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# ON THE MAPPING DEGREE SETS FOR 3-MANIFOLDS

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ABSTRACT. This note records the recent results on the following questions: Let $M$ and $N$ be a closed orientable 3-manifolds, $D(M,N)$ be the set of degrees of maps from $M$ to $N$, denote $D(M,M)$ by $D(M)$.

(1) For which $N$, is the set $\mathcal{D}(M,N)$ finite for any $M$?

(2) If $D(M)$ is unbounded, what is $D(M)$?

(3) When is a self-map of degree $\pm 1$ on $M$ homotopic to a homeomorphims?

Some of those results were presented at the RIMS Seminar at Akita Shirakami during September 13-17, 2010. For the proofs of those results, see [DeW2], [DeSW], [Wa1], [Du], [SWW], [SWWZ], [Sun].

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## 1. INTRODUCTION

Let $M$ and $N$ be two closed oriented 3-dimensional manifolds. Let $D(M,N)$ be the set of degrees of maps from $M$ to $N$, that is

$$D(M,N)=\{d\in\mathbb{Z}\,|\,f\colon\ M\to N,\ \ \deg(f)=d\}.$$

We will simply use $D(N)$ to denote $D(N,N)$, the set of self-mapping degrees of $N$.

The calculation of $D(M,N)$ is a classical topic which often appeared in the literatures. According to [CT], Gromov thought it is a fundamental problem in topology to determine the set $D(M,N)$ for any dimension $n$.

Specially the calculation of $D(M)$, the integer set naturally associated to each closed orientable manifold $M$ which presents an interesting connections between topology and number theory.

The result is simple and well-known for dimension $n=1,2$. For dimension $n>3$, there are some interesting special results (See [DW] for recent ones and references therein), but it is difficult to get general results, since there are no classification results for manifolds of dimension $n>3$.

The case of dimension 3 becomes the most attractive in this topic. Since Thurston's geometrization conjecture, which has been confirmed, implies that closed orientable 3-manifolds can be classified in reasonable sense.

A basic property of $D(M,N)$ is reflected in the following:

**Question 1.1.** *(see* [Wa2, Question 1.3] *and* [Re, Problem A]*): For which closed orientable 3-manifolds $N$, is the set $D(M,N)$ finite for any given closed oriented 3-manifold $M$?*

It is clear if $D(N)$ is unbounded, then $D(M,N)$ is unbounded for some $M$. For each $M$, it is clear $\{0,1\}\subset D(M)$, and if $D(M)$ is bounded then $D(M)\subset\{0,1,-1\}$.

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**Question 1.2.** *Let $M$ be a closed orientable 3-manifold.*
*(1) When is $D(M)$ bounded?*
*(2) If $D(M)$ is unbounded, what is $D(M)$?*

**Remark 1.3.** The still unknown part for $D(M)$ is that if $D(M)$ is bounded, when does $-1 \subset D(M)$?

The following related question is also natural and interesting.

**Question 1.4.** *For which closed orientable 3-manifolds $M$, whether there is a selfmap of degree $\pm 1$ on $M$ which is not homotopic to a homeomorphism on $M$?*

Under Thurston's picture of 3-manifold, which is confirmed now, Question 1.2 (1) is answered 20 years ago; Question 1.1 and Question 1.2 (2) were answered very recently; the answer of Question 3 is known for Haken manifold and hyperboloic manifolds long times ago, and the answer is complete now for prime 3-manifolds. In Sections 2, 3 and 4, we will present those answers as well as how those answers are developed.

To end this section, we present the picture of 3-manifold which will be used to present the answers. All terminologies not defined are standard, see [He], [Sc] and [IR].

