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EPIMORPHISMS BETWEEN 2-BRIDGE LINK GROUPS: ESSENTIAL SIMPLE LOOPS ON 2-BRIDGE SPHERES

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1. INTRODUCTION

The purpose of this note is to explain some of the ideas in [4] which gives an answer to a question on certain word problems on 2-bridge link groups raised in [10]. The key tool used in the proof is small cancellation theory, applied to two-generator and one-relatior presentations of 2-bridge link groups. We note that it has been proved by Weinbaum [16] and Appel and Schupp [2] that the word and conjugacy problems for prime alternating link groups are solvable, by using small cancellation theory (see also [3] and references in it). Moreover, it was also shown by Sela [14] and Préaux [11] that the word and conjugacy problems for any link group are solvable. A characteristic feature of [4] is that we give a complete answer to a special (but also natural) word problem for the groups of 2-bridge links, which form a special (but also important) family of prime alternating links. In the sequels [5, 6, 7] of [4], we give a complete answer to certain natural conjugacy problems, and the solutions will be used in [8] to establish a variation of McShane's identity for 2-bridge link groups, which had been conjectured by [13].

2. MAIN RESULTS

Consider the discrete group, H, of isometries of the Euclidean plane $\mathbb{R}^2$ generated by the π-rotations around the points in the lattice $\mathbb{Z}^2$. Set $(\boldsymbol{S}^2, \boldsymbol{P}) := (\mathbb{R}^2, \mathbb{Z}^2)/H$ and call it the *Conway sphere*. Then $\boldsymbol{S}^2$ is homeomorphic to the 2-sphere, and $\boldsymbol{P}$ consists of four points in $\boldsymbol{S}^2$. We also call $\boldsymbol{S}^2$ the Conway sphere. Let $\boldsymbol{S} := \boldsymbol{S}^2 - \boldsymbol{P}$ be the complementary 4-times punctured sphere. For each $r \in \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$, let α_r be the simple loop in $\boldsymbol{S}$ obtained as the projection of a line in $\mathbb{R}^2 - \mathbb{Z}^2$ of slope r. Then α_r is *essential* in $\boldsymbol{S}$, i.e., it does not bound a disk in $\boldsymbol{S}$ and is not homotopic to a loop around a puncture. Conversely, any essential simple loop in $\boldsymbol{S}$ is isotopic to α_r for a unique $r \in \hat{\mathbb{Q}}$. Then r is called the *slope* of the simple loop. Similarly, any simple arc δ in $\boldsymbol{S}^2$ joining two different points in $\boldsymbol{P}$ such that $\delta \cap \boldsymbol{P} = \partial\delta$ is isotopic to the image of a line in $\mathbb{R}^2$ of some slope $r \in \hat{\mathbb{Q}}$ which intersects $\mathbb{Z}^2$. We call r the *slope* of δ.

A *trivial tangle* is a pair (B^3, t), where B^3 is a 3-ball and t is a union of two arcs properly embedded in B^3 which is parallel to a union of two mutually disjoint arcs in ∂B^3. Let τ be the simple unknotted arc in B^3 joining the two components of t as illustrated in Figure 1. We call it the *core tunnel* of the trivial tangle. Pick a base point x_0 in int τ, and let (μ_1, μ_2) be the generating pair of the fundamental group $\pi_1(B^3 - t, x_0)$ each of which is represented by a based loop consisting of a small peripheral simple loop around a component of t and a subarc of τ joining the circle to x_0. For any base point $x \in B^3 - t$, the generating pair of $\pi_1(B^3 - t, x)$ corresponding to the generating pair (μ_1, μ_2) of $\pi_1(B^3 - t, x_0)$ via a

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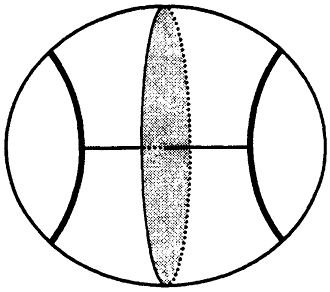


FIGURE 1. A trivial tangle

path joining x to x_0 is denoted by the same symbol. The pair (μ_1, μ_2) is unique up to (i) reversal of the order, (ii) replacement of one of the members with its inverse, and (iii) simultaneous conjugation. We call the equivalence class of (μ_1, μ_2) the *meridian pair* of the fundamental group $\pi_1(B^3 - t)$.

By a *rational tangle*, we mean a trivial tangle (B^3, t) which is endowed with a homeomorphism from $\partial(B^3, t)$ to $(\boldsymbol{S}^2, \boldsymbol{P})$. Through the homeomorphism we identify the boundary of a rational tangle with the Conway sphere. Thus the slope of an essential simple loop in $\partial B^3 - t$ is defined. We define the *slope* of a rational tangle to be the slope of an essential loop on $\partial B^3 - t$ which bounds a disk in B^3 separating the components of t. (Such a loop is unique up to isotopy on $\partial B^3 - t$ and is called a *meridian* of the rational tangle.) We denote a rational tangle of slope r by $(B^3, t(r))$. By van Kampen's theorem, the fundamental group $\pi_1(B^3 - t(r))$ is identified with the quotient $\pi_1(\boldsymbol{S})/\langle\langle \alpha_r \rangle\rangle$, where $\langle\langle \alpha_r \rangle\rangle$ denotes the normal closure.

