

TWISTED ALEXANDER POLYNOMIAL REVISITED

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1. IDEAS BEHIND THE DEFINITION

We first would like to explain the heuristic ideas behind the definition of the twisted Alexander polynomial. For a formal treatment of the subject, the reader is referred to [3].

1.1. **Represented knot diagram.** Let $\Gamma = \pi_1(S^3 - K)$ be a knot group, and $\rho : \Gamma \rightarrow GL(n, R)$ a representation of Γ over a field R .

Suppose that a specific diagram D of the knot K is given, and let

$$\Gamma = \langle x_1, \dots, x_s \mid r_1, \dots, r_{s-1} \rangle$$

be the Wirtinger presentation of Γ . Recall that we associate to each overpass of D a generator x_i , and to each crossing of D a relation of the form:

$$x_i x_j = x_j x_k$$

A representation ρ can then be thought of as a way of associating to each overpass of D a matrix $X_i = \rho(x_i)$ so that the equation

$$X_i X_j = X_j X_k$$

holds at each crossing of D . We call a knot diagram with associated matrices X_i satisfying the Wirtinger relations a represented knot diagram.

1.2. **Affine deformations.** Now, let us ask if the given representation extends to an affine representation, or, equivalently, if the represented knot diagram extends to an affine represented knot diagram by matrices of the form:

$$(1) \quad \begin{pmatrix} X_i & dX_i \\ 0 & 1 \end{pmatrix} \quad (i = 1, \dots, s)$$

Note that the Wirtinger relation

$$\begin{pmatrix} X_i & dX_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X_j & dX_j \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} X_j & dX_j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X_k & dX_k \\ 0 & 1 \end{pmatrix}$$

is equivalent to the condition

$$dX_i + (X_i - 1)dX_j - X_j dX_k = 0.$$

This may remind the reader that the free differential of the Wirtinger relator

$$r = x_i x_j - x_j x_k$$

Received March 1, 2011.

is given by

$$\begin{aligned} dr &= \frac{\partial r}{\partial x_i} dx_i + \frac{\partial r}{\partial x_j} dx_j + \frac{\partial r}{\partial x_k} dx_k \\ &= dx_i + (x_i - 1) dx_j - x_j dx_k. \end{aligned}$$

In fact, the matrices (1) define an affine representation if and only if the set of vectors dX_1, \dots, dX_s satisfy the following equation.

$$(2) \quad \begin{pmatrix} \dots & 1 & \dots & (X_i - 1) & \dots & -X_j & \dots \\ \dots & & & & & & \dots \\ \dots & & & \tilde{\rho} \left(\frac{\partial r_i}{\partial x_j} \right) & & & \dots \\ \dots & & & & & & \dots \end{pmatrix} \begin{pmatrix} dX_1 \\ \vdots \\ dX_s \end{pmatrix} = 0$$

Let us denote by M the matrix on the left hand side, and the affine deformations correspond to the kernel of M .

One obtains certain solutions of (2) by translating the origin of the linear representation. Namely, the matrices

$$\begin{pmatrix} X_i & dX_i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X_i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} X_i & (1 - X_i)v \\ 0 & 1 \end{pmatrix}$$

define an affine representation for each vector $v \in R^n$. These are non-interesting ones; let us call them inessential affine deformations. The real question is if there exists an essential affine representation that is not a mere translation of the linear one. For simplicity of the argument, let us assume that $1 - X_s$ is non-singular. Then, the question is equivalent to ask if there is a non-zero solution of (2) such that $dX_s = 0$. Let M_s denote the square matrix obtained from M by removing the s -th "column". Then, there is an essential affine deformation if and only if the kernel of M_s is non-trivial, that is, if and only if $\det M_s = 0$.

1.3. Parameterized representations. Now, let $\alpha : \Gamma \rightarrow R^\times$ be a one-dimensional representation of Γ . Since R^\times is commutative, α factors through the abelianization $\Gamma \rightarrow H_1(S^3 - K) = \langle t \rangle$, and is determined by the image of the meridian t , which we denote by t again. Thus, we have $\alpha(x_i) = t \in R^\times$ ($i = 1, \dots, s$).

By taking the tensor product of ρ and α , we obtain a one-parameter family of representations $\rho_t = \rho \otimes \alpha : \Gamma \rightarrow GL(n, R)$. From the viewpoint of represented knot diagram, this amounts to considering matrices of the form:

$$\begin{pmatrix} tX_i & dX_i \\ 0 & 1 \end{pmatrix}$$

If we replace ρ by ρ_t and repeat the argument of the previous section, we obtain the following:

Claim 1.1. *There exists an interesting affine deformation of ρ_t if and only if $\det M_s(t) = 0$.*

This argument does not show that $\det M_s(t)$ is independent of the choice of the knot diagram, but at least its roots are.

Proof of the invariance of $\det M_s(t)$ and its generalization to link groups and more general groups require some labor, and the reader finds the result of our endeavor in [3].

2. PROBLEMS

2.1. Surjective homomorphism. Let $\varphi : \Gamma \rightarrow \Gamma'$ be a surjective homomorphism. Then, every representation $\rho' : \Gamma' \rightarrow GL(n, R)$ induces a representation $\rho = \rho' \circ \varphi : \Gamma \rightarrow GL(n, R)$. We raised the question in around 2004 about the relationship between the twisted Alexander polynomial of Γ associated to ρ and that of Γ' associated to ρ' , and soon obtained the following ([1]).

Theorem 2.1. *The twisted Alexander polynomial of Γ associated to ρ is divisible by the twisted Alexander polynomial of Γ' associated to ρ' .*

2.2. Tensor product. Suppose that we have a second representation $\rho' : \Gamma \rightarrow GL(m, R)$ of Γ . It is easy to show the following.

