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On a free boundary problem related to the motion of an amoeba

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1 Introduction

Amoeba motion is one of cellular motions, and is common among many cells, for example, white blood cell, cancer cells, slime mold, keratocyte and so on. The locomotion is caused by some chemical reactions within a cell and substances outside the cell. However the detailed mechanism to produce such motion structure is not well understood so that its mathematical study is an interesting question.

Meanwhile, there are some mathematical models proposed by a lot of biologist and physicist who desire to understand the locomotion. In this paper we consider a mathematical model, which is proposed by T. Umeda [6], of them.

We consider the following free boundary problem:

$$
(\mathrm{P})\begin{cases}
u_t = \Delta u + k_1\left(C_0 - \displaystyle\int_{\Omega(t)} u dx\right) - u + k_2 & \text{in } Q, \qquad \cdots(\mathrm{i})\\
u = 1 + A\kappa + BV & \text{on } \Gamma, \qquad \cdots(\mathrm{ii})\\
V = -\epsilon\nabla u\cdot\mathbf{n} + g\left(C_0 - \displaystyle\int_{\Omega(t)} u dx\right) - u & \text{on } \Gamma, \qquad \cdots(\mathrm{iii})\\
u = \phi \geq k_2 & \text{in } \Omega(0),
\end{cases}
$$

where $\Omega(t)$ is an unknown bounded domain in $\mathbb{R}^2$ at time t with the boundary $\partial\Omega(t)$, Q and Γ are a non-cylindrical domain and a non-cylindrical surface, respectively, defined by

$$
Q := \bigcup_{t>0}\Omega(t)\times\{t\},\quad \Gamma := \bigcup_{t>0}\partial\Omega(t)\times\{t\},
$$

$\kappa = \kappa(x,t)$ is an inward curvature at $x \in \partial\Omega(t)$, $V = V(x,t)$ is a scalar function representing the outer normal velocity of $\partial\Omega(t)$, $\mathbf{n} = \mathbf{n}(x,t)$ is an outer normal unit vector at $x \in \partial\Omega(t)$. Coefficients k_1, k_2, C_0, A, g are positive constants and B, ϵ are non-negative constants.

(P) is related to a mathematical model describing the motion of an amoeba, which is based on the density of F-actin $u(x,t)$ and actin layer $\Omega(t)$ contained in a cell (see Figure 1). Since the actin layer is almost same as the shape of the cell, we

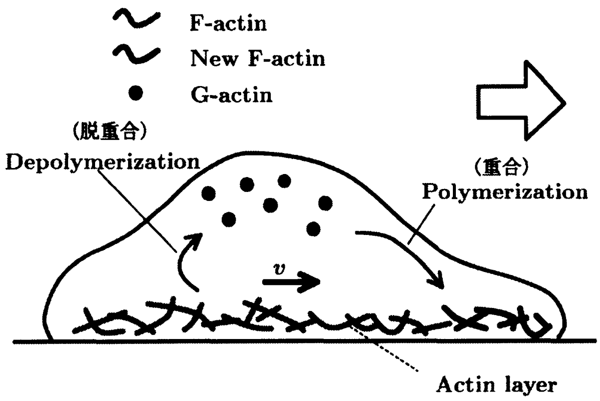

Figure 1: Cross section of a cell

can regard the domain as the shape of the cell.

On the other hand, (P) is regarded mathematically as a one-phase Stefan problem with reaction terms. In terms of Stefan problems, the interior condition (i) and boundary conditions (ii), (iii) correspond to the heat equation, the Gibbs-Thomson effect and the Stefan condition, respectively. In general, for free boundary problems, the topology of domains may change at a finite time so that the existence time of classical solutions is local. Therefore, some additional conditions are necessary for the existence of the time global classical solutions. In what follows, we consider the time local and time global existence of solutions for (P) under some assumptions.