For each $r \in \hat{\mathbb{Q}}$, the *2-bridge link* $K(r)$ *of slope* r is defined to be the sum of the rational tangles of slopes ∞ and r, namely, $(S^3, K(r))$ is obtained from $(B^3, t(\infty))$ and $(B^3, t(r))$ by identifying their boundaries through the identity map on the Conway sphere $(\boldsymbol{S}^2, \boldsymbol{P})$. (Recall that the boundaries of rational tangles are identified with the Conway sphere.) $K(r)$ has one or two components according as the denominator of r is odd or even. We call $(B^3, t(\infty))$ and $(B^3, t(r))$, respectively, the *upper tangle* and *lower tangle* of the 2-bridge link.

Let $\mathcal{D}$ be the *Farey tessellation*, whose ideal vertex set is identified with $\hat{\mathbb{Q}}$. For each $r \in \hat{\mathbb{Q}}$, let Γ_r be the group of automorphisms of $\mathcal{D}$ generated by reflections in the edges of $\mathcal{D}$ with an endpoint r, and let $\hat{\Gamma}_r$ be the group generated by Γ_r and Γ_∞. Then the region, R, bounded by a pair of Farey edges with an endpoint ∞ and a pair of Farey edges with an endpoint r forms a fundamental domain of the action of $\hat{\Gamma}_r$ on $\mathbb{H}^2$ (see Figure 2). Let I_1 and I_2 be the closed intervals in $\hat{\mathbb{R}}$ obtained as the intersection with $\hat{\mathbb{R}}$ of the closure of R. Suppose that r is a rational number with $0 < r < 1$. (We may always assume this except when we treat the trivial knot and the trivial 2-component link.) Write

$$r = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \ddots + \cfrac{1}{m_k}}} =: [m_1, m_2, \dots, m_k],$$

where $k \geq 1$, $(m_1, \ldots, m_k) \in (\mathbb{Z}_+)^k$, and $m_k \geq 2$. Then the above intervals are given by $I_1 = [0, r_1]$ and $I_2 = [r_2, 1]$, where

$$r_1 = \begin{cases} [m_1, m_2, \ldots, m_{k-1}] & \text{if } k \text{ is odd,} \\ [m_1, m_2, \ldots, m_{k-1}, m_k - 1] & \text{if } k \text{ is even,} \end{cases}$$

$$r_2 = \begin{cases} [m_1, m_2, \ldots, m_{k-1}, m_k - 1] & \text{if } k \text{ is odd,} \\ [m_1, m_2, \ldots, m_{k-1}] & \text{if } k \text{ is even.} \end{cases}$$

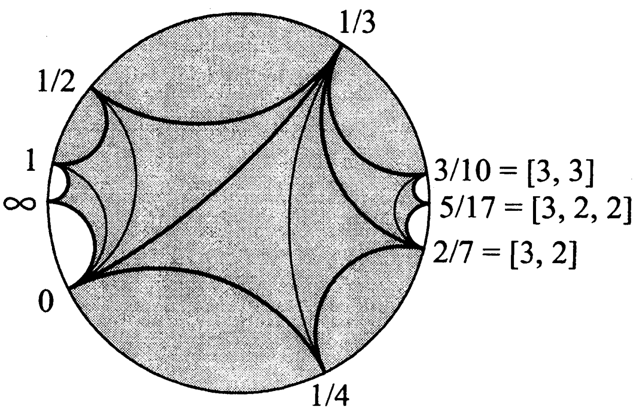


FIGURE 2. A fundamental domain of $\hat{\Gamma}_r$ in the Farey tessellation (the shaded domain) for $r = 5/17 = \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{2}}} =: [3, 2, 2]$.

We recall the following fact ([10, Proposition 4.6 and Corollary 4.7] and [4, Lemma 7.1]) which describes the role of $\hat{\Gamma}_r$ in the study of 2-bridge link groups.

Proposition 2.1. (1) *If two elements s and s' of $\hat{\mathbb{Q}}$ belong to the same orbit $\hat{\Gamma}_r$-orbit, then the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$.*

(2) *For any $s \in \hat{\mathbb{Q}}$, there is a unique rational number $s_0 \in I_1 \cup I_2 \cup \{\infty, r\}$ such that s is contained in the $\hat{\Gamma}_r$-orbit of s_0. In particular, α_s is homotopic to α_{s_0} in $S^3 - K(r)$. Thus if $s_0 \in \{\infty, r\}$ then α_s is null-homotopic in $S^3 - K(r)$.*

Thus the following question naturally arises (see [10, Question 9.1(2)]).

Question 2.2. (1) Which essential simple loops on S are null-homotopic in $S^3 - K(r)$?

(2) For two distinct rational numbers $s, s' \in I_1 \cup I_2$, when are the unoriented loops α_s and $\alpha_{s'}$ homotopic in $S^3 - K(r)$?

A complete answer to Question 2.2(1) is given by [4, Main Theorem 2.3] as follows.

Theorem 2.3. *The loop α_s is null-homotopic in $S^3 - K(r)$ if and only if s belongs to the $\hat{\Gamma}_r$-orbit of ∞ or r. In other words, if $s \in I_1 \cup I_2$ then α_s is not null-homotopic in $S^3 - K(r)$.*

This theorem implies the following theorem [4, Main Theorem 2.4], which gives a partial answer to [10, Question 9.1(1)].

