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A local solution for some PDEs with hysteresis and some problems

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1 Introduction

In this paper, we report that the existence of a solution of the following nonlinear PDE system with hysteresis operator:

$$w_t - \varepsilon\Delta\tilde{g}(w) - \mathrm{div}[g_1(w)\nabla h] = 0 \text{ in } \Omega\times(0,T), \tag{1.1}$$

$$h_t - \nu\Delta h + \partial I_w h \ni 0 \text{ in } \Omega\times(0,T), \tag{1.2}$$

$$h(x,-L) = h(x,L) = w(x,-L) = w(x,L) = 0 \text{ for } t\in(0,T), \tag{1.3}$$

$$w(0) = w_0,\ h(0) = h_0 \text{ in } \Omega, \tag{1.4}$$

where $\Omega = (-L, L)$ is a one dimensional interval and L is a positive constant. In (1.1) and (1.2), ε, ν are positive constants and $\tilde{g}$ and $g_1$is a smooth function on $\mathbf{R}$, Here, I_w is the indicater function on a closed interval $[f_a(w), f_d(w)]$ for given non-decreasing functions $f_a, f_d \in C^2(\mathbf{R})$ with $f_a \le f_d$ on $\mathbf{R}$ (see Fig.1) given by

$$I_w(h) = \begin{cases} 0 & \text{if } f_a(w) \le h \le f_d(w), \\ +\infty & \text{if } h < f_a(w) \text{ and } f_d(w) < h. \end{cases} \tag{1.6}$$

Upper Function f_a and lower function f_d represents the relation between the input function w and output function h. Also, the subdifferential ∂I_w of the indicater function I_w describes a hysteresis effects, and is a multivalued mapping given by

$$\partial I_w(h) = \begin{cases} \phi & \text{if } h < f_a(w) \text{or } f_d(w) < h, \\ (-\infty, 0] & \text{if } h = f_a(w) < f_d(w), \\ 0 & \text{if } f_a(w) < h < f_d(w), \\ [0, +\infty) & \text{if } f_a(w) < f_d(w) = h, \\ \mathbf{R} & \text{if} w = f_a(h) = f_d(h). \end{cases} \tag{1.7}$$

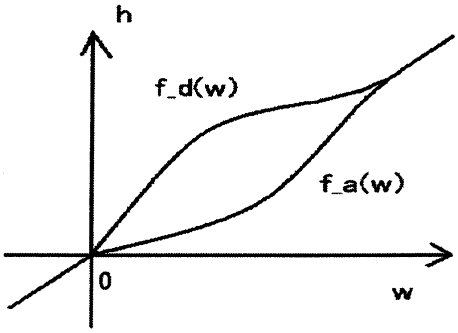

図 1: the graph of f_a and f_d

We are interested in concrete carbonation process, and focused on moisture flow in this process. The problem (P) is the first step to consider moisture flow in concrete, and the system with $\varepsilon = \nu = 0$ is a model of the moisture flow under simplified several conditions.

First, we explain concrete carbonation briefly. Concrete has innumerical small spaces, since concrete is harding of sand, gravel, aggregate with cement paste. In particular, there exists small spaces such that liquid water still remains because of liquids of cement paste. In the case of exposure concrete, by carbon dioxide in air and water in small space, the following chemical reaction occurs in the small space:

$$CO_2 + Ca(OH)_2 \to CaCO_3 + H_2O \quad \text{in water.}$$

$Ca(OH)_2$, which shows alkalinity, is main ingredients of concrete. By H_2O generated by this reaction flow in concrete, the potential of hydrogen of the whole concrete changes from alkalinity to acidly, which is called concrete carbonation.

In this process, H_2O is an important element in order to evolute carbonation. Focused on the flow of H_2O, the hysteresis effect that small space is drying or wetting by interaction between vapor and liquid water is observed. More detail of this point is noted by [7] and [6]. To our best knowledge, the initial work for modeling of concrete carbonation and mathematical analysis thereof is Muntean [8] and Aiki-Muntean [1]. They derive a model in one dimension case, and show the existence of a solution. However, in this model, this hysteresis effects does not contain. Attention to this hysteresis effect, Maekawa, Ishida and Kishi [7] and Maekawa Chaube and Kishi [6] derive a model of concrete carbonation. The following equation is the one attention to moisture flow in their model (In fact, we arrange this model):

$$w_t - \mathrm{div}[g_1(w)\nabla h] = 0, \quad w = \Lambda(h), \tag{1.8}$$

where h and w represent respectively, vapor pressure in small pores and the quantity of liquid water corresponding vapor pressure, and Λ describes the hysteresis effects.

In order to deal with this equation, first, by the inverse function Λ^{-1} of Λ, we consider $h = \Lambda^{-1}(w)$, since the actual relation is difficult to deal with from mathematical point of view. Figure 1 represents this inverse function Λ^{-1}. Then, it is well known that

$h = \Lambda^{-1}(w)$ is equivalent to $h_t + \partial I_w(h) \ni 0$, where $I_w(h)$ and $\partial I_w(h)$ are the same as in (1.6) and (1.7). By this property, (1.8) can be written as

$$\begin{cases} w_t - \mathrm{div}[g_1(w)\nabla h] = 0, \\ h_t + \partial I_w(h) \ni 0. \end{cases}$$

The system $\{(1.1) - (1.2)\}$ is an approximate by spatial diffusive for the above system. In this paper, we consider the diffusive coefficient of liquid water depending on liquid water itself, which is more natural case.