**The picture of 3-manifolds:** Each closed orientable 3-manifold $N$ has a unique prime decomposition $N_1\#.....\#N_k$, the prime factors are unique up to the order and up to homeomorphisms. Each closed orientable prime 3-manifold $N$ has a unique geometric decomposition such that each geometric piece supports one of the following eight geometries: $H^3$, $\widetilde{PSL}(2,R)$, $H^2 \times E^1$, Sol, Nil, $E^3$, $S^3$ and $S^2 \times E^1$ (where $H^n$, $E^n$ and $S^n$ are n-dimensional hyperbolic space, Euclidean space and sphere respectively), for details see [Th] and [Sc]. Moreover each geometric piece of $N$ with non-trivial geometric decomposition supports either $H^3$-geometry or $H^2 \times E^1$-geometry, hence each 3-manifold supporting one of the remaining six geometry is closed. Furthermore each 3-manifold supporting geometries of either $H^2 \times E^1$, or $E^3$, or $S^2 \times E^1$ is covered by a trivial circle bundle, and each 3-manifold supporting geometries of either Sol, or Nil, or $E^3$ is covered by a torus bundle. Call prime closed orientable 3-manifold $N$ a *non-trivial graph manifold* if $N$ has non-trivial geometric decomposition but contains no hyperbolic piece.

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## 2. About $D(M,N)$

This section is based on [DeW2] and [DeSW].
The answer of Question 1.1 is the following

**Theorem 2.1.** *Let $N$ be a closed orientable 3-manifold. Then there is a closed orientable 3-manifold $M$ such that $|D(M,N)| = \infty$ if and only if $|\mathcal{D}(R)| = \infty$ for each prime factor $R$ of $N$.*

In the following we will make a brief recall of the development of Theorem 2.1.

**The development of Theorem 2.1:** It is a common sense for many people that $|D(N)| = \infty$ for 3-manifold $N$ which is either a product of a surface and the circle, or $N$ is covered by the 3-sphere. The first significant result in this direction is due to Milnor and Thurston in the later 1970's. By using the minimum integer number of 3-simplices to build $N$ [MT, Theorem 2], they proved

**Theorem 2.2.** *For each given hyperbolic 3-manifold $N$, $|D(M,N)| < \infty$ for any $M$.*

Gromov [G] introduced the simplicial volume $||N||$ for a manifold $N$, which is approximately the minimum real number of 3-simplices to build $N$. Gromov and Thurston proved that $||N||$ is proportional to the hyperbolic volume of $N$ in the case of $N$ is a hyperbolic 3-manifold, and then Soma proved $||N||$ is proportional to the sum of the hyperbolic volume of the hyperbolic pieces in the geometric decomposition of $N$ (see [G], [Th], [So]). $|| * ||$ respects the mapping degrees, i.e. for any map $f\colon M \to N$ then $||M|| \geq |\deg(f)| \cdot ||N||$. Then it is deduced that

**Theorem 2.3.** *Suppose $N$ is a closed orientable 3-manifold. If a prime factor of $N$ has a hyperbolic piece in its geometric decomposition, then $|D(M,N)| < \infty$ for any $M$.*

Brooks and Goldman [BG1] [BG2] introduced the Seifert volume $SV(*)$ for closed orientable 3-manifolds which also respects the mapping degrees and is non-zero for each 3-manifold supporting the $\widetilde{PSL}(2,R)$ geometry. Then it is deduced that

**Theorem 2.4.** *Suppose $N$ is a closed orientable 3-manifold. If a prime factor of $N$ supports $\widetilde{PSL}(2,R)$ geometry. Then $|D(M,N)| < \infty$ for any $M$.*

Both Theorems 2.3 and 2.4 were already known in the early 1980's. The following result is known no later than early 1990's (see [Wal] for example).