Theorem 2.4. *There is an upper-meridian-pair-preserving epimorphism from $G(K(s))$ to $G(K(r))$ if and only if s or $s+1$ belongs to the $\hat{\Gamma}_r$-orbit of r or ∞.*

The following theorem, established in the series of papers [5, 6, 7], gives a complete answer to Question 2.2(2).

Theorem 2.5. (1) *Suppose $r = 1/p$, where $p \geq 2$ is an integer. Then, for any two distinct $s, s' \in I_1 \cup I_2$, the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$ if and only if $s = q_1/p_1$ and $s' = q_2/p_2$ satisfy $q_1 = q_2$ and $q_1/(p_1+p_2) = 1/p$, where (p_i, q_i) is a pair of relatively prime positive integers.*

(2) *Suppose $r = 3/8$, namely $K(r)$ is the Whitehead link. Then, for any two distinct $s, s' \in I_1 \cup I_2$, the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$ if and only if the set $\{s, s'\}$ equals either $\{1/6, 3/10\}$ or $\{3/4, 5/12\}$.*

(3) *Suppose $r \neq 1/p$ and $r \neq 3/8$. Then, for any two distinct $s, s' \in I_1 \cup I_2$, the unoriented loops α_s and $\alpha_{s'}$ are never homotopic in $S^3 - K(r)$.*

These results will be used in [8] to prove the following variation of McShane's identity, which had been conjectured in [13].

Theorem 2.6. *Suppose $r = q/p$ satisfies the condition $q \not\equiv \pm 1 \pmod{p}$, and let ρ be the holonomy representation of the complete hyperbolic structure of $S^3 - K(r)$. Then the following identity holds:*

$$2\sum_{s\in \mathrm{int} I_1} \frac{1}{1+e^{l_\rho(\alpha_s)}} + 2\sum_{s\in \mathrm{int} I_2} \frac{1}{1+e^{l_\rho(\alpha_s)}} + \sum_{s\in \partial I_1\cup\partial I_2} \frac{1}{1+e^{l_\rho(\alpha_s)}} = -1.$$

Further the modulus $\lambda(L(r))$ of the cusp torus of the cusped hyperbolic manifold $S^3 - K(r)$ with respect to a suitable choice of a longitude is given by the following formula:

$$\lambda(K(r)) = 2\sum_{s\in \mathrm{int} I_1} \frac{1}{1+e^{l_\rho(\alpha_s)}} + \sum_{r\in \partial I_1} \frac{1}{1+e^{l_\rho(\alpha_s)}}.$$

In the above theorem, $l_\rho(\alpha_s)$ is an element of $\mathbb{C}/2\pi\sqrt{-1}\mathbb{Z}$ defined as follows. The $PSL(2,\mathbb{C})$-representation of $\pi_1(S)$ induced by ρ extends to a representation, denoted by the same symbol ρ, of the orbifold fundamental group of the $(2,2,2,\infty)$-orbifold, $\mathcal{O}$, obtained as the quotient of S by the natural $\mathbb{Z}/(2\mathbb{Z}) \oplus \mathbb{Z}/(2\mathbb{Z})$-action (see e.g., [1, Proposition 2.2.2]). Each simple loop α_s in S doubly covers a simple loop in $\mathcal{O}$. Let $\sqrt{u_s}$ be (a conjugacy class of) an element of $\pi_1(\mathcal{O})$ represented by the simple loop. Then $l_\rho(\alpha_s)$ denotes the complex translation length of the hyperbolic isometry $\rho(\sqrt{u_s}) \in PSL(2,\mathbb{C}) \cong \mathrm{Isom}(\mathbb{H}^3)$.

We also obtain the following theorem concerning the set of end invariants $\mathcal{E}(\rho)$, defined by Tan, Wong and Zhang [15], of the $PSL(2,\mathbb{C})$-representation of $\pi_1(T)$ induced by the representation ρ in Theorem 2.6, where T is the once-punctured torus obtained as the double covering of the orbifold $\mathcal{O}$.

Theorem 2.7. *Let $r = q/p$ be a rational number. If $q \not\equiv \pm 1 \pmod{p}$, then let ρ be the holonomy representation of the complete hyperbolic structure of $S^3 - K(r)$. If $q \equiv \pm 1 \pmod{p}$, then let ρ be the faithful discrete $PSL(2,\mathbb{R})$-representation of the quotient of $G(K(r))$ by the infinite cyclic center. In both cases, we continue to denote by the same symbol ρ the $PSL(2,\mathbb{C})$-representation of $\pi_1(T)$ induced by ρ. Then the set of end invariants $\mathcal{E}(\rho)$ of ρ is equal to the limit set $\Lambda(\hat{\Gamma}_r)$ of $\hat{\Gamma}_r$.*

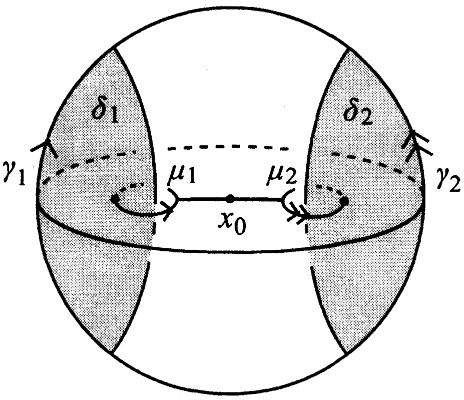


FIGURE 3. $\pi_1(B^3 - t(\infty), x_0) = F(a, b)$, where a and b are represented by μ_1 and μ_2, respectively.