Early works for this system $\{(1.1) - (1.2)\}$ is Kenmochi, Koyama and Meyer [5] and Colli, Kenmochi and Kubo [4]. Compaired with these works, we emphasize that in this problem, diffusive coefficient depends on the unknown function, and the divergence term of the gradient of the unknown function in second equation appears in first equation. In order to prove the existence of a solution, we consider the approximate problem with Yosida approximation, and by using time discritization method, we prove the existence of an approximate solution (Section 3). Next, we derive the uniform estimate independing on the approximate parameter for this approximate solution (Section 4), and by limiting process, we show the existence of a solution of problem (P).

1.1 Notations and assumptions

For a Hilbert space H, $(\cdot,\cdot)_H$ and $||\cdot||_H$ represent the inner product and norm, respectively. Through out this paper, we denote simply by $(\cdot,\cdot)$ and $||\cdot||$ the inner product. and norm of $L^2(\Omega)$. $H^1(\Omega)$ and $H_0^1(\Omega)$ are the usual Sobolev spaces, and $H_0^1(\Omega)$ is equipped with the following inner product

$$(z_1, z_2)_{H_0^1(\Omega)} := \int_\Omega \nabla z_1 \cdot \nabla z_2 dx \quad \text{for } z_1, z_2 \in H_0^1(\Omega),$$

Then, $H_0^1(\Omega)$ is a Hilbert space. Also, for a proper, lower semi-continuous convex function ϕ on a Hilbert space H, the effective domain $D(\phi)$ is defined by $\{z \in H; \phi(z) < +\infty\}$, and the subdifferential $\partial_H\phi$ of ϕ on H is a multivalued mapping from H to itself defined by the following:

$$z^* \in \partial_H\phi(z) \quad \text{if and only if } z \in D(\phi) \text{ and}(z^*, u - z) \le \phi(u) - \phi(z) \quad \text{for all } u \in H.$$

Next, we state our assumptions.

(A1) $f_a, f_d \in C^2(\mathbf{R})$ are non-decreasing function. Also, there exists

$$c_0 := \max\{||f_a'||_{L^\infty}, ||f_d'||_{L^\infty}\} < 1, \quad \text{and} \quad \tilde{c}_0 := \max\{||f_a'||_{L^\infty}, ||f_d'||_{L^\infty}\}.$$

(A2) $\tilde{g} \in C^2(\mathbf{R})$ with $\tilde{g}(0) = 0$ satisfies the following properties:

$$|\tilde{g}(r_1) - \tilde{g}(r_2)|^2 \le L_g(\tilde{g}(r_1) - \tilde{g}(r_2))(r_1 - r_2) \quad \text{for } r_1, r_2 \in \mathbf{R}, \tag{1.9}$$

$$\tilde{\delta}|r_1 - r_2| \le |\tilde{g}(r_1) - \tilde{g}(r_2)| \qquad \text{for } r_1, r_2 \in \mathbf{R}, \tag{1.10}$$

where δ and L_g are positive constants. Also, we set $L_{\tilde{g}} := \sup_{r\in\mathbf{R}}\left|\frac{d^2}{dr^2}\tilde{g}(r)\right|$. Moreover, $g_1 \in C^2(\Omega)$ and we set

$$L_{g_1} := \sup_{r\in\mathbf{R}} |g_1(r)| = \frac{1}{L_g}, \quad \text{and} \quad L_{g_1'} := \sup_{r\in\mathbf{R}}\left|\frac{d}{dr}g_1(r)\right|. \tag{1.11}$$

(A3) ε, ν are positive constants with $\varepsilon < 1$.

(A4) $w_0 \in H_0^1(\Omega)$ and $h_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ with $f_a(w_0) \le h_0 \le f_d(w_0)$ a.e. in Ω.

By (1.8) and (1.9), we see that $0 < \delta \le g(r) := \frac{d}{dr}\tilde{g}(r) \le L_g$. Also, condition (1.8), which shows the strictly monotone of $\tilde{g}$, implies that $\tilde{g}$ is surgective, and there exists the inverse function $\tilde{g}^{-1}$. we can see that function $\tilde{g}^{-1}$ fulfills the same property (1.8), (1.9). These properties are important to construct the solution of problem (P). In the rest of this paper, we denoted by (P) as the initial boundary value problem $\{(1.1)-(1.4)\}$. Now, we define a notion of solution of problem (P).

Definition 1.1 A pair of function $(w, h) : [0, T] \to L^2(\Omega) \times L^2(\Omega)$ is called *a solution* of (**P**), if the following items are satisfied.

(S1) $w \in W^{1,2}(0,T; L^2(\Omega)) \cap L^\infty(0,T; H_0^1(\Omega))$ and
$h \in W^{1,2}(0,T; L^2(\Omega)) \cap L^\infty(0,T; H_0^1(\Omega)) \cap L^2(0,T; H^2(\Omega))$

(S2) $w_t - \varepsilon\Delta\tilde{g}(w) - \mathrm{div}[g_1(w)\nabla h] = 0$ in $L^2(\Omega)$ a.e. in $(0, T)$.