Theorem 2.2. *The twisted Alexander polynomial of Γ associated to $\rho \oplus \rho'$ is the product of those associated to ρ and to ρ' .*

However, we know nothing about the twisted Alexander polynomial of the tensor product representation.

Problem 2.3. *Can we say anything about the twisted Alexander polynomial of Γ associated to $\rho \otimes \rho'$ in terms of those associated to ρ and to ρ' ?*

In particular:

Problem 2.4. *Does the twisted Alexander polynomial of Γ associated to $\rho^{\otimes k}$ contain more information about the representation than that associated to ρ ?*

2.3. Generalized derivation and relative Alexander polynomial. Let us write $X_i = \rho(x_i) \in GL(n, R)$ and $X'_i = \rho'(x_i) \in GL(m, R)$, and consider represented knot diagrams with matrices of the form

$$\begin{pmatrix} tX_i & dX_i \\ 0 & X'_i \end{pmatrix},$$

where $dX_i \in M(n, m; R)$ are $n \times m$ matrices.

We may introduce a generalized derivation by the property

$$d(uv) = du\rho'(v) + \rho(u)dv \quad (u, v \in \Gamma),$$

and extend the definition of the Alexander matrix $M(t)$ accordingly. Then, we foresee that an essential deformation exists if and only if $\det M_s(t) = 0$.

Problem 2.5. *Formalize the above argument, and define a relative twisted Alexander polynomial of Γ associated to ρ and ρ' .*

2.4. twisted Alexander polynomial associated to the holonomy representation of hyperbolic knot. Let K be a hyperbolic knot. Then, the complement of K admits a unique complete hyperbolic metric of finite volume, and we have a holonomy representation

$$\mu : \Gamma \rightarrow \text{Isom}^+(\mathbf{H}^3) = PSL(2, \mathbf{C}).$$

It is known that this representation lifts to

$$\tilde{\mu} : \Gamma \rightarrow SL(2, \mathbf{C}),$$

and the twisted Alexander polynomial of Γ associated to $\tilde{\mu}$ becomes an invariant of the hyperbolic knot K . We once studied this invariant, but could not find its geometric meaning. The difficulty lies in the fact that the holonomy representation is not linear by nature. It may be more natural to consider μ as an $SO(3, 1)$ -representation.

Let us review how $PSL(2, \mathbf{C})$ is related to $SO(3, 1)$ according to [2]. First, recall that the group $PSL(2, \mathbf{C})$ acts on the hyperbolic 3-space

$$\mathbf{H}^3 = \{ z = z_0 + z_1i + z_2j \in \mathbf{H} \mid z_2 > 0 \}$$

by Möbius transformations. Here, \mathbf{H} denotes the quaternions. For a matrix

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{C}),$$

we denote the corresponding Möbius transformation by the same symbol:

$$g(z) = (az + b)(cz + d)^{-1}$$

Now, consider the transformation

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -j \\ -j & 1 \end{pmatrix} : z \mapsto (z - j)(-jz + 1)^{-1}$$

which maps \mathbf{H}^3 onto

$$\mathbf{B}^3 = \{ z = z_0 + z_1i + z_2j \in \mathbf{H} \mid z_0^2 + z_1^2 + z_2^2 < 1 \}.$$

The composition

$$\begin{aligned} \phi g \phi^{-1} &= \frac{1}{2} \begin{pmatrix} 1 & -j \\ -j & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & j \\ j & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} (a + \bar{d}) + (b - \bar{c})j & (b + \bar{c}) + (a - \bar{d})j \\ (\bar{b} + c) - (\bar{a} - \bar{c})j & (\bar{a} + d) - (\bar{b} - c)j \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} e & f \\ \bar{f} & \bar{e} \end{pmatrix} \end{aligned}$$

defines a transformation of \mathbf{B}^3 .

Next, we consider the transformation

$$\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \ell \\ -\ell & 1 \end{pmatrix} : z \mapsto (z + \ell)(-\ell z + 1)^{-1}.$$

Here, the symbol ℓ is assumed to satisfy $\ell^2 = 1$ and anti-commutes with i and j . This ψ maps \mathbf{B}^3 onto a sheet of hyperboloid

$$Q_+^3 = \{ z_0 + z_1i + z_2j + z_3\ell \mid z_0^2 + z_1^2 + z_2^2 - z_3^2 = -1, z_3 > 0 \}.$$

The composition given by the matrix

$$\begin{aligned} \psi \phi g \phi^{-1} \psi^{-1} &= \frac{1}{2} \begin{pmatrix} 1 & \ell \\ -\ell & 1 \end{pmatrix} \begin{pmatrix} e & f \\ \bar{f} & \bar{e} \end{pmatrix} \begin{pmatrix} 1 & -\ell \\ \ell & 1 \end{pmatrix} \\ &= \begin{pmatrix} e + f\ell & 0 \\ 0 & \bar{e} - \bar{f}\ell \end{pmatrix} \end{aligned}$$

maps z to $(e + f\ell)z(\bar{e} - \bar{f}\ell)^{-1}$. This not only defines a transformation of \mathbb{Q}_+^3 , but also gives a global transformation of $\mathbb{R}^{3,1}$ preserving the Minkowski form $z_0^2 + z_1^2 + z_2^2 - z_3^2$. Thus, it defines an element of $SO(3, 1)$. See [2] for the detail.

Problem 2.6. *Let $\mu : \Gamma \rightarrow SO(3, 1)$ be the holonomy representation defined by the hyperbolic structure of the complement of K . Study the twisted Alexander polynomial of Γ associated to μ , and find its geometric meaning.*

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