**Theorem 2.5.** *Suppose $N$ is a closed orientable 3-manifold. Then $|\mathcal{D}(N)| = \infty$ if and only if either $N$ is covered by a torus bundle or a trivial circle bundle, or each prime factor of $N$ is covered by $S^3$ or $S^2 \times E^1$.*

After Theorems 2.3, 2.4 and 2.5, the remaining unknown cases for Question 1.1 are: either a prime factor of $N$ is a non-trivial graph manifold; or $N$ is a non-prime 3-manifold, and $|\mathcal{D}(R)| = \infty$ for each prime factor $R$ of $N$, but some $R$ is not covered by either $S^3$ or $S^2 \times E^1$.

In 2009 it is proved in [DeW2] that each closed orientable non-trivial graph manifold $N$ has a finite covering $\widetilde{N}$ with positive Seifert volume (it is still unknown weather $SV(\widetilde{N}) > 0$ implies $SV(N) > 0$ for a finite cover $\tilde{N} \to N$)), and therefore it is deduced that

**Theorem 2.6.** *Let $N$ be closed orientable non-trivial graph manifold. Then $|D(M,N)| < \infty$ for any closed orientable 3-manifold $M$.*

**Remark 2.7.** Two years before [DeW2], Theorem 2.6 is proved under the restriction that $M$ are also graph manifolds [DeW1], by using a standard form of maps between graph manifolds[De1], and the estimation of the PSL(2,R)-volume for a certain special class of graph manifolds.

In 2010 it is proved in [DeSW]

**Theorem 2.8.** *Let $N$ be a given closed oriented 3-manifold $N$. If $|\mathcal{D}(R)| = \infty$ for each prime factor $R$ of $N$, then there is a closed orientable 3-manifold $M$ such that $|D(M,N)| = \infty$.*

Theorems 2.3 2.4, 2.6 and 2.8 (and Theorem 2.5) imply Theorem 2.1.

**Remark 2.9.** Theorem 2.8 follows from an explicit result [DeSW, Theorem 2.5], which provides the concrete $M$ and the infinite set in $D(M,N)$ for the given $N$. The proof of Theorem 2.8 is essentially elementary, which does not appear until now mainly due to three reasons:

(1) $|\mathcal{D}(N)|$ may be finite even if $|\mathcal{D}(R)| = \infty$ for each prime factor $R$ of $N$; for example $|\mathcal{D}(T^3)| = \infty$ but $|\mathcal{D}(T^3\#T^3)| < \infty$ for 3-dimensional torus $T^3$ [Wa1]. Such phenomena puzzled us to wonder if Theorem 2.8 was always true [Wa2, page 460].

(2) The target concerned in Theorem 2.8 became the only unknown case for Question 1.1 after the work [DeW2].

(3) The proof of Theorem 2.8 uses the result of $\mathcal{D}(N)$ which was just completely determined for each $N$ recently ([Du], [SWW], [SWWZ]).

## 3. About $D(M)$

This section is based on [Wa1], [SWW], [SWWZ] and [Du].

### 3.1. Finer classes for calculate $D(N)$ when $D(N)$ is unbounded.

To make this section to be complete, we allow it to have some light repeat with Section 2. The following result, which is a re-statement of Theorem 2.5, is known in early 1990's and answered Question 1.2 (1).

**Theorem 3.1.** *Suppose $M$ is a geometrizable 3-manifold. Then $M$ admits a self-map of degree larger than 1 if and only if $M$ is either*

*(a) covered by a torus bundle over the circle, or*

*(b) covered by $F \times S^1$ for some compact surface $F$ with $\chi(F) < 0$, or*

*(c) each prime factor of $M$ is covered by $S^3$ or $S^2 \times E^1$.*

Hence for any 3-manifold $M$ not listed in (a)-(c) of Theorem 3.1, $D(M)$ is either $\{0,1,-1\}$ or $\{0,1\}$, which depends on whether $M$ admits a self map of degree $-1$. To determine $D(M)$ for geometrizable 3-manifolds listed in (a)-(c) of Theorem 1.0, let's have a close look of them.