3. PRESENTATIONS OF 2-BRIDGE LINK GROUPS

In this section, we introduce the upper presentation of a 2-bridge link group which we shall use throughout this paper. By van Kampen's theorem, the link group $G(K(r)) = \pi_1(S^3 - K(r))$ is identified with $\pi_1(\boldsymbol{S})/\langle\langle \alpha_\infty, \alpha_r \rangle\rangle$. We call the image in the link group of the meridian pair of the fundamental group $\pi_1(B^3 - t(\infty))$ (resp. $\pi_1(B^3 - t(r))$ the *upper meridian pair* (resp. *lower meridian pair*). The link group is regarded as the quotient of the rank 2 free group, $\pi_1(B^3 - t(\infty)) \cong \pi_1(\boldsymbol{S})/\langle\langle \alpha_\infty \rangle\rangle$, by the normal closure of α_r. This gives a one-relator presentation of the link group.

To find the presentation of $G(K(r))$ explicitly, let a and b, respectively, be the elements of $\pi_1(B^3 - t(\infty), x_0)$ represented by the oriented loops μ_1 and μ_2 based on x_0 as illustrated in Figure 3. Then $\{a, b\}$ forms the meridian pair of $\pi_1(B^3 - t(\infty))$, which is identified with the free group $F(a, b)$. Note that μ_i intersects the disk, δ_i, in B^3 bounded by a component of $t(\infty)$ and the essential arc, γ_i, on $\partial(B^3, t(\infty)) = (\boldsymbol{S}^2, \boldsymbol{P})$ of slope $1/0$, in Figure 3. Obtain a word u_r in $\{a, b\}$ by reading the intersection of the (suitably oriented) loop α_r with $\gamma_1 \cup \gamma_2$, where a positive intersection with γ_1 (resp. γ_2) corresponds to a (resp. b). Then the word u_r represents the free homotopy class of α_r. It then follows that

$$\begin{aligned} G(K(r)) = \pi_1(S^3 - K(r)) &\cong \pi_1(B^3 - t(\infty))/\langle\langle \alpha_r \rangle\rangle \\ &\cong F(a, b)/\langle\langle u_r \rangle\rangle \cong \langle a, b \,|\, u_r \rangle. \end{aligned}$$

If $r \neq \infty$, then α_r intersects γ_1 and γ_2 alternately, and hence a and b appear in (u_r) alternately.

By using the universal abelian covering $\mathbb{R}^2 - \mathbb{Z}^2 \to \boldsymbol{S}$, we can write down the word u_r explicitly. Note that the inverse image of γ_1 (resp. γ_2) in $\mathbb{R}^2 - \mathbb{Z}^2$ is the union of the single arrowed (resp. double arrowed) vertical edges in Figure 4. Let $L(r)$ be the line in $\mathbb{R}^2$ of slope r passing through the origin, and let $L^+(r)$ be the line obtained by translating $L(r)$ by the vector $(0, \eta)$ for sufficiently small positive real number η. Then $L^+(r)$ lies in $\mathbb{R}^2 - \mathbb{Z}^2$ and projects to the simple loop α_r. Pick a base point, z, from the intersection of $L^+(r)$ with the second quadrant, and consider the sub-line-segment of $L^+(r)$ bounded by z and $z + (2p, 2q)$. Then it forms a fundamental domain of the covering $L^+(r) \to \alpha_r$, and the word u_r is obtained by reading the intersection of the line-segment with the vertical lattice lines. To be precise, for each integer $0 \leq i \leq 2p - 1$, let P_i^+ be the intersection of

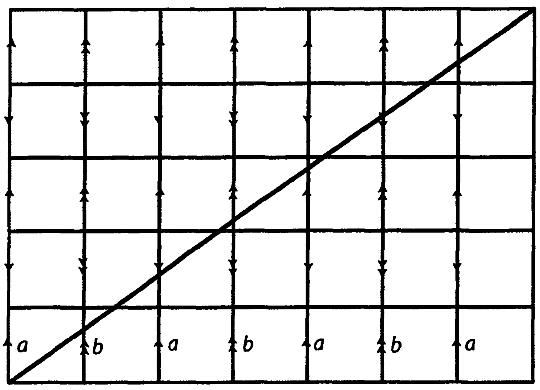


FIGURE 4. The line of slope 5/7 gives $\hat{u}_{5/7} = ba^{-1}bab^{-1}a$, so the free homotopy class of $\alpha_{5/7}$ is represented by the cyclic word $(u_{5/7}) = (a\hat{u}_{5/7}b^{-1}\hat{u}_{5/7}^{-1}) = (aba^{-1}bab^{-1}ab^{-1}a^{-1}ba^{-1}b^{-1}ab^{-1})$. Since the inverse image of γ_1 (resp. γ_2) in $\mathbb{R}^2$ is the union of the single arrowed (resp. double arrowed) vertical edges, a positive intersection with a single arrowed (resp. double arrowed) edge corresponds to a (resp. b).

the line-segment with the vertical lattice line $x = i$. We define the *letter* at P_i^+ to be a or b according as P_i^+ lies on a vertical edge with a single arrow or double arrow in Figure 4, namely according as i is even or odd. We define the *sign* of P_i^+ to be $+1$ or -1 according as the corresponding arrow is upward or downward. Then the letter and the sign of P_i^+, respectively, give the letter and the exponent of the $(i+1)$-th term of the word u_r for each $0 \le i \le 2p-1$. This gives the following formula for the word u_r (see Figure 4).

$$u_r = a^{\varepsilon_1}b^{\varepsilon_2}\cdots a^{\varepsilon_{2p-1}}b^{\varepsilon_{2p}},$$

where $\varepsilon_i = (-1)^{\lceil (i-1)q/p \rceil^* - 1}$. Here $\lceil t \rceil^*$ denotes the smallest integer greater than t.