Assumption 1. *The initial domain $\partial\Omega(0)$ is a Jordan curve such that $\partial\Omega(0) \in C^{3+\alpha}$, $\partial\Omega(0) = \{X^0(s) + \Lambda_0(s)N(s) \mid s \in [0,l]\}$, where α is a Hölder index $(0 < \alpha < 1)$, X^0 is a regular Jordan curve, in $\mathbb{R}^2$, parameterized by s. Here $N(s)$ is the outer normal unit vector at $X^0(s)$, and $\Lambda_0 \in C^{3+\alpha}([0,l])$.*

Definition 1. *In the case of $\epsilon > 0$, we call "(P) has a time local solution", if there exists a finite time $T > 0$ such that $(u, \Omega(t))$ satisfies (P) and have a regularity*

$$u \in C^{2+\alpha,(2+\alpha)/2}(\overline{Q_T}), \qquad \Gamma_T \in C^{3+\alpha,(3+\alpha)/2},$$

where $Q_T = \bigcup_{0<t<T} \Omega(t) \times \{t\}$ and $\Gamma_T = \bigcup_{0<t<T} \partial\Omega(t) \times \{t\}$. Also, if $T = \infty$, we call "(P) has a unique time global solution".

Considering the viscosity effect $(B > 0)$, we have the following result:

Theorem 1. *Let $B, \epsilon > 0$. Suppose that $\partial\Omega(0)$ satisfies Assumption 1. If the initial datum ϕ belongs to $C^{2+\alpha}(\overline{\Omega(0)})$ and satisfies a compatibility condition, then (P) has a unique time local classical solution.*

In Theorem 1, as in X. Chen and F. Reitch [1], W. Merz and P. Rybka [4], we regard the boundary condition (ii) as a parabolic problem on a curve, and an approximate sequence of the boundary can be found by solving the problem. As a result, we have a unique solution with the aid of the Hanzawa diffemophism (E. Hanzawa [2]) and Banach's fixed point theorem. On the other hand, in the case of $B = 0$, if we try to apply the same method as in the case of $B > 0$, we can not find an approximate sequence of the boundary with the time evolution. However, under the special condition (spherically symmetric case), we can show the existence of classical solutions since we can construct an approximate sequence of the boundary by the boundary condition (iii). From now on, we consider the case where $(\phi, \Omega(0))$ is spherically symmetric.

Assumption 2. *Initial data are spherically symmetric, i.e.*

$$\Omega(0) = \{x \in \mathbb{R}^2 \mid 0 \leq |x| < s_0\}, \quad \phi(x) = \psi(|x|),$$

where s_0 is a positive constant and $\psi \in C^{2+\alpha}([0, s_0])$ with $\psi_r(0) = 0$.

From now on, we suppose that initial data satisfy Assumption 2.

Definition 2. *In the case of $\epsilon = 0$, we call "(P) has a time local classical solution", if there exists $T > 0$ such that $(u, \Omega(t))$ satisfies (P) and have a regularity*

$$u \in C^{2+\alpha,\ (2+\alpha)/2}(\overline{Q_T}), \qquad \Gamma_T \in C^{4+\alpha,(4+\alpha)/2}.$$

Also, if $T_* = \infty$, we call "(P) has a unique time global solution".

Theorem 2. *Let $B = 0$ and $\epsilon \geq 0$. If the initial datum ϕ satisfies Assumption 2 and compatibility conditions, then (P) has a unique time local classical solution.*

From the viewpoint of mathematical modeling, it is preferable that u and $C_0 - \int_{\Omega(t)} u\, dx$ are positive. With this view in mind, we consider the time global existence of solutions in the case of $\epsilon = 0$.

Assumption 3. *Coefficients satisfy the following condition:*

$$k_1 C_0 - (1 - A\pi g)(1 - k_2) < 0, \quad 1 - k_2 > 0.$$

Assumption 4. *There exists $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (0, (-A + \sqrt{A^2 + 4C_0/\pi})/2]$ such that following conditions hold:*

$$g(C_0 - \pi\alpha_i(A + \alpha_i)) = 1 + \frac{A}{\alpha_i}, \quad g(C_0 - \pi k_2 \beta_i^2) = 1 + \frac{A}{\beta_i}, \quad (i = 1, 2),$$

where $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$.

Theorem 3. *Let $B = 0$ and $\epsilon = 0$. Suppose that initial data and coefficients satisfy Assumption 2, 3, 4. If initial data satisfy compatibility conditions and*

$$k_2 < \phi|_{\Omega(0)} < \phi|_{\partial\Omega(0)}, \quad C_0 - \int_{\Omega(0)} \phi\, dx > 0, \quad \alpha_2 \leq s_0 \leq \beta_2,$$

then (P) has a unique time global solution $(u, \Omega(t))$ such that

$$k_2 < u|_{\Omega(t)} < u|_{\partial\Omega(t)}, \quad C_0 - \int_{\Omega(t)} u(t, \cdot)\, dx > 0, \quad \alpha_2 \leq s(t) \leq \beta_2,$$

where $s(t)$ is the radius of $\Omega(t)$.