For short, we often call a 3-manifold supporting Nil geometry *a Nil 3-manifold, and so on.* Among Thurston's eight geometries, six of them belong to the list (a)-(c) in Theorem 1.0. 3-manifolds in (a) are exactly those supporting either $E^3$, or Sol or Nil geometries. $E^3$ 3-manifolds, Sol 3-manifolds, and some Nil 3-manifolds are torus bundle or semi-bundles; Nil 3-manifolds which are not torus bundles or semi-bundles are Seifert fibered spaces having Euclidean orbifolds with three singular points. 3-manifolds in (b) are exactly those supporting $H^2 \times E^1$ geometry; 3-manifolds supporting $S^3$ or $S^2 \times E^1$ geometries form a proper subset of (3). Now we divide all 3-manifolds in the list (a)-(c) in Theorem 3.1 into the following five classes:

Class 1. $M$ supporting either $S^3$ or $S^2 \times E^1$ geometries;

Class 2. each prime factor of $M$ supporting either $S^3$ or $S^2 \times E^1$ geometries, but $M$ is not in Class 1;

Class 3. torus bundles and torus semi-bundles;

Class 4. Nil 3-manifolds not in Class 3;

Class 5. $M$ supporting $H^2 \times E^1$ geometry. We will present $D(M)$ for $M$ in all those five classes. To do this, we need first to coordinate 3-manifolds in each class, then state the results of $D(M)$ in term of those coordinates. This is carried in the next subsection.

3.2. **Main Results. Class 1.** According to [Or] or [Sc], the fundamental group of a 3-manifold supporting $S^3$-geometry is among the following eight types: $\mathbb{Z}_p$, $D^*_{4n}$, $T^*_{24}$, $O^*_{48}$, $I^*_{120}$, $T'_{8\cdot 3^q}$, $D'_{n'\cdot 2^q}$ and $\mathbb{Z}_m \times \pi_1(N)$, where $N$ is a 3-manifold supporting $S^3$-geometry, $\pi_1(N)$ belongs to the previous seven ones, and $|\pi_1(N)|$ is coprime to $m$. The cyclic group $Z_p$ is realized by lens space $L(p,q)$, each group in the remaining types is realized by a unique 3-manifold supporting $S^3$-geometry. Note also the sub-indices of those seven types groups are exactly their orders, and the order of the groups in the last type is $m|\pi_1(N)|$. There are only two closed orientable 3-manifolds supporting $S^2 \times \mathbb{E}^1$ geometry: $S^2 \times S^1$ and $RP^3 \# RP^3$ .

**Theorem 3.2.** *(1) $D(M)$ for $M$ supporting $S^3$-geometry are listed below:*

| $\pi_1(M)$ | $D(M)$ |
|---|---|
| $\mathbb{Z}_p$ | $\{k^2 \mid k \in \mathbb{Z}\} + p\mathbb{Z}$ |
| $D^*_{4n}$ | $\{h^2 \mid h \in \mathbb{Z};\ 2 \nmid h \ or\ h = n \ or\ h = 0\} + 4n\mathbb{Z}$ |
| $T^*_{24}$ | $\{0, 1, 16\} + 24\mathbb{Z}$ |
| $O^*_{48}$ | $\{0, 1, 25\} + 48\mathbb{Z}$ |
| $I^*_{120}$ | $\{0, 1, 49\} + 120\mathbb{Z}$ |
| $T'_{8\cdot 3^q}$ | $\begin{cases} \{k^2 \cdot (3^{2q-2p} - 3^q) \mid 3 \nmid k, q \geq p > 0\} + 8 \cdot 3^q\mathbb{Z} & (2 \mid q) \\ \{k^2 \cdot (3^{2q-2p} - 3^{q+1}) \mid 3 \nmid k, q \geq p > 0\} + 8 \cdot 3^q\mathbb{Z} & (2 \nmid q) \end{cases}$ |
| $D'_{n'\cdot 2^q}$ | $\{k^2 \cdot [1 - (n')^{2^q - 1}]^i \cdot [1 - 2^{(2p-q)(n'-1)}]^j \mid i, j, k, p \in \mathbb{Z},$ $q \geq p > 0\} + n' 2^q \mathbb{Z}$ |
| $\mathbb{Z}_m \times \pi_1(N)$ | $\left\{ d \in \mathbb{Z} \,\middle\vert\, \begin{array}{l} d = h + \vert\pi_1(N)\vert\mathbb{Z},\ h \in D(N) \\ d = k^2 + m\mathbb{Z},\ k \in \mathbb{Z} \end{array} \right\}$ |

