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COBORDISM ON HOMOLOGY CYLINDERS AND COMBINATORIAL TORSIONS

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1. Introduction

In this extended abstract, we survey the results in [CFK09] of the authors. This extended abstract contains no original results. Let $\Sigma_{g,n}$ be an oriented compact surface of genus g with n boundary components. The *mapping class group* $\mathcal{M}_{g,n}$ is the group of orientation-preserving isotopy classes of automorphisms of $\Sigma_{g,n}$ which reduce to the identity on $\partial\Sigma_{g,n}$. The mapping class group has been one of the central research subjects in many mathematical areas, and we refer the reader to [FM11] for more details about the mapping class group.

Recently an enlargement of the mapping class group has been introduced. Goussarov [Go99] and Habiro [Ha00] introduced the notion of *homology cylinder* and Garoufalidis and Levine [GL05][Le01] introduced the homology cobordism group of homology cylinders. Roughly speaking, a homology cylinder over $\Sigma_{g,n}$ is a cobordism between surfaces equipped with markings (diffeomorphisms) to $\Sigma_{g,n}$ where the cobordism is required to be homologically a product. The isotopy classes of homology cylinders over $\Sigma_{g,n}$ form a monoid under juxtaposition and this monoid is denoted by $\mathcal{C}_{g,n}$. By considering categorical differences, we obtain a group $\mathcal{H}^{\text{smooth}}_{g,n}$ (resp. $\mathcal{H}^{\text{top}}_{g,n}$) of smooth (resp. topological) homology cobordism classes of homology cylinders. Henceforth, we will use the notation $\mathcal{H}_{g,n}$ for both $\mathcal{H}^{\text{smooth}}_{g,n}$ and $\mathcal{H}^{\text{top}}_{g,n}$ when the concerned results hold in both categories. See Section 2 for the precise definitions of homology cylinders and their cobordism groups.

We say that $\mathcal{C}_{g,n}$ and $\mathcal{H}_{g,n}$ are enlargements of $\mathcal{M}_{g,n}$ since $\mathcal{M}_{g,n}$ injects into $\mathcal{C}_{g,n}$ and $\mathcal{H}_{g,n}$. More precisely, there is a map $\mathcal{M}_{g,n} \to \mathcal{C}_{g,n} \to \mathcal{H}_{g,n}$ which is injective [CFK09, Proposition 2.4]. (Also see [GL05, Section 2.4] and [Le01, Section 2.1] for the case $n = 1$.) Therefore some natural questions arise regarding the comparison between the structures of $\mathcal{M}_{g,n}$ and $\mathcal{H}_{g,n}$. For instance, it is known that the mapping class group is finitely presented [BH71, Mc75] and perfect if $g \geq 3$ [Po78]. Regarding the homology cylinder, Goda and Sakasai [GS09] ask if $\mathcal{H}^{\text{smooth}}_{g,1}$ is a perfect group and Garoufalidis and Levine [GL05] ask if $\mathcal{H}^{\text{smooth}}_{g,1}$ is infinitely generated. Finally in [CFK09] the authors showed that if $b_1(\Sigma_{g,n}) > 0$ then $\mathcal{H}_{g,n}$ is not a perfect group and not finitely generated, answering the questions of Goda-Sakasai and Garoufalidis-Levine.

Theorem 1.1. [CFK09, Theorem 1.2 and Theorem 1.3]

(1) *If* $b_1(\Sigma_{g,n}) > 0$*, then there exists an epimorphism*

$$\mathcal{H}_{g,n} \to (\mathbb{Z}/2)^\infty$$

which splits (i.e., there is a right inverse). In particular, the abelianization of $\mathcal{H}_{g,n}$ *contains a direct summand isomorphic to* $(\mathbb{Z}/2)^\infty$*.*

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(2) *If $n > 1$, then there exists an epimorphism*

$$\mathcal{H}_{g,n} \to \mathbb{Z}^\infty.$$

Furthermore, the abelianization of $\mathcal{H}_{g,n}$ contains a direct summand isomorphic to $(\mathbb{Z}/2)^\infty \bigoplus \mathbb{Z}^\infty$.

