\ 1 & 0 \end{pmatrix} z + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z^2$.

$x_1x_2x_3 + x_1 + x_2 + 1 = 0$.

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