(S3) $h_t - \nu\Delta h + \partial I_w(h) \ni 0$ in $L^2(\Omega)$ a.e. in $(0, T)$.

(S4) $w(0) = w_0, h(0) = h_0$

Now, we state the result obtained in this time.

Theorem 1.2 (Local existence) *Assume that conditions (A1)-(A4) hold and* δ, c_0 *and* ε *fulfill with*

$$0 < \gamma := \varepsilon\delta - 1 < 1 \quad \text{and} \quad 0 < \zeta := \frac{c_0}{\delta} < \frac{1}{2}, \tag{1.12}$$

and

$$0 < \nu^* := \frac{3}{4(\gamma - \zeta^2\delta)} < \delta - c_0. \tag{1.13}$$

Then, there exists $T^* > 0$ *such that for each* $\nu \in (v^*, \delta - c_0)$, *problem* (P) *has at least one solution on* $[0, T^*]$.

Our motivation is to solve the problem (P) under ε is small. The former of condition (1.12) implies that δ is sufficiently large compairing with small number ε. Therefore, if δ is large, then L_{g_1} is small while L_g is large by condition (1.11).

2 Proof of Main Theorem

In This section, we note the outline of the proof of Theorem 1.2 briefly. First of all, for each $\lambda > 0$, we consider the following approximation problem $(\mathbf{P})_\lambda$:

$$w_t - \varepsilon\Delta\tilde{g}(w) - \operatorname{div}[g_1(w)\nabla h] = 0 \text{ in } \Omega \times (0,T), \tag{2.1}$$

$$h_t - \nu\Delta h + \partial I_w^\lambda(h) = 0 \text{ in } \Omega \times (0,T), \tag{2.2}$$

$$h(x,-L) = h(x,L) = w(x,-L) = w(x,L) = 0 \text{ for } t \in (0,T), \tag{2.3}$$

$$h(0) = h_0,\ w(0) = w_0 \text{ in } \Omega. \tag{2.4}$$

Here, Also, the subdifferential ∂I_w^λ of I^λ, which coincides with the Yosida approximation of the subdifferential of the indicater function I_w is given by

$$\partial I_w^\lambda(h) = \frac{1}{\lambda}[h - f_d(w)]^- \frac{1}{\lambda}[f_a(w) - h]^+,$$

where $[h - f_d(w)]^+$ and $[f_a(w) - h]^+$ represent the positive parts of $(h - f_d(w))$ and $(f_a(w) - h)$. In particular, it is well-known that ∂I_w^λ converges to ∂I as λ tends to 0. Now, we define a notion of solutions of $(\mathbf{P})_\lambda$.

Definition 2.1 A pair of function $(w_\lambda, h_\lambda) : [0,T] \to L^2(\Omega) \times L^2(\Omega)$ is called *a solution* of $(\mathbf{P})_\lambda$, if the following items are satisfied.

$(S_\lambda 1)$ $w_\lambda \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega))$ and
$h_\lambda \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$

$(S_\lambda 2)$ $w_t - \varepsilon\Delta\tilde{g}(w) - \operatorname{div}[g_1(w)\nabla h] = 0$, in $L^2(\Omega)$ a.e. in $(0,T)$.

$(S_\lambda 3)$ $h_t - \nu\Delta h + \partial I_w^\lambda(h) = 0$ in $L^2(\Omega)$ a.e. in $(0,T)$.

Then, we obtain the following lemma that the existence of the solution of the approximate problem $(\mathbf{P})_\lambda$ for each λ and the uniform estimate independent of λ.

Lemma 2.2 *Assume that conditions (A1)-(A4) hold. Then, $(\mathbf{P})_\lambda$ has at least one time global solution. Also, let (w_λ, h_λ) be any solution of the problem of $(\mathbf{P})_\lambda$. If (1.12) and (1.13) hold in addition to (A1)-(A4), then there exists $T^* > 0$ and $K > 0$ depends on ε, δ, L_g and c_0 such that for each $\nu \in (v^*, \delta - c_0)$,*

$$\begin{aligned}
&||\tilde{g}(w_\lambda)||_{W^{1,2}(0,T^*;L^2(\Omega))} + ||h_\lambda||_{W^{1,2}(0,T^*;L^2(\Omega))} + ||\tilde{g}(w_\lambda)||_{L^\infty(0,T^*;H_0^1(\Omega))} \\
&+||h_\lambda||_{L^\infty(0,T^*;H_0^1(\Omega))} + |I_{w_\lambda}^\lambda(h_\lambda)|_{L^\infty(0,T^*)} + ||\Delta\tilde{g}(w_\lambda)||_{L^2(0,T^*;L^2(\Omega))} \\
&+\nu||\Delta h_\lambda||_{L^2(0,T^*;L^2(\Omega))} + ||\partial I_{w_\lambda}^\lambda(h_\lambda)||_{L^2(0,T^*;L^2(\Omega))} \le K, \quad \textit{for} \quad \forall\lambda \in (0,1).
\end{aligned}$$

By Lemma 2.2, since we can construct suitable convergence sequences, by limiting process $n \to \infty$, we see that Theorem 1.2 holds. In this process, the relation that $\xi \in \partial_{w(t)}(h(t))$ for a.e. in $t \in (0,T)$ is proved by employing the idea of [5].