In particular, this shows that the structure of $\mathcal{H}_{g,n}$ is much different from that of $\mathcal{M}_{g,n}$.
In this article, we will survey the proof of Theorem 1.1(1) in [CFK09] and the related materials such as the *torsion* invariant (see Section 2). The reader is referred to [CFK09] for more results such as the proof of Theorem 1.1(2), the difference between $\mathcal{H}^{\text{top}}_{g,n}$ and $\mathcal{H}^{\text{smooth}}_{g,n}$, and the Torelli group analogue for $\mathcal{H}_{g,n}$. We note that the proof of Theorem 1.1(2) also uses the torsion invariant but the argument is more elaborate than in the proof of Theorem 1.1(1).

2. Homology Cylinders and torsion invariants

2.1. Homology cylinders.

Definition 2.1. (1) A *homology cylinder (M, i_+, i_-) over a compact surface* Σ is defined to be a 3–manifold M together with injections (markings) i_+, $i_-\colon \Sigma \to \partial M$ satisfying the following:
   (a) i_+ is orientation preserving and i_- is orientation reversing.
   (b) $\partial M = i_+(\Sigma) \cup i_-(\Sigma)$ and $i_+(\Sigma) \cap i_-(\Sigma) = i_+(\partial\Sigma) = i_-(\partial\Sigma)$.
   (c) $i_+|_{\partial\Sigma} = i_-|_{\partial\Sigma}$.
   (d) $i_+, i_-\colon H_*(\Sigma) \to H_*(M)$ are isomorphisms.

(2) Two homology cylinders (M, i_+, i_-) and (N, j_+, j_-) over $\Sigma = \Sigma_{g,n}$ are called *isomorphic* if there exists an orientation-preserving diffeomorphism $f\colon M \to N$ satisfying $j_\pm = f \circ i_\pm$.

An example of homology cylinder is given using a mapping class $\varphi \in \mathcal{M}_{g,n}$ as follows: Let $M(\varphi) = (\Sigma_{g,n} \times [0,1]/\sim, i_+ = \mathrm{id} \times 0, i_- = \varphi \times 1)$ where $\sim$ is given by $(x,s) \sim (x,t)$ for $x \in \partial\Sigma_{g,n}$ and $s,t \in [0,1]$. Then $M(\varphi)$ is a homology cylinder. In particular, when $\varphi = \mathrm{id}$, we call the resulting homology cylinder the *product homology cylinder.* The isotopy classes of homology cylinders over $\Sigma_{g,n}$ form a monoid under juxtaposition: $(M, i_+, i_-)\cdot(N, j_+, j_-) := (M \cup_{i_- \circ (j_+)^{-1}} N, i_+, j_-)$. We denote by $\mathcal{C}_{g,n}$ the resulting monoid. Note that the data from the markings i_- and j_+ is used in the definition of the monoid operation.

Definition 2.2. Let (M, i_+, i_-) and (N, j_+, j_-) be homology cylinders over $\Sigma_{g,n}$. Then (M, i_+, i_-) and (N, j_+, j_-) are called *smoothly homology cobordant* (resp. topologically homology cobordant) if there exists a compact oriented smooth 4-manifold (resp. topological 4-manifold) W such that

$$\partial W = M \cup (-N) \,/\, i_+(x) = j_+(x),\, i_-(x) = j_-(x) \quad (x \in \Sigma),$$

and such that the inclusion induced maps $H_*(M) \to H_*(W)$ and $H_*(N) \to H_*(W)$ are isomorphisms.

The smooth (resp. topological) homology cobordism classes form a group under juxtaposition and we denote the resulting group by $\mathcal{H}^{\text{smooth}}_{g,n}$ (resp. $\mathcal{H}^{\text{top}}_{g,n}$). In particular, the monoid structure of $\mathcal{C}_{g,n}$ descends to a group structure of $\mathcal{H}_{g,n}$ and we have the surjection

$\mathcal{C}_{g,n} \to \mathcal{H}_{g,n}^{\mathrm{smooth}} \to \mathcal{H}_{g,n}^{\mathrm{top}}$. In $\mathcal{H}_{g,n}$, the identity is the class of the product homology cylinder and the inverse of (M, i_+, i_-) is $(-M, i_-, i_+)$. In [CFK09, Theorem 1.1], the authors showed that the kernel of the epimorphism $\mathcal{H}_{0,n}^{\mathrm{smooth}} \to \mathcal{H}_{0,n}^{\mathrm{top}}$ maps onto an abelian group of infinite rank.