*(2) $D(S^2 \times S^1) = D(RP^3 \# RP^3) = \mathbb{Z}$.*

**Class 2.** We assume that each 3-manifold $P$ supporting $S^3$-geometry has the canonical orientation induced from the canonical orientation on $S^3$. When we change the orientation of $P$, the new oriented 3-manifold is denoted by $\bar{P}$. Moreover, lens space $L(p,q)$ is orientation reversed homeomorphic to $L(p, p-q)$, so we can write all the lens spaces connected summands as $L(p,q)$. Now we can decompose each 3-manifold in Class 2 as

$$M = (mS^2 \times S^1) \# (m_1 P_1 \# n_1 \bar{P}_1) \# \cdots \# (m_s P_s \# n_s \bar{P}_s)$$
$$\#(L(p_1, q_{1,1}) \# \cdots \# L(p_1, q_{1,r_1})) \# \cdots \# (L(p_t, q_{t,1}) \# \cdots \# L(p_t, q_{t,r_t})),$$

where all the $P_i$ are 3-manifolds with finite fundamental group different from lens spaces, all the $P_i$ are different from each other, and all the positive integer $p_i$ are different from each other. Define

$$D_{iso}(M) = \{deg(f) \mid f:\ M \to M,\ f \text{ induces an isomorphism on } \pi_1(M)\}.$$

**Theorem 3.3.** *(1)* $D(M) = D_{iso}(m_1P_1\#n_1\bar{P}_1)\bigcap\cdots\bigcap D_{iso}(m_sP_s\#n_s\bar{P}_s)\bigcap D_{iso}(L(p_1,q_{1,1})\#\cdots\#L(p_1,q_{1,r_1}))\bigcap\cdots\bigcap D_{iso}(L(p_t,q_{t,1})\#\cdots\#L(p_t,q_{t,r_t}))$;

*(2)* $D_{iso}(mP\#n\bar{P}) = \begin{cases} D_{iso}(P) & if\ m \neq n, \\ D_{iso}(P)\bigcup(-D_{iso}(P)) & if\ m = n; \end{cases}$

*(3)* $D_{iso}(L(p,q_1)\#\cdots\#L(p,q_n)) = H^{-1}(C)$.

The notions $H$ and $C$ in Theorem 3.3 (3) is defined as below:

Let $U_p = \{\text{all units in ring } \mathbb{Z}_p\}$, $U_p^2 = \{a^2 \mid a \in U_p\}$, which is a subgroup of $U_p$. We consider the quotient $U_p/U_p^2 = \{a_1, \cdots, a_m\}$, every $a_i$ corresponds with a coset $A_i$ of $U_p^2$. For the structure of $U_p$, see [IR] page 44. Define $H$ to be the natural projection from $\{n \in \mathbb{Z} \mid gcd(n,p) = 1\}$ to $U_p/U_p^2$.

Define $\bar{A}_s = \{L(p,q_i) \mid q_i \in A_s\}$ (with repetition allowed). In $U_p/U_p^2$, define $B_l = \{a_s \mid \#\bar{A}_s = l\}$ for $l = 1, 2, \cdots$, there are only finitely many $l$ such that $B_l \neq \emptyset$. Let $C_l = \{a \in U_p/U_p^2 \mid a_i a \in B_l,\ \forall a_i \in B_l\}$ if $B_l \neq \emptyset$ and $C_l = U_p/U_p^2$ otherwise. Define $C = \bigcap_{l=1}^{\infty} C_l$.