3 Proof of the existence results of $(\mathbf{P})_\lambda$

In this section, we state the surgery of the proof of of the existence of a solution of $(\mathbf{P})_\lambda$ in Lemma 2.2. First, The basic idea is time discretization problem (cf. [3]). For given $T > 0$ and $\lambda > 0$, we set $\tau = \frac{T}{m}(m \in \mathbf{z})$ and consider the following time discretization problem of $(\mathbf{P})_\lambda$: Find (w_m^n, h_m^n) such that

$$\frac{w_m^n - w_m^{n-1}}{\tau} - \varepsilon\Delta\tilde{g}(w_m^n) - \operatorname{div}[g_1(w_m^{n-1})\nabla h_m^{n-1}] = 0 \quad \text{in} L^2(\Omega). \tag{3.1}$$

$$\frac{h_m^n - h_m^{n-1}}{\tau} - \nu\Delta h_m^n + \partial I^\lambda_{w_m^{n-1}}(h_m^{n-1}) = 0 \quad \text{in} L^2(\Omega). \tag{3.2}$$

Now, we define a function ψ on $L^2(\Omega)$ given by

$$\psi(z) = \begin{cases} \frac{\nu}{2}||\nabla z||^2 & \text{for } z \in H_0^1(\Omega), \\ +\infty & \text{for } z \in L^2(\Omega) \setminus H_0^1(\Omega) \end{cases}$$

Since w^{n-1} and h^{n-1} are known data, this solution (w_m^n, h_m^n) can be obtained from the facts that $I + \tau\partial_{L^2}\psi$ and $\tilde{g}^{-1} + \tau\partial_{L^2}\psi$ are surjective on $L^2(\Omega)$, where I is the identity mapping from $L^2(\Omega)$ to itself and $\partial_{L^2}\psi$ is the subdifferential of ψ. The surjective property of $\tilde{g}^{-1} + \tau\partial_{L^2}\psi$ comes from the abstract theory of maximal monotone in Banach spaces [2, Corollary 1.3], so that by applying to the following problem : Find $\tilde{w} \in H_0^1(\Omega)$ such that

$$\frac{\tilde{g}^{-1}(\tilde{w}) - w_{n-1}^m}{\tau} - \varepsilon\Delta\tilde{w} - \operatorname{div}[g_1(w_m^{n-1})\nabla h_m^{n-1}] = 0 \quad \text{in} L^2(\Omega). \tag{3.3}$$

we have w_m^n by setting $w_m^n := \tilde{g}^{-1}(\tilde{w})$. The important tool to show the existence of a solution of problem $(P)_\lambda$ is the following two lemmas.

Lemma 3.1 *there exists $M = M(\varepsilon, \delta, \nu) > 0$ such that*

$$\tau\sum_{n=1}^{k}\left|\left|\frac{h_m^n - h_m^{n-1}}{\tau}\right|\right|^2 + ||w_m^k||^2 + \tau\sum_{n=1}^{k}||\nabla w_m^n||^2 + \tau\sum_{n=0}^{k}||\nabla h_m^n - \nabla h_m^{n-1}||^2$$

$$+||\nabla h_m^k||^2 + \tau\sum_{n=1}^{k}||\nabla w_m^n - \nabla w_m^{n-1}||^2 \le M, \quad \textit{for } 1 \le \forall k \le m,\ \forall\tau \le 1.$$

Lemma 3.2 *for any $\tau \le 1$,*

$$\tau\sum_{n=1}^{m}||\Delta h_m^n||^2 \quad \textit{and} \quad \tau\sum_{n=1}^{m}\left|\left|\frac{w_m^n - w_m^{n-1}}{\tau}\right|\right|^2 \quad \textit{is bounded.}$$

As to Lemma 3.1, first, we multiply w_m^n and $h_m^n - h_m^{n-1}$to (3.1) and (3.2) respectively, and consider the summation from 1 to $k < m$ for the addition of two analogous test. Lemma 3.1 is a direct consequence of discrete Gronwall′ s lemma for this sum. In this paper, we omit this proof. Lemma 3.2 is obtained from Lemma 3.1. From now on, we note the outline of the proof of Lemma 3.2. For the former results of Lemma 3.2, it follows from applying (3.14) after multiplying $-\Delta h_m^n \in L^2(\Omega)$ and suming up from from 1 to m. Next, by multiplying $\frac{\tilde{g}(w_\tau(t))-\tilde{g}(w_\tau(t-\tau))}{\tau}$ to (3.12), and summing up from 1 to k $(1 \le k \le M)$ for n,

$$\frac{\tau}{2L_g}\sum_{n=1}^{k}\left|\left|\frac{\tilde{g}(w_m^n)-\tilde{g}(w_m^{n-1})}{\tau}\right|\right|^2 + \frac{\varepsilon}{2}\left\{||\nabla\tilde{g}(w_m^k)||^2 - ||\nabla\tilde{g}(w_0)||^2\right\}$$

$$\le L_g C\tau\sum_{n=1}^{k}||h_m^{n-1}||_{H^2(\Omega)}^2||\nabla(g_1(w_m^{n-1}))||^2 + L_g L_{g_1}^2\tau\sum_{n=1}^{k}||\Delta h_m^{n-1}||^2. \tag{3.4}$$