Since $M(\varphi) \cdot M(\psi) = M(\varphi \circ \psi)$ for φ, $\psi \in \mathcal{M}_{g,n}$, we have a monoid morphism $\mathcal{M}_{g,n} \to \mathcal{C}_{g,n}$ which sends $\varphi \mapsto M(\varphi)$. Furthermore, the composition $\mathcal{M}_{g,n} \to \mathcal{C}_{g,n} \to \mathcal{H}_{g,n}$ is injective [CFK09, Proposition 2.4]. (Also see [GL05, Section 2.4] and [Le01, Section 2.1] for the case $n = 1$.)

2.2. **The torsion invariant of homology cylinders.** Let (M, N) be a manifold pair such that $H_*(M, N; \mathbb{Z}) = 0$ and $\varphi\colon \pi_1(M) \to H$ be an epimorphism to a free abelian group. Let $Q(H)$ be the quotient field of the group ring $\mathbb{Z}[H]$. Let $p\colon \tilde{M} \to M$ be the universal covering map of M and $\tilde{N} := p^{-1}(N)$. Then the *torsion invariant* $\tau(M, N; Q(H))$ is defined using the chain complex $C_*(\tilde{M}, \tilde{N}) \otimes_{\mathbb{Z}[\pi_1(M)]} Q(H)$. (See [Mi66] and [Tu01] for the definition of the torsion.)

Let (M, i_+, i_-) be a homology cylinder over $\Sigma_{g,n}$. Let $\Sigma_\pm := i_\pm(\Sigma)$ in M and $H := H_1(\Sigma; \mathbb{Z})$. We define a homomorphism $\varphi = \varphi(M) = \varphi((M, i_+, i_-))$ as follows:

$$\varphi\colon \pi_1(M) \to H_1(M; \mathbb{Z}) \xleftarrow{\cong} H_1(\Sigma_+; \mathbb{Z}) \xleftarrow{i_+} H = H_1(\Sigma; \mathbb{Z}).$$

Since $H_1(M, \Sigma_+; \mathbb{Z}) = 0$, we have $H_1(M, \Sigma_+; Q(H)) = 0$ (see [CFK09, Lemma 3.1]). Now we define the *torsion* of (M, i_+, i_-) as follows:

Definition 2.3. For the homomorphism $\varphi\colon \pi_1(M) \to H_1(M; \mathbb{Z}) \xleftarrow{\cong} H_1(\Sigma_+; \mathbb{Z}) \xleftarrow{i_+} H = H_1(\Sigma; \mathbb{Z})$, we define the *torsion* of (M, i_+, i_-) to be

$$\tau(M) := \tau(M, \Sigma_+; Q(H)) \in Q(H)^\times.$$

The torsion of a homology cylinder is well-defined up to multiplication by $\pm h$ $(h \in H)$ and was first studied by Sakasai [Sa06]. In fact, the torsion of a homology cylinder is computed easily as below. Note that $\mathbb{Z}[H]$ is a unique factorization domain, and hence for any finitely generated module over $\mathbb{Z}[H]$, we can define its *order*, which is an element of $\mathbb{Z}[H]$.

Lemma 2.4. [CFK09, Lemma 3.2] *For a homology cylinder (M, i_+, i_-), the torsion $\tau(M)$ is the order of $H_1(M, \Sigma_+; \mathbb{Z}[H])$ as a $\mathbb{Z}[H]$-module.*

3. A homomorphism to an abelian group

3.1. **Construction of a homomorphism.** To show Theorem 1.1, we will construct a homomorphism from $\mathcal{H}_{g,n}$ to an abelian group whose image is isomorphic to $(\mathbb{Z}/2)^\infty$.

Abusing the notation, for a homology cylinder (M, i_+, i_-) over $\Sigma_{g,n}$ and $H = H_1(\Sigma_{g,n}; \mathbb{Z})$, we denote the automorphism

$$(i_+)_*^{-1}(i_-)_*\colon H \xrightarrow[(i_-)_*]{\cong} H_1(M; \mathbb{Z}) \xrightarrow[(i_+)_*^{-1}]{\cong} H$$

by $\varphi(M)$. And for H_∂, the image of $H_1(\partial\Sigma_{g,n}; \mathbb{Z})$ in H, we define

$$\mathrm{Aut}^*(H) = \{\varphi \in \mathrm{Aut}(H) \mid \varphi \text{ fixes } H_\partial \text{ and preserves the intersection form of } \Sigma\}.$$

By [GS08, Proposition 2.3 and Remark 2.4], it is known that $\varphi(M) \in \mathrm{Aut}^*(H)$. Furthermore, $(i_+)_*^{-1}(i_-)_*$ induces an automorphism of $\mathbb{Z}[H]$ and we also denote it by $\varphi(M)$. For $a, b \in \mathbb{Z}[H]$, we write $a \doteq b$ if a and b differ by a unit in $\mathbb{Z}[H]$.