In order to simplify this formula, let $\hat{u}_r$ be the subword of u_r corresponding to the set $\{P_i^+ \mid 1 \le i \le p-1\}$. Then $\hat{u}_r$ is obtained from the open interval in $L(r)$ bounded by $(0,0)$ and (p,q) by reading its intersection with the vertical lattice lines, and so we obtain the following formula.

$$\hat{u}_r = \begin{cases} b^{\epsilon_1}a^{\epsilon_2}\cdots b^{\epsilon_{p-2}}a^{\epsilon_{p-1}} & \text{if } p \text{ is odd,} \\ b^{\epsilon_1}a^{\epsilon_2}\cdots a^{\epsilon_{p-2}}b^{\epsilon_{p-1}} & \text{if } p \text{ is even,} \end{cases}$$

where $\epsilon_i = (-1)^{\lfloor iq/p \rfloor}$. By using the symmetry around (p,q) of $\mathbb{R}^2 - \mathbb{Z}^2$, we can observe that the subword of u_r corresponding to the set $\{P_i^+ \mid p+1 \le i \le 2p-1\}$ is equal to $\hat{u}_r^{-1}$. Hence we obtain the following formula (see [12, Proposition 1]).

$$u_r = \begin{cases} a\hat{u}_{q/p}b^{(-1)^q}\hat{u}_{q/p}^{-1} & \text{if } p \text{ is odd,} \\ a\hat{u}_{q/p}a^{-1}\hat{u}_{q/p}^{-1} & \text{if } p \text{ is even,} \end{cases}$$

We now describe a natural decomposition of the word u_r, which plays a key role in this paper. Let $r_i = q_i/p_i$ $(i = 1,2)$ be the rational number introduced in Section 2. Consider the infinite broken line, B, obtained by joining the lattice points

$$\cdots,(0,0),(p_2,q_2),(p,q),(p+p_2,q+q_2),(2p,2q),\cdots$$

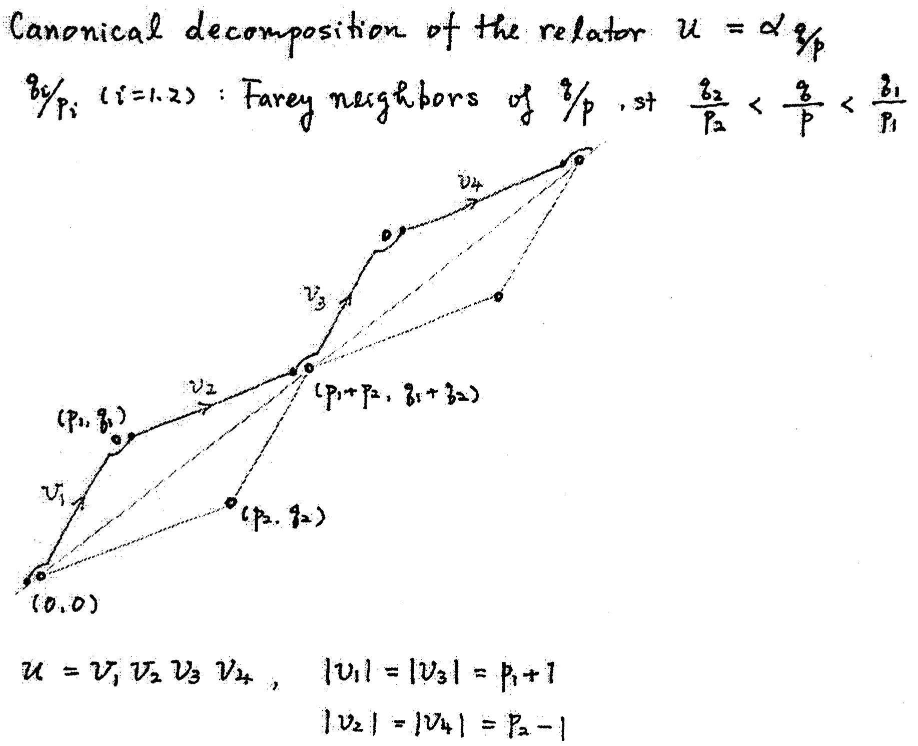


FIGURE 5. The decomposition of the relator $u_r = v_1 v_2 v_3 v_4$

which is invariant by the translation $(x, y) \mapsto (x + p, y + q)$. By slightly modifying B near the lattice points, we obtain a (topological) line, B^+, in $\mathbb{R}^2 - \mathbb{Z}^2$, invariant by the translation, which is homotopic to the line $L^+(r)$. Pick a point, $z_0 \in B^+$ in the second quadrant, and consider the sub-path of B^+ bounded by z_0 and $z_4 := z_0 + (2p, 2q)$. Then the word u_r is also obtained by reading the intersection of the sub-path with the vertical lattice lines. Pick a point $z_1 \in B^+$ whose x-coordinate is p_2 + (small positive number), and set $z_2 := z_0 + (p, q)$ and $z_3 := z_1 + (p, q)$. Let B_i^+ be the sub-path of B^+ joining z_{i-1} with z_i $(i = 1, 2, 3, 4)$. Let v_i be the subword of u_r corresponding to B_i^+. Then we have the decomposition

$$u_r = v_1 v_2 v_3 v_4.$$

The subword v_i is non-empty except when $r = 1/p$ $(p \in \mathbb{N})$ and $i \in \{1, 3\}$. The importance of this decomposition is described in the following section.