Next, we multiply $U_k := \tau||h_m^k||_{H^2(\Omega)}^2$ and sum up from 1 to i for k $(1 \le i \le m)$. Then, by putting $Z_i := \sum_{k=0}^{i}\tau||h_m^k||_{H^2(\Omega)}^2||\nabla\tilde{g}(w_m^k)||^2$, we derive that

$$\frac{\varepsilon}{2}Z_i \le \frac{L_g L_{g_1}^2 C}{\delta^2}\sum_{k=0}^{i-1}U_{k+1}Z_k$$

$$+L_g L_{g_1}^2\left(\sum_{k=1}^{m}U_k\right)^2 + \frac{\varepsilon}{2}\left(\sum_{k=1}^{m}U_k\right)||\nabla\tilde{g}(w_0)||^2 + \frac{\varepsilon}{2}Z_0,$$

where $Z_0 = \tau||h_0||_{H^2(\Omega)}^2||\nabla\tilde{g}(w_0)||^2$. Then by using discrete Gronwall lemma it turns out that $\sum_{k=0}^{i}\tau||h_m^k||_{H^2(\Omega)}^2||\nabla\tilde{g}(w_m^k)||^2$ is bounded. By applying this results to (3.4) with $k = M$, we see that $||\tilde{w}_\tau'||_{L^2(0,T;L^2)}$ is bounded. Thus, Lemma 3.2 is proved. ◇

Now, we introduce the following new variable :

$$w_\tau(t) = w_0, \quad h_\tau(t) = h_0 \quad \text{for } t \le 0,$$

$$w_\tau(t) = w_m^n, \quad h_\tau(t) = h_m^n \quad \text{for } t \in ((n-1)\tau, n\tau] \text{ and } n = 1, \cdots, m,$$

and

$$\tilde{w}_\tau(t) = w_m^n + \frac{w_m^n - w_m^{n-1}}{\tau}(t - n\tau), \quad \tilde{h}_\tau(t) = h_m^n + \frac{h_m^n - h_m^{n-1}}{\tau}(t - n\tau)$$

for $t \in [(n-1)\tau, n\tau]$ and $n = 1, \cdots, m$. Then, system$\{(3.1) - (3.2)\}$ can be written as follows by this variable:

$$\tilde{w}_\tau'(t) - \varepsilon\Delta\tilde{g}(w_\tau(t)) - \mathrm{div}[g_1(w_\tau(t-\tau))\nabla h_\tau(t)] = 0 \quad \text{in} L^2(\Omega), \tag{3.5}$$

$$\tilde{h}_\tau'(t) - \nu\Delta h_\tau(t) + \partial I_{w_\tau(t-\tau)}^\lambda(h_\tau(t-\tau)) = 0 \quad \text{in} L^2(\Omega). \tag{3.6}$$

By applying Lemma 3.1 and Lemma 3.2, there exists a positive constant $\tilde{M}$ such that

$$||\tilde{w}_\tau||_{W^{1,2}(0,T;L^2(\Omega))} + ||w_\tau||_{L^\infty(0,T;L^2(\Omega))} + ||w_\tau||_{L^2(0,T;H_0^1(\Omega))} + ||\tilde{h}_\tau'||_{L^2(0,T;L^2(\Omega))}$$

$$+||\tilde{h}_\tau||_{L^2(0,T;H_0^1(\Omega))} + ||\tilde{w}_\tau||_{L^2(0,T;H_0^1(\Omega))} + ||h_\tau||_{L^\infty(0,T;H_0^1(\Omega))} \le \tilde{M}.$$

From Lemma 3.1 and 3.2, we can construct suitable convergence sequences with respect to τ. In these convergences, a pair $(\tilde{w}_\tau, \tilde{h}_\tau)$ and a pair (w_τ, h_τ) have respectively, a limit pair $(\tilde{w}, \tilde{h})$ and (w, h). However, we can see that as $\tau \to 0$,

$$||\tilde{w}_\tau - w_\tau||_{L^2(0,T;L^2(\Omega))} \le \tau ||\tilde{w}_\tau'||_{L^2(0,T;L^2(\Omega))} \to 0.$$

$$||\tilde{h}_\tau - h_\tau||_{C([0,T];L^2(\Omega))} = \max_t \tau^2 ||\tilde{h}_\tau'(t)||^2_{L^2(\Omega)} \le \tau ||\tilde{h}_\tau'||_{L^2(0,T;L^2(\Omega))} \to 0.$$

Therefore, we have that $\tilde{h} = h, \quad \tilde{w} = w$ in $L^2(\Omega)$ a.e. $t \in (0,T)$. Also, we note that on account of the above convergences, we can see that $(w_\tau(\cdot - \tau), h_\tau(\cdot - \tau)$ converges (w,h) strongly in $L^2(0,T;L^2(\Omega)$, in $C([0,T];L^2(\Omega))$ Finally, by limiting process $\tau \to 0$, we can prove that $(S_\lambda 2)$ and $(S_\lambda 3)$ hold.