**Class 3.** To simplify notions, for a diffeomorphism $\phi$ on torus $T$, we also use $\phi$ to present its isotopy class and its induced 2 by 2 matrix on $\pi_1(T)$ for a given basis.

A *torus bundle* is $M_\phi = T \times I/(x,1) \sim (\phi(x),0)$ where $\phi$ is a diffeomorphism of the torus $T$ and $I$ is the interval $[0,1]$. Then the coordinates of $M_\phi$ is given as below:

(1) $M_\phi$ admits $E^3$ geometry, $\phi$ conjugates to a matrix of finite order $n$, where $n \in \{1,2,3,4,6\}$;

(2) $M_\phi$ admits Nil geometry, $\phi$ conjugates to $\pm\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, where $n \neq 0$;

(3) $M_\phi$ admits Sol geometry, $\phi$ conjugates to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $|a+d| > 2, ad - bc = 1$.

A *torus semi-bundle* $N_\phi = N\bigcup_\phi N$ is obtained by gluing two copies of $N$ along their torus boundary $\partial N$ via a diffeomorphism $\phi$, where $N$ is the twisted $I$-bundle over the Klein bottle. We have the double covering $p: S^1 \times S^1 \times I \to N = S^1 \times S^1 \times I/\tau$, where $\tau$ is an involution such that $\tau(x,y,z) = (x+\pi, -y, 1-z)$.

Denote by $l_0$ and $l_\infty$ on $\partial N$ be the images of the second $S^1$ factor and first $S^1$ factor on $S^1 \times S^1 \times \{1\}$. *A canonical coordinate* is an orientation of $l_0$ and $l_\infty$, hence there are four choices of canonical coordinate on $\partial N$. Once canonical coordinates on each $\partial N$ are chosen, $\phi$ is identified with an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $GL_2(\mathbb{Z})$ given by $\phi$ $(l_0, l_\infty)$= $(l_0, l_\infty)$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

With suitable choice of canonical coordinates of $\partial N$, $N_\phi$ has coordinates as below:

(1) $N_\phi$ admits $E^3$ geometry, $\phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;

(2) $N_\phi$ admits Nil geometry, $\phi = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}$ or $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, where $z \neq 0$;

(3) $N_\phi$ admits Sol geometry, $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $abcd \neq 0, ad - bc = 1$.

**Theorem 3.4.** *$D(M_\phi)$ is in the table below for torus bundle $M_\phi$, where $\delta(3) = \delta(6) = 1, \delta(4) = 0$.*

| $M_\phi$ | $\phi$ | $D(M_\phi)$ |
|---|---|---|
| $E^3$ | *finite order* $k = 1,2$ | $\mathbb{Z}$ |
| $E^3$ | *finite order* $k = 3,4,6$ | $\{(kt+1)(p^2 - \delta(k)pq + q^2) \mid t,p,q \in \mathbb{Z}\}$ |
| *Nil* | $\pm\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}, n \neq 0$ | $\{l^2 \mid l \in \mathbb{Z}\}$ |
| *Sol* | $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \lvert a+d\rvert > 2$ | $\{p^2 + \frac{(d-a)pr}{c} - \frac{br^2}{c} \mid p,r \in \mathbb{Z}$, *either* $\frac{br}{c}, \frac{(d-a)r}{c} \in \mathbb{Z}$ *or* $\frac{p(d-a)-br}{c} \in \mathbb{Z}\}$ |