4. SEQUENCES ASSOCIATED WITH THE SIMPLE LOOP α_r

In this section, we define a sequence $S(r)$ of slope r and a cyclic sequence $CS(r)$ of slope r all of which arise from the single word u_r representing the simple loop α_r, and observe several important properties of these sequences, so that we can adopt small cancellation theory in the succeeding sections.

We fix some definitions and notation. Let X be a set. By a *word* in X, we mean a finite sequence $x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$ where $x_i \in X$ and $\epsilon_i = \pm 1$. Here we call $x_i^{\epsilon_i}$ the *i-th letter* of the word. For two words u, v in X, by $u \equiv v$ we denote the *visual equality* of u and v, meaning that if $u = x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$ and $v = y_1^{\delta_1} \cdots y_m^{\delta_m}$ ($x_i, y_j \in X$; $\epsilon_i, \delta_j = \pm 1$), then $n = m$ and $x_i = y_i$ and $\epsilon_i = \delta_i$ for each $i = 1, \dots, n$. The length of a word v is denoted by $|v|$. A word v in X is said to be *reduced* if v does not contain xx^{-1} or $x^{-1}x$ for any $x \in X$. A word is said to be *cyclically reduced* if all its cyclic permutations are reduced. A *cyclic word* is defined to be the set of all cyclic permutations of a cyclically reduced word. By (v) we denote the cyclic word associated with a cyclically reduced word v. Also by $(u) \equiv (v)$ we mean the *visual equality* of two cyclic words (u) and (v). In fact, $(u) \equiv (v)$ if and only if v is visually a cyclic shift of u.

Definition 4.1. (1) Let v be a nonempty reduced word in $\{a, b\}$. Decompose v into

$$v \equiv v_1 v_2 \cdots v_t,$$

where, for each $i = 1, \dots, t-1$, all letters in v_i have positive (resp. negative) exponents, and all letters in v_{i+1} have negative (resp. positive) exponents. Then the sequence of positive integers $S(v) := (|v_1|, |v_2|, \dots, |v_t|)$ is called the *S-sequence of* v.

(2) Let (v) be a nonempty reduced cyclic word in $\{a, b\}$ represented by a word v. Decompose (v) into

$$(v) \equiv (v_1 v_2 \cdots v_t),$$

where all letters in v_i have positive (resp. negative) exponents, and all letters in v_{i+1} have negative (resp. positive) exponents (taking subindices modulo t). Then the *cyclic* sequence of positive integers $CS(v) := ((|v_1|, |v_2|, \dots, |v_t|))$ is called the *cyclic S-sequence of* (v). Here the double parentheses denote that the sequence is considered modulo cyclic permutations.

Definition 4.2. For a rational number r with $0 < r \leq 1$, let u_r be the word in $\{a, b\}$ defined in Section 3. Then the symbol $S(r)$ (resp. $CS(r)$) denotes the S-sequence $S(u_r)$ of u_r (resp. cyclic S-sequence $CS(u_r)$ of (u_r)), which is called the *S-sequence of slope* r (resp. the *cyclic S-sequence of slope* r).

In the remainder of this paper unless specified otherwise, we suppose that r is a rational number with $0 < r \leq 1$, and write r as a continued fraction:

$$r = [m_1, m_2, \dots, m_k],$$

where $k \geq 1$, $(m_1, \dots, m_k) \in (\mathbb{Z}_+)^k$ and $m_k \geq 2$ unless $k = 1$. For brevity, we write m for m_1.

The following proposition plays a key role in the proof of Lemma 5.4 and Theorem 5.2.

Proposition 4.3 ([4, Proposition 4.10]). *The sequence $S(r)$ has a decomposition (S_1, S_2, S_1, S_2) which satisfies the following.*

1. *Each S_i is symmetric, i.e., the sequence obtained from S_i by reversing the order is equal to S_i. (Here, S_1 is empty if $k = 1$.)*
2. *Each S_i occurs only twice in the cyclic sequence $CS(r)$.*
3. *S_1 begins and ends with $m + 1$.*
4. *S_2 begins and ends with m.*

The above decomposition corresponds to the decomposition $u_r = v_1 v_2 v_3 v_4$ introduced in Section 3. To be precise, we have $S_1 = S(v_1) = S(v_3)$ and $S_2 = S(v_2) = S(v_4)$. The following proposition plays a key role in the proof of the main theorem.

Proposition 4.4. *Let $S(r) = (S_1, S_2, S_1, S_2)$ be as in Proposition 4.3. For a rational number s with $0 < s \le 1$, suppose that the cyclic S-sequence $CS(s)$ contains both S_1 and S_2 as a subsequence. Then $s \notin I_1 \cup I_2$.*

5. Small cancellation conditions for 2-bridge link groups

Let $F(X)$ be the free group with basis X. A subset R of $F(X)$ is called *symmetrized*, if all elements of R are cyclically reduced and, for each $w \in R$, all cyclic permutations of w and w^{-1} also belong to R.