2 Positive invariant region

Local existence of a classical solution for (P) is shown by Hanzawa diffe.[2], a parabolic standard existence theory [3] and Banach's fixed point theorem (see [5]). The proof is based on the paper of Chen and Reitch [1].

In this section, we prove the existence of a positive invariant region, for $\partial\Omega(t)$, to show the existence of a time global classical solution. To this end, we examine some properties such that

$$\phi > k_2 (> 0), \quad C_0 - \int_{\Omega(0)} \phi \, dx > 0 \implies u > k_2, \quad C_0 - \int_{\Omega(t)} u \, dx > 0.$$

These properties come from the fact that area density of F-actin and concentration of G-actin are positive in the biological view point. Likewise, they are of help to prove the boundedness of u and $\Omega(t)$. As a result, we can find global solutions with the help of these properties and initial conditions.

Here, for simplicity, we rewrite (P) as an one-dimensional problem

$$\text{(SP)} \begin{cases} v_t = v_{rr} + \dfrac{v_r}{r} + k_1 \left(C_0 - 2\pi \displaystyle\int_0^{s(t)} rv \, dr \right) - v + k_2 & \text{in } Q((0,T); s(t)), \\ v = 1 + \dfrac{A}{s(t)} & \text{on } \Gamma((0,T); s(t)), \\ \dot{s}(t) = g \left(C_0 - 2\pi \displaystyle\int_0^{s(t)} rv \, dr \right) - v & \text{on } \Gamma((0,T); s(t)), \\ v_r = 0 & \text{on } \{0\} \times [0,T], \\ v = \psi > k_2 & \text{in } (0, s_0), \end{cases}$$

where T is a positive constant,

$$Q((a,b); s(t)) := \bigcup_{a<t<b} [0, s(t)) \times \{t\}, \quad \Gamma((a,b); s(t)) := \bigcup_{a<t<b} \{s(t)\} \times \{t\}$$

and $v(r,t)$ is equal to $u(x,t)$ for (P) with $r = |x|$.

2.1 Boundedness of $s(t)$ and $v(r,t)$

To prove Theorem 3, we prepare some Lemmas. From now on, we suppose that T is a time such that (SP) has the unique time local solution $(v, s(t))$ in $[0,T]$.

Lemma 1. *Assume that $\dot{s}(t) \leq 0$ for any $t \in [t_0, t_1] \subset [0,T]$. If $k_2 < v(r,t_0) < v(s(t_0), t_0)$ for any $r \in [0, s(t_0))$ and $C_0 - \pi(A + s(t_0))s(t_0) > 0$, then*

$$k_2 < v(r,t) < v(s(t),t), \quad C_0 - 2\pi \int_0^{s(t)} rv \, dr > 0$$

for any $t \in [t_0, t_1]$ and $r \in [0, s(t))$.

Proof. It is clear in the case of $t_0 = t_1$, so we consider the case of $t_0 \neq t_1$. Suppose that there exists a point $(r^*, t^*) \in Q((t_0, t_1]; s(t))$ such that $v(r^*, t^*) = k_2$ and $k_2 < v(r, t) < v(s(t), t)$ in $Q((t_0, t_1); s(t))$, where $Q((a, b]; s(t)) := \bigcup_{a<t\leq b}[0, s(t)) \times \{t\}$. Since $\dot{s}(t) \leq 0$ and $v \leq v(s(t), t)$ for $t \in [t_0, t^*]$,

$$C_0 - 2\pi \int_0^{s(t)} rv\,dr > C_0 - \pi(A + s(t))s(t) > C_0 - \pi(A + s(t_0))s(t_0) > 0.$$

Then, from the interior condition of (SP),

$$\begin{aligned} 0 &\geq v_t(r^*, t^*) \\ &= v_{rr}(r^*, t^*) + \frac{v_r(r^*, t^*)}{r^*} + k_1 \left(C_0 - 2\pi \int_0^{s(t^*)} rv\,dr \right) - v(r^*, t^*) + k_2 > 0. \end{aligned}$$