4 Proof of uniform estimate independing on λ

In this section, we note outline of the derivation of the uniform estimate independing on λ in Lemma 2.2. Also, we denote the approximate solution (w_λ, h_λ) as (w,h) for the sake of simplicity. First of all, we calculate the following items:

[I] (2.5) $\times \tilde{g}(w)_t$, [II] (2.5) $\times(-\Delta \tilde{g}(w)(g(w)))$,

[III] (2.6) $\times h_t$, [IV] (2.6) $\times(-\Delta h)$, [V] (2.6) $\times \partial I_w^\lambda(h)$.

Then, In [I] and [II], we use the following estimate :

$$\begin{aligned}
&||\nabla w \nabla h||\ ||\tilde{g}(w)_t|| \le \frac{1}{\delta} ||\nabla \tilde{g}(w) \nabla h||\ ||\tilde{g}(w)_t|| \\
\le\ & \frac{1}{2\delta\varepsilon_1} ||\nabla \tilde{g}(w)\nabla h||^2 + \frac{\varepsilon_1}{2}||\tilde{g}(w)_t||^2 \\
\le\ & \frac{1}{2\delta\varepsilon_1} ||\nabla \tilde{g(w)}||^2_{L^4(\Omega)}\ ||\nabla h||^2_{L^4(\Omega)} + \frac{\varepsilon_1}{2}||\tilde{g}(w)_t||^2 \\
\le\ & \frac{1}{2\delta\varepsilon_1}\left\{\frac{1}{2}||\nabla \tilde{g}(w)||^4_{L^4(\Omega)} + \frac{1}{2}||\nabla h||^4_{L^4(\Omega)}\right\} + \frac{\varepsilon_1}{2}||\tilde{g}(w)_t||^2 \\
\le\ & \frac{1}{2\delta\varepsilon_1}\left\{\frac{1}{2}\left[C_0^4||\nabla \tilde{g}(w)||^3||\Delta\tilde{g}(w)||\right] + \frac{1}{2}\left[C_0^4||\nabla h||^3||\Delta h||\right]\right\} + \frac{\varepsilon_1}{2}||\tilde{g}(w)_t||^2 \\
\le\ & \frac{1}{2\delta\varepsilon_1}\left\{\frac{C_0^4}{2}\left[\frac{1}{2\xi}||\nabla\tilde{g}(w)||^6 + \frac{\xi}{2}||\Delta\tilde{g}(w)||^2\right]\right. \\
&\qquad\left. + \frac{C_0^4}{2}\left[\frac{1}{2\xi}||\nabla h||^6 + \frac{\xi}{2}||\Delta h||^2\right]\right\} + \frac{\varepsilon_1}{2}||\tilde{g}(w)_t||^2,
\end{aligned} \tag{4.1}$$

where C_0 is a positive constant by a Gagliard Nirenburg's inequality with one dimensional case:

$$||z||_{L^4(\Omega)} \le C_0 ||z||^{\frac{3}{4}} ||\nabla z||^{\frac{1}{4}} \quad \text{for } z \in H^1(\Omega), \tag{4.2}$$

and ε_1 and ξ are arbitrary positive constants. Now, we set $C_1 = C_0^4 L_{g_1'}/8\varepsilon_1\delta\xi$ and $C_2 = C_0^4 L_{g_1'}/8\varepsilon_1\delta$. Also, (4.1) replaced $\tilde{g}(w)_t$ by $\Delta\tilde{g}(w)$ holds. Also, for [III], [IV] and [V], we use the following lemmas.

Lemma 4.1(cf. [4, Lemma4.1]) *Let (w,h) be any solution of problem $(P)_\lambda$. Then,*

$$I_w^\lambda(h) = \frac{1}{2\lambda}||[h - f_d(w)]^+||^2 + \frac{1}{2\lambda}||[f_a(w) - h]^+||^2$$

is absolutely continuous on $[0,T]$ and the following inequality holds:

$$\frac{d}{dt}I_w^\lambda(h) \le (\partial I_w^\lambda(h), h_t) + c_0||\partial I_w^\lambda(h)||\ ||w_t|| \quad a.e.\ in\ (0,T),$$

Lemma 4.2(cf. [4]) *Let (w,h) be any solution of problem $(P)_\lambda$. Then, the following inequality holds:*

$$(\partial I_w^\lambda(h), -\Delta h) \ge -2\zeta_0||\partial I_w^\lambda(h)||\ ||\nabla\tilde{g}(w)||^2_{L^4(\Omega)} - 2\zeta||\partial I_w^\lambda(h)||\ ||\Delta\tilde{g}(w)||,$$

where $\zeta = c_0/\delta$ and $\zeta_0 = \tilde{c}_0/\delta^2 + c_0 L_{\tilde{g}}$