Definition 5.1. Suppose that R is a symmetrized subset of $F(X)$. A nonempty word b is called a *piece* if there exist distinct $w_1, w_2 \in R$ such that $w_1 \equiv bc_1$ and $w_2 \equiv bc_2$. Small cancellation conditions $C(p)$ and $T(q)$, where p and q are integers such that $p \ge 2$ and $q \ge 3$, are defined as follows (see [9]).

(1) Condition $C(p)$: If $w \in R$ is a product of n pieces, then $n \ge p$.
(2) Condition $T(q)$: For $w_1, \dots, w_n \in R$ with no successive elements w_i, w_{i+1} an inverse pair (i mod n), if $n < q$, then at least one of the products $w_1w_2, \dots, w_{n-1}w_n, w_nw_1$ is freely reduced without cancellation.

The following key theorem enables us to apply small cancellation theory to the groups presentation $\langle a, b \,|\, u_r \rangle$ of $G(K(r))$.

Theorem 5.2. *Let r be a rational number such that $0 < r < 1$. Recall the presentation $\langle a, b \,|\, u_r \rangle$ of $G(K(r))$ given in Section 3, and let R be the symmetrized subset of $F(a, b)$ generated by the single relator u_r. Then R satisfies $C(4)$ and $T(4)$.*

Definition 5.3. For a positive integer n, a non-empty subword w of the cyclic word (u_r) is called a *maximal n-piece* if w is a product of n pieces and if any subword w' of u_r which properly contains w as an *initial* subword is not a product of n-pieces.

Theorem 5.2 actually follows from the following complete characterizations of the maximal n-pieces for $n = 1, 2, 3$. (For simplicity, we describe the result only for generic case.)

Lemma 5.4. *Suppose that r is a rational number such that $0 < r < 1$ and $r \ne 1/p$ for any integer $p \ge 2$. Let v_{ib}^* be the maximal proper initial subword of v_i, i.e., the initial subword of v_i such that $|v_{ib}^*| = |v_i| - 1$ ($i = 1, 2, 3, 4$). Then the following hold, where v_{ib} and v_{ie} are nonempty initial and terminal subwords of v_i with $|v_{ib}|, |v_{ie}| \le |v_i| - 1$, respectively.*

(1) *The following is the list of all maximal 1-pieces of (u_r), arranged in the order of the position of the initial letter:*

$$v_{1b}^*, v_{1e}v_2, v_2v_{3b}^*, v_{2e}v_{3b}^*, v_{3b}^*, v_{3e}v_4, v_4v_{1b}^*, v_{4e}v_{1b}^*.$$

(2) *The following is the list of all maximal 2-pieces of (u_r), arranged in the order of the position of the initial letter:*

$$v_1v_2, v_{1e}v_2v_{3b}^*, v_2v_3v_4, v_{2e}v_3v_4, v_3v_4, v_{3e}v_4v_{1b}^*, v_4v_1v_2, v_{4e}v_1v_2.$$

(3) *The following is the list of all maximal 3-pieces of (u_r), arranged in the order of the position of the initial letter:*

$$v_1v_2v_{3b}^*, v_{1e}v_2v_3v_4, v_2v_3v_4v_{1b}^*, v_{2e}v_3v_4v_{1b}^*, v_3v_4v_{1b}^*, v_{3e}v_4v_1v_2, v_4v_1v_2v_{3b}^*, v_{4e}v_1v_2v_{3b}^*.$$

Corollary 5.5. (1) *A subword w of the cyclic word $(u_r^{\pm 1})$ is a piece if and only if $S(w)$ does not contain S_1 as a subsequence and does not contain S_2 in its interior, i.e., $S(w)$ does not contain a subsequence (ℓ_1, S_2, ℓ_2) for some $\ell_1, \ell_2 \in \mathbb{Z}_+$.*

(2) *For a subword w of the cyclic word $(u_r^{\pm 1})$, w is not a product of two pieces if and only if $S(w)$ either contains (S_1, S_2) as a proper initial subsequence or contains (S_2, S_1) as a proper terminal subsequence.*

6. Outline of the proof of Theorem 2.3

Let R be the symmetrized subset of $F(a,b)$ generated by the single relator u_r of the group presentation $G(K(r)) = \langle a, b \mid u_r \rangle$. Suppose on the contrary that α_s is null-homotoic in $S^3 - K(r)$, i.e., $u_s = 1$ in $G(K(r))$, for some $s \in I_1 \cup I_2$. Then there is a *van Kampen diagram* M over $G(K(r)) = \langle a, b \,|\, R \rangle$ such that the boundary label is u_s. Here M is a simply connected 2-dimensional complex embedded in $\mathbb{R}^2$, together with a function ϕ assigning to each oriented edge e of M, as a *label*, a reduced word $\phi(e)$ in $\{a, b\}$ such that the following hold.

(1) If e is an oriented edge of M and e^{-1} is the oppositely oriented edge, then $\phi(e^{-1}) = \phi(e)^{-1}$.
(2) For any boundary cycle δ of any face of M, $\phi(\delta)$ is a cyclically reduced word representing an element of R. (If $\alpha = e_1, \dots, e_n$ is a path in M, we define $\phi(\alpha) \equiv \phi(e_1) \cdots \phi(e_n)$.)