This is a contradiction, and we see that $v > k_2$ for $t \in [t_0, t_1]$. On the other hand, suppose that there exists a point $(r^*, t^*) \in Q((t_0, t_1]; s(t))$ such that $v(r^*, t^*) = 1 + A/s(t^*)$ and $k_2 < v(r, t) < 1 + A/s(t)$ in $Q((t_0, t_1); s(t))$. From easy calculations, $C_0 - 2\pi \int_0^{s(t)} rv\,dr$ and $\int_0^{s(t)} rv\,dr$ are positive for $t \in [t_0, t^*]$. Then

$$0 \leq v_t(r^*, t^*) \leq k_1 C_0 - 2k_1\pi \int_0^{s(t^*)} rv\,dr - (1 - k_2) - \frac{A}{s(t^*)} < 0$$

from Assumption 4. This is a contradiction, and we see that $v < v(s(t), t)$ for $t \in [t_0, t_1]$.

□

Similarly, for the case of $\dot{s}(t) \geq 0$, we will show the boundedness. Here we remark that the assumption of boundedness for $s(t)$ differ slightly between $\dot{s}(t) \geq 0$ and $\dot{s}(t) \leq 0$.

Lemma 2. *Assume that $\dot{s}(t) \geq 0$ for $t \in [t_0, t_1] \subset [0, T]$, and $s(t_0) \geq \alpha_2$. If $k_2 < v(r, t_0) < v(s(t_0), t_0)$ for any $r \in [0, s(t_0))$ and $C_0 - \pi(A + s(t_i))s(t_i) > 0$ for $i = 0, 1$, then*

$$k_2 < v < v(s(t), t), \quad C_0 - 2\pi \int_0^{s(t)} rv\,dr > 0$$

for any $t \in [t_0, t_1]$ and $r \in [0, s(t))$.

Proof. We show this Lemma in the case of $t_0 \neq t_1$ only. From the boundary condition (iii) and $\dot{s}(t) \geq 0$, it follows that $C_0 - 2\pi \int_0^{s(t)} rv\,dr > 0$ for any $t \in [t_0, t_1]$. As the argument of Lemma 1, we have the property $v > k_2$. To prove that $v < v(s(t), t)$ in $(0, s(t))$, we use a super-solution $1 + A/s(t)$.

Let $X(r, t) = 1 + A/s(t) - v(r, t)$. From directly calculations, $X(r, t)$ satisfies the following problem:

$$\begin{cases} X_t = X_{rr} + \dfrac{X_r}{r} - k_1 \left(C_0 - 2\pi \displaystyle\int_0^{s(t)} rv\,dr \right) - X \\ \qquad\qquad - k_2 + \left(-\dfrac{A\dot{s}(t)}{s^2(t)} + 1 + \dfrac{A}{s(t)} \right) \qquad \text{in } Q((t_0, t_1); s(t)), \\ X(s(t), t) = 0, \quad X_r(0, t) = 0, \quad X(r, 0) \geq 0 \ \text{ in } (0, s_0). \end{cases} \tag{1}$$

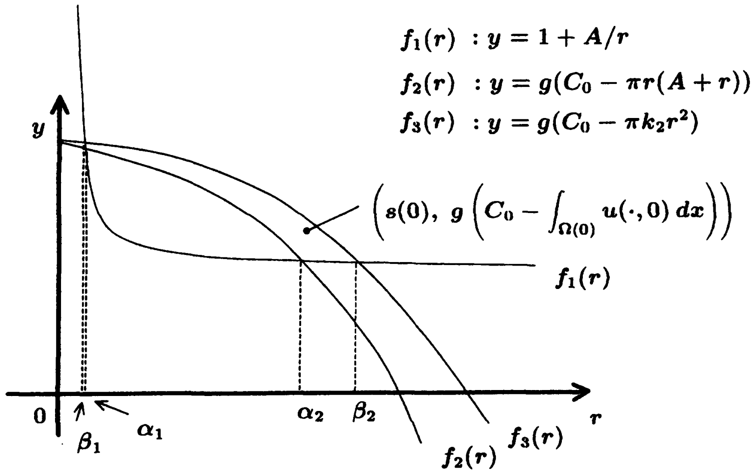

Figure 2: Assumption 4

Suppose that there exists a point $(r^*, t^*) \in Q((t_0, t_1]; s(t))$ such that $X(r^*, t^*) = 0$ and $X(r, t) > 0$ in $Q((t_0, t^*); s(t))$. The left hand side $X_t(r^*, t^*) \leq 0$. On the other hand, since $\dot{s}(t) \geq 0$, $s(t_0) \geq \alpha_2$ and $C_0 - \pi(A + s(t_1))s(t_1) > 0$,