Lemma 4.1 can be proved by the same argument in [4] because of $w_t \in L^2(\Omega)$. Therefore, we omit this proof. For Lemma 4.2, we define $\tilde{f}_a := f_a \circ \tilde{g}^{-1}$. and $\tilde{f}_d := f_d \circ \tilde{g}^{-1}$. Then, by computing $(\partial I_w^\lambda(h), -\Delta h)$, Lemma 5 is proved. By calculating $r_1[I] + r_2[II] + r_3[III] + r_4[IV] + r_5[V]$ through (4.4), Lemma 4.1 and Lemma 4.2, we can obtain that

$$\begin{aligned}
&\left[r_1\left(\frac{1}{2L_g} - \frac{\varepsilon_1 L_{g_1'}}{2}\right) - \frac{c_0}{2\delta}(r_3 + r_5)\right]||\tilde{g}(w)_t||^2 + r_3||h_t||^2 \\
&+\frac{d}{dt}\left[\left(\frac{r_2}{2} + \frac{\varepsilon r_1}{2}\right)||\nabla\tilde{g}(w)||^2 + \left(\frac{\nu r_3}{2} + \frac{r_4}{2}\right)||\nabla h||^2 + (r_3 + r_5)I_w^\lambda(h)\right] \\
&+\left[r_2\left(\varepsilon\delta - 1 - \frac{\varepsilon_1 L_{g_1'} L_g}{2}\right) - \left((2\zeta^2)r_4 + (2\zeta^2)\delta\nu r_5\right)\right]||\Delta\tilde{g}(w)||^2 \\
&+\left[\nu r_4 - \left(\frac{L_{g_1}^2 L_g}{2}r_1 + \frac{1}{4}r_2\right)\right]||\Delta h||^2 + \left[r_5 - \left(\frac{c_0}{2\delta}(r_3 + r_5) + \frac{r_4}{2} + \frac{r_5\nu}{2\delta}\right)\right]||\partial I_w^\lambda(h)||^2 \\
&\le C_1(r_1 + r_2 L_g)\left[||\nabla\tilde{g}(w)||^6 + ||\nabla h||^6\right] + C_2(r_1 + r_2 L_g)\xi\left(||\Delta\tilde{g}(w)||^2 + ||\Delta h||^2\right) \\
&\qquad +2\zeta_0(r_4 + \nu r_5)||\partial I_w^\lambda(h)||\ ||\nabla\tilde{g}(w)||^2_{L^4(\Omega)}. \qquad (4.3)
\end{aligned}$$

Now, we have to check that each coefficients are positive. For this aim, it is enough to check that

$$C_3 := \frac{r_1}{2L_g} - \frac{c_0 r_5}{2\delta} > 0, \quad C_4 := r_2(\varepsilon\delta - 1) - \left(2\zeta^2 r_4 + (2\zeta^2)\delta\nu r_5\right) > 0,$$

$$C_5 := \nu r_4 - \left(\frac{L_{g_1}^2 L_g}{2} r_1 + \frac{1}{4} r_2\right) > 0, \quad C_6 := r_5 - \left(\frac{c_0 r_5}{2\delta} + \frac{r_4}{2} + \frac{r_5 \nu}{2\delta}\right) > 0.$$

First, let ν, ε, c_0 and δ be satisfied

$$0 < \frac{3}{4(\gamma - \zeta^2 \delta)} < \nu < \delta - c_0, \quad \gamma := \varepsilon\delta - 1 < 1 \quad \text{and} \quad \zeta := \frac{c_0}{\delta} < \frac{1}{2}.$$

Then, by taking $r_4 = r_5 = 1$, it easily seen that C_6 is positive. Next, by choosing $r_1 = 2L_g \frac{c_0}{\delta}$ and $r_2 = \frac{2}{\gamma}(2\zeta^2)(1 + \nu_0\delta)$, we see that C_3, C_4 and C_5 are positive. Thus, $C_i (3 \le i \le 6)$ are positive whenever ε, δ, c_0 and ν satisfy the above condition. Then, we can take ε_1, r_3 and ξ such that the corresponding coefficients are positive. because of by the arbitrariness of ξ, ε and r_3. Finally, it still remains to estimate the last line of (4.3). By Gagliard Nirenburg's inequality in the form of (4.2), we deduce that

$$\begin{aligned}
& 2\zeta_0(r_4 + \nu r_5)||\partial I_w^\lambda(h)||\ ||\nabla \tilde{g}(w)||_{L^4(\Omega)}^2 \\
\le\ & \frac{(2\zeta_0(r_4 + \nu r_5))^2}{2\varepsilon_2}||\nabla \tilde{g}(w)||_{L^4(\Omega)}^4 + \frac{\varepsilon_2}{2}||\partial I_w^\lambda(h)||^2 \\
\le\ & \frac{(2\zeta_0(r_4 + \nu r_5))^2}{2\varepsilon_2} C_0 ||\nabla \tilde{g}(w)||^3\ ||\Delta \tilde{g}(w)|| + \frac{\varepsilon_2}{2}||\partial I_w^\lambda(h)||^2 \\
\le\ & \frac{(2\zeta_0(r_4 + \nu r_5))^2}{2\varepsilon_2} C_0 \left[\frac{1}{\xi_2}||\nabla \tilde{g}(w)||^6 + \frac{\xi_2}{2}||\Delta \tilde{g}(w)||^2\right] + \frac{\varepsilon_2}{2}||\partial I_w^\lambda(h)||^2,
\end{aligned}$$