The proposition below shows that the torsion invariant induces not a homomorphism, but a *crossed* homomorphism on $\mathcal{C}_{g,n}$.

Proposition 3.1. [CFK09, Proposition 3.5] *Let (M, i_+, i_-) and (N, j_+, j_-) be homology cylinders over $\Sigma_{g,n}$. Then*

$$\tau(M \cdot N) \doteq \tau(M) \cdot \varphi(M)(\tau(N)).$$

Moreover, the torsion invariant is not a homology cobordism invariant under this setting. That is, the map $\tau\colon \mathcal{C}_{g,n} \to \mathbb{Z}[H]/\pm H$ does not factor through $\mathcal{H}_{g,n}$. For the group H, we equip $\mathbb{Z}[H]$ with the standard involution taking $g \mapsto g^{-1}$ for $g \in H$ and extend it to $Q(H)$ by setting $\overline{p \cdot q} = \overline{p} \cdot \overline{q}^{-1}$.

Theorem 3.2. [CFK09, Theorem 3.10] *Let $M = (M, i_+, i_-)$ and $N = (N, j_+, j_-)$ be homology cylinders over $\Sigma_{g,n}$ which are homology cobordant. Then $\tau(M)$ and $\tau(N)$ differ by a norm in $Q(H)$:*

$$\tau(M) \doteq \tau(N) \cdot q \cdot \bar{q} \in Q(H)^\times$$

for some $q \in Q(H)^\times$.

From Proposition 3.1 and Theorem 3.2, it seems that the torsion invariant does not work for our purpose, which is to construct a homomorphism on $\mathcal{H}_{g,n}$ with abelian image. From now on we will take an appropriate quotient group of $Q(H)^\times$ to make the torsion give a homomorphism on $\mathcal{H}_{g,n}$ with abelian image.

Define the subgroup $A = A(H)$ of $Q(H)^\times$ to be the subgroup of $Q(H)^\times$ generated by the set $\{\pm h \cdot p^{-1} \cdot \varphi(p) \mid h \in H,\ p \in Q(H)^\times, \text{ and } \varphi \in \mathrm{Aut}^*(H)\}$, and the subgroup $N = N(H)$ to be the subgroup $N(H) = \{\pm h \cdot q \cdot \bar{q} \mid q \in Q(H)^\times,\ h \in H\}$. From Proposition 3.1 and Theorem 3.2. we obtain the following theorem:

Theorem 3.3. [CFK09, Corollary 3.12] *The torsion invariant gives rise to a group homomorphism*

$$\tau\colon \mathcal{H}_{g,n} \to Q(H)^\times/AN,$$

where $H = H_1(\Sigma_{g,n}; \mathbb{Z})$.

Since for a homology cylinder (M, i_+, i_-) the torsion invariant is defined by using only the base manifold M, one can easily see that the above homomorphism descends to a homomorphism of the quotient of $\mathcal{H}_{g,n}$ modulo the normal subgroup $\langle \mathcal{M}_{g,n} \rangle$ generated by the mapping class group $\mathcal{M}_{g,n}$.

3.2. **Proof of Theorem 1.1.** In this subsection we give a proof of Theorem 1.1. We define

$$Q(H)^{sym} = \{p \in Q(H)^\times \mid p = \bar{p} \text{ in } Q(H)^\times/A\}.$$

Then $AN \subset Q(H)^{sym}$. For $p, q \in \mathbb{Z}[H]^\times$, define $p \sim q$ if $p \doteq \varphi(q)$ for some $\varphi \in \mathrm{Aut}^*(H)$. One easily sees that this gives an equivalence relation on $\mathbb{Z}[H]^\times$. Then we say that $p \in \mathbb{Z}[H]^\times$ is *self-dual* if $p \sim \overline{p}$. For each irreducible element $\lambda \in \mathbb{Z}[H]$, define $e_\lambda\colon Q(H)^\times \to \mathbb{Z}$ where for $p \in Q(H)$, $e_\lambda(p)$ is the sum of exponents of distinct irreducible factors μ of p such that $\mu \sim \lambda$. Now we have the following proposition.