$$
\begin{aligned}
0 &\leq g\left(C_0 - 2\pi \int_0^{s(t)} rv\, dr\right) - 1 - A/s(t) \\
&< g\left(C_0 - 2\pi \int_0^{s(t)} rv\, dr\right) - g(C_0 - \pi s(t)(A + s(t)))
\end{aligned} \tag{2}
$$

for any $t \in [t_0, t_1]$ (see Figure 2). Moreover, by normalizing free boundary,

$$
2\pi \int_0^{s(t)} rv\, dr > k_2 \pi s^2(t) \tag{3}
$$

for any $t \in [t_0, t_1]$, where $v(r, t) = w(\rho, t)$ and $r = \rho s(t)$. By (2), (3) and Assumption 3, we see that

$$
\begin{aligned}
&\left\{-k_1\left(C_0 - 2\pi \int_0^{s(t)} rv\, dr\right) + 1 - k_2\right\} s^2(t) \\
&\qquad + A\left\{s(t) - g\left(C_0 - 2\pi \int_0^{s(t)} rv\, dr\right) + 1 + A/s(t)\right\} \\
&> \{-k_1 C_0 + (1 - Ag\pi)(1 - k_2)\} s^2(t) + A(1 - Ag\pi)s(t) > 0.
\end{aligned}
$$

Hence we see that the right hand side of interior condition is positive at the point (r^*, t^*). This is a contradiction for the assumption of (r^*, t^*), and we have the Lemma. □

Remark 2.1. *Assumption 4 just means that, in Figure 2, $f_1(r)$ and $f_2(r)$ have two intersections in the interval $(0, r_*)$, where $f_2(r_*) = 0$. This relation of $f_1(r)$ and $f_2(r)$ play a key part of the existence of the time global solution.*

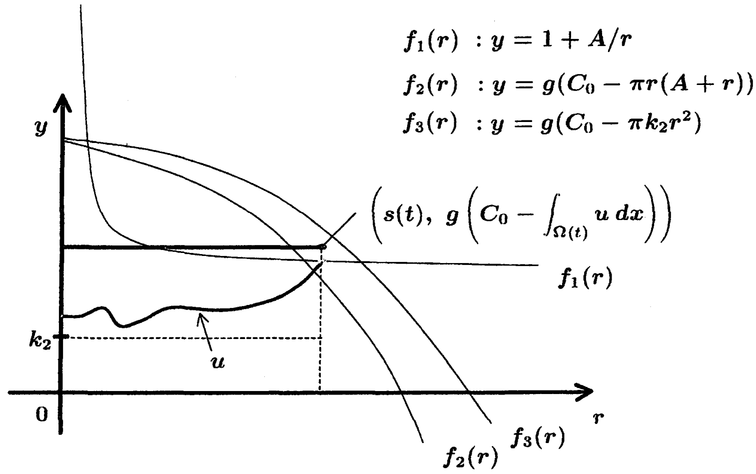

Figure 3: Cross section of a cell

From these Lemmas 1, 2, we see that

$$\alpha_2 \leq s(t) \leq \beta_2.$$

Lemma 3. *Initial data satisfy following conditions:*

$$k_2 < \psi|_{[0,s_0)} < \psi(s_0), \quad \alpha_2 \leq s_0 \leq \beta_2, \quad C_0 - 2\pi \int_0^{s_0} r\psi\, dr > 0,$$

then

$$k_2 < v < v(s(t), t), \quad \alpha_2 \leq s(t) \leq \beta_2, \quad C_0 - 2\pi \int_0^{s(t)} rv\, dr > 0,$$

for any $t \in [0, T]$ *and* $r \in [0, s(t))$.

By using the result of boundedness for u and $s(t)$, we obtain the boundedness for the Hölder norm of $s(t)$. As a result, we have the exsictence of the time global solution for (P). Here we remark that the profile of u is that the value in a neighborhood of the boundary is larger than one of the inside (see Figure 3). Actually, we can make sure of the truth that there exists some living things such that the density of F-actin in the cell is similar to the solution for (P).

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