where C_0 is the same as in (4.2) and ξ_2 and ε_2 are arbitrary positive numbers. Therefore, it is enough to choose ξ_2 and ε_2 such that corresponding coefficients are positive. Consequently, we conclude that there exists positive numbers $C_i > 0 (7 \le i \le 15)$ such that

$$\begin{aligned}
& C_7||\tilde{g}(w)_t||^2 + C_8||h_t||^2 + C_9||\Delta \tilde{g}(w)||^2 + C_{10}||\Delta h||^2 + C_{11}||\partial I_w^\lambda(h)||^2 \\
& + \frac{d}{dt}\left[C_{12}||\nabla \tilde{g}(w)||^2 + C_{13}||\nabla h||^2 + C_{14} I_w^\lambda(h)\right] \le C_{15}\left[||\nabla \tilde{g}(w)||^6 + ||\nabla h||^6\right],
\end{aligned}$$

which yields the following the ordinary differential equation :

$$\frac{d}{dt}L(t) \le C_{15}\left(\frac{1}{C_{12}^3} + \frac{1}{C_{13}^3}\right) L^3(t) + C_{12}||\nabla \tilde{g}(w_0)||^2 + C_{13}||\nabla h_0||^2 \quad \text{for } \forall t \in [0, T].$$

where $L(t) := \int_0^t \left\{ \text{the left hand side of (4.24)} \right\} ds + C_{12}||\nabla \tilde{g}(w_0)||^2 + C_{13}||\nabla h_0||^2$ for any $t \in (0, T)$. Therefore, by solving this equation, we derive the uniform estimate independing on λ time locally. Thus, the proof of Lemma 2.2 is complete.

5 Future Works

For this problem (P), we are interested in the existence and uniqueness of a solution of this problem (P) with the following various condition:

[I] the uniquness of the local solution in one dimension

[II] a global solution in one dimension

[III] a local / global solution in higher dimension

[IV] a local / global solution of the problem imposed on inhomogenious Dirichlet boundary condition

[V] a local / global solution of the problem coupled with other equation

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[I] At this point, Coll, Kenmochi and Kubo [4] point out that when $\nu > 0$, there does not exist a unique solution of the following system:

$$\begin{cases} h_t + w_t - \varepsilon\Delta w = 0 \\ h_t - \nu\Delta h + g(h,w) + \partial I_h(w) \ni 0, \end{cases}$$

where g is given smooth function on $\mathbf{R}$, while $\nu = 0$, the above problem has a unique solution. In this problem (P), since ν is belong to the interval $(\nu^*, \delta - c_o)$, which is not contained 0, we can not obtain a solution of problem (P) with $\nu = 0$ as ν tends to 0. Concerning this point, we want to consider the relaxation of assumption. In particular, the range of ν is seem to be narrow. When δ is large, the range of ν expands while upper bound of g_1 is small. However, this implies that g_1 hardly has width. In future, we consider about such respect as well as the uniqueness results.

[II] and [III] Now, we attempt to prove the existence of a global solution in one dimension, however, we does not obtain it yet. In proving the local solution of problem (P), (4.1) is a key estimate, however, by employing (4.1), we show only the existence of the local solution. Also, regarding with [III], we have to consider three dimensional case, however, (4.1) only holds in one dimension, and it does not holds in higher dimension. In order to overcome this difficulty, we considered that [II] and [III] may be proved if it is obtained the L^∞ bound of the solution, however, it does not works well. In fact, it is difficult to have L^∞ bound of the solution only condition (A1)-(A4) because of lacking the upper bound of f_a and f_d. (In [4], they succeed to derive L^∞ bound of the solution, since f_a and f_d have a upper bound.) However, it is possible to show the L^∞ bound of the solution provided that, for instance,

$$g_1(r) = 0 \quad \text{on } r > 1.$$

From physical point of view, since we does not need to consider the case of $w > 1$, without loss of generality, this condition can be assumed. In this case, the following estimate hold instead of (4.1) in any dimension:

$$||\nabla w \nabla h||^2 \le \frac{1}{\delta}\left\{\frac{9}{2}||\tilde{g}(w)||^2_{L^\infty}||\Delta\tilde{g}(w)||^2 + \frac{9}{2}||h||^2_{L^\infty}||\Delta h||^2\right\}.$$

However, the term of $||\Delta \tilde{g}(w)||^2$ and $||\Delta h||^2$ with large constant increase compaired with the local solution, and therefore, there does not exists suitable condition like (1.12) and (1.13) such that the uniform estimate independent of λ can be obtained. Now, we consider to overcome this difficulty.

[IV] and [V] Inhomogenious Dirichlet boundary case is more natural compaired with problem (P), and we consider that we can prove a local solution of problem (P)in this case. Also, this system $\{(1.1) - (1.2)\}$ with $\varepsilon = \nu = 0$ describes moisture flow in concrete, and problem (P) is an approximate problem by spatial diffusion. Actual concrete carbonation is more complex by various chemical elements, and therefore, we have to consider the problem coupled with other evolution equation, for example, flows of carbon dioxide, hydrogen, various ion, and so on. We want to consider the full system include the restructure of a model of concrete carbonation.

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