We may assume M is *reduced*, namely it satisfies the following condition: Let D_1 and D_2 be faces (not necessarily distinct) of M with an edge $e \subseteq \partial D_1 \cap \partial D_2$, and let $e\delta_1$ and $\delta_2 e^{-1}$ be boundary cycles of D_1 and D_2, respectively. Set $\phi(\delta_1) = f_1$ and $\phi(\delta_2) = f_2$. Then we have $f_2 \neq f_1^{-1}$.

Moreover, we may assume the following conditions:

(1) $d_M(v) \geq 3$ for every vertex $v \in M - \partial M$.
(2) For every edge e of ∂M, the label $\phi(e)$ is a piece.
(3) For a path $e_1, \cdots, e_n$ in ∂M of length $n \geq 2$ such that the vertex $\bar{e}_i \cap \bar{e}_{i+1}$ has degree 2 for $i = 1, 2, \cdots, n-1$, $\phi(e_1)\phi(e_2)\cdots\phi(e_n)$ cannot be expressed as a product of less than n pieces.

Since R satisfies the conditions $C(4)$ and $T(4)$ by Theorem 5.2, M is a $[4,4]$-map, i.e.,

(1) $d_M(v) \geq 4$ for every vertex $v \in M - \partial M$.
(2) $d_M(D) \geq 4$ for every face $D \in M$.

Here, $d_M(v)$, the *degree of* v, denotes the number of oriented edges in M having v as initial vertex, and $d_M(D)$, the *degree of* D, denotes the number of oriented edges in a boundary cycle of D.

Now, for simplicity, assume that M is homeomorphic to a disk. (In general, we need to consider an extremal disk of M.) Then by the Curvature Formula of Lyndon and Schupp (see [9, Corollary V.3.4]), we have

$$\sum_{v \in \partial M} (3 - d_M v)) \geq 4.$$

By using this formula, we see that there are three edges e_1, e_2 and e_3 in ∂M such that $e_1 \cap e_2 = \{v_1\}$ and $e_2 \cap e_3 = \{v_2\}$, where $d_M(v_i) = 2$ for each $i = 1, 2$. Since $\phi(e_1)\phi(e_2)\phi(e_3)$ is not expressed as a product of two pieces, we see by Corollary 5.5 that the boundary label of M contains a subword, w, with $S(w) = (S_1, S_2, \ell)$ or (ℓ, S_2, S_1). This in turn implies that the S-sequence of the boundary label contains both S_1 and S_2 as subsequences. Hence, by Proposition 4.4, we have $s \notin I_1 \cup I_2$, a contradiction.

7. Outline of the proof of Theorem 2.5

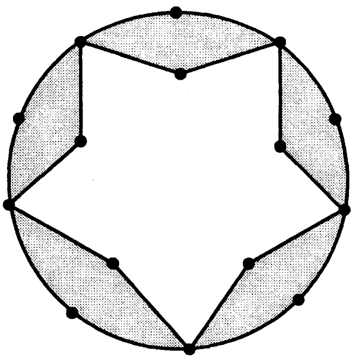
Suppose, for two distinct $s, s' \in I_1 \cup I_2$, the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$. Then there is a reduced annular R-diagram, such that u_s is an outer boundary label and $u_{s'}^{\pm 1}$ is an inner boundary label of M. Again we can see that M is a $[4, 4]$-map and hence we have the following curvature formula.

$$0 \leq \sum_{v \in \partial M} (3 - d_M(v)).$$

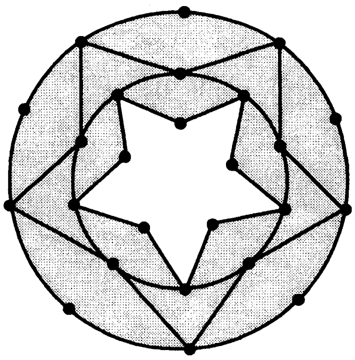
By using this formula, we obtain the following very strong structure theorem for M, which plays key roles throughout the series of papers [5, 6, 7].

Theorem 7.1. *Figure 6(a) illustrates the only possible type of the outer boundary layer of M, while Figure 6(b) illustrates the only possible type of whole M. (The number of faces per layer and the number of layers are variable.)*

In the above theorem, the *outer boundary layer* of the annular map M is a submap of M consisting of all faces D such that the intersection of ∂D with the outer boundary of M contains an edge, together with the edges and vertices contained in ∂D.



(a)



(b)

Figure 6.

The first paper [5] of the series treates the case when the 2-bridge link is a $(2, p)$-torus link, the second paper [6], treats the case of 2-bridge links of slope $n/(2n + 1)$ and $(n + 1)/(3n + 2)$, where $n \geq 2$ is an arbitrary integer, and the third paper [7] treats the general case. The two families treated in the second paper play special roles in the project in the sense that the treatment of these links form a base step of an inductive proof of the theorem for general 2-bridge links. We note that both a 2-bridge link of slope $n/(2n + 1)$ with $n = 2$ and a 2-bridge link of slope $(n + 1)/(3n + 2)$ with $n = 1$ are the figure-eight knot. It is a bit surprising that the treatment of the figure-eight knot is the most complicated. This reminds us of the phenomenon in the theory of exceptional Dehn filling that the figure-eight knot attains the maximal number of exceptional Dehn fillings.

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