Proposition 3.4. [CFK09, Proposition 5.1] *For a self-dual irreducible element $\lambda \in \mathbb{Z}[H]$, the map*

$$\Psi_\lambda \colon Q(H)^{sym}/AN \to \mathbb{Z}/2$$

defined by $\Psi_\lambda(p \cdot AN) = e_\lambda(p) + 2\mathbb{Z}$ is a surjective group homomorphism. Furthermore,

$$\Psi = \bigoplus_{[\lambda]} \Psi_\lambda \colon Q(H)^{sym}/AN \to \bigoplus_{[\lambda]} \mathbb{Z}/2,$$

is an isomorphism, where $[\lambda]$ runs over the equivalence classes of self-dual irreducible λ.

Proof of Theorem 1.1(1). Choose a knot K_i for each $i \in \mathbb{N}$ such that K_i are negative amphicheiral knots with irreducible Alexander polynomials $\Delta_i(t) := \Delta_{K_i}(t)$ and the multisets C_i of nonzero coefficients of $\Delta_i(t)$ are mutually distinct up to sign. We can find such knots, for instance using the knots in [Ch07, p. 60] whose Alexander polynomials are of the form $a^2t^2 - (2a^2+1)t + a^2$.

Let E_0 be the exterior of the trivial (string) knot in $D^2 \times [0,1]$ and $X = D^2 \times 0 \cap E_0$. Let $M = \Sigma_{g,n} \times [0,1]$ and $\iota \colon X \to \operatorname{int}(\Sigma_{g,n})$ be an embedding which induces a nontrivial homomorphism on homology groups, and let $f \colon E_0 \to \operatorname{int}(M)$ be the embedding defined by $f(x,t) = \iota(x, t/2 + 1/4)$. Now for each K_i denote by E_{K_i} the exterior of K_i in S^3 and define

$$M_i = (M - f(\operatorname{int}(E_0))) \underset{f(\partial E_0) = \partial E_{K_i}}{\cup} E_{K_i}.$$

Since E_{J_i} and E_0 have isomorphic homology groups, $M_i = (M_i, \mathrm{id}, \mathrm{id})$ is a homology cylinder. By [CFK09, Proposition 4.3], the M_i generate an abelian subgroup of $\mathcal{H}_{g,n}$ and we denote it by $\mathcal{S}$. Furthermore, if we denote by h the image of the generator $H_1(E_0;\mathbb{Z}) \cong \mathbb{Z}$ under the homomorphism induced from f, then by [CFK09, Proposition 4.3] $\tau(M_i) = \Delta_i(h)$. Since h is an indivisible element in $H = H_1(\Sigma_{g,n};\mathbb{Z})$, one can see that $\Delta_i(h)$ is irreducible and self-dual. Moreover $\Delta_i(h) \nsim \Delta_j(h)$ if $i \neq j$ since the multisets C_i are mutually distinct and invariants of $\Delta_i(h)$ under the equivalence relation $\sim$. Therefore we deduce that

$$\Psi_{\Delta_i(h)}(\tau(M_j)) = \Psi_{\Delta_i(h)}(\Delta_j(h)) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

This implies that the image of $\Psi \circ \tau \colon \mathcal{H}_{g,n} \to \bigoplus_{[\lambda]} \mathbb{Z}/2$ is isomorphic with $(\mathbb{Z}/2)^\infty$. Moreover for an irreducible and self-dual element $\lambda \in \mathbb{Z}[H]^\times$, if $\lambda \nsim \Delta_i(h)$, then $\Psi_\lambda(\Delta_i(h)) = 0$. Therefore we have a homomorphism $\mathcal{H}_{g,n} \to (\mathbb{Z}/2)^\infty$ whose restriction to the abelian group $\mathcal{S}$ is an isomorphism. Now the homomorphism splits and the abelian group $\mathcal{S}$, which is isomorphic to $(\mathbb{Z}/2)^\infty$, descends to a summand of the abelianization of $\mathcal{H}_{g,n}$. □

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