*(2) $D(N_\phi)$ is listed in the table below for torus semi-bundle $N_\phi$, where $\delta(a,d) = \frac{ad}{gcd(a,d)^2}$.*

| $N_\phi$ | $\phi$ | $D(N_\phi)$ |
|---|---|---|
| $E^3$ | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | $\mathbb{Z}$ |
| $E^3$ | $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ | $\{2l+1 \mid l \in \mathbb{Z}\}$ |
| *Nil* | $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, z \neq 0$ | $\{l^2 \mid l \in \mathbb{Z}\}$ |
| *Nil* | $\begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}$ *or* $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, z \neq 0$ | $\{(2l+1)^2 \mid l \in \mathbb{Z}\}$ |
| *Sol* | $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, abcd \neq 0, ad - bc = 1$ | $\{(2l+1)^2 \mid l \in \mathbb{Z}\}$, *if* $\delta(a,d)$ *is even or* $\{(2l+1)^2 \mid l \in \mathbb{Z}\} \bigcup \{(2l+1)^2 \cdot \delta(a,d) \mid l \in \mathbb{Z}\}$, *if* $\delta(a,d)$ *is odd* |

To coordinate 3-manifolds in Class 4 and Class 5, we first recall the well known coordinates of Seifert fibered spaces.

Suppose an oriented 3-manifold $M'$ is a circle bundle with a given section $F$, where $F$ is a compact surface with boundary components $c_1, ..., c_n$ with $n > 0$. On each boundary component of $M'$, orient $c_i$ and the circle fiber $h_i$ so that the product of their orientations match with the induced orientation of $M'$ (call such pairs $\{(c_i, h_i)\}$ a section-fiber coordinate system). Now attach $n$ solid tori $S_i$ to the $n$ boundary tori of $M'$ such that the meridian of $S_i$ is identified with slope $r_i = c_i^{\alpha_i} h_i^{\beta_i}$ where $\alpha_i > 0, (\alpha_i, \beta_i) = 1$. Denote the resulting manifold by $M(\pm g; \frac{\beta_1}{\alpha_1}, \cdots, \frac{\beta_s}{\alpha_s})$ which has the Seifert fiber structure extended from the circle bundle structure of $M'$, where $g$ is the genus of the section $F$ of $M$, with the sign $+$ if $F$ is orientable and $-$ if $F$ is nonorientable, here 'genus' of nonorientable surfaces means the number of $RP^2$ connected summands. Call $e(M) = \sum_{i=1}^s \frac{\beta_i}{\alpha_i} \in \mathbb{Q}$ the Euler number of the Seifert fiberation.

**Class 4.** If a Nil manifold $M$ is not a torus bundle or torus semi-bundle, then $M$ has one of the following Seifert fibreing structures: $M(0; \frac{\beta_1}{2}, \frac{\beta_2}{3}, \frac{\beta_3}{6})$, $M(0; \frac{\beta_1}{3}, \frac{\beta_2}{3}, \frac{\beta_3}{3})$, or $M(0; \frac{\beta_1}{2}, \frac{\beta_2}{4}, \frac{\beta_3}{4})$, where $e(M) \in \mathbb{Q} - \{0\}$.

**Theorem 3.5.** *For 3-manifold $M$ in Class 4, we have*
*(1)* $D(M(0;\frac{\beta_1}{2},\frac{\beta_2}{3},\frac{\beta_3}{6}))=\{l^2 | l=m^2+mn+n^2, l\equiv 1 \mod 6, m,n\in\mathbb{Z}\}$;
*(2)* $D(M(0;\frac{\beta_1}{3},\frac{\beta_2}{3},\frac{\beta_3}{3}))=\{l^2 | l=m^2+mn+n^2, l\equiv 1 \mod 3, m,n\in\mathbb{Z}\}$;
*(3)* $D(M(0;\frac{\beta_1}{2},\frac{\beta_2}{4},\frac{\beta_3}{4}))=\{l^2 | l=m^2+n^2, l\equiv 1 \mod 4, m,n\in\mathbb{Z}\}$.

**Class 5.** All manifolds supporting $H^2\times E^1$ geometry are Seifert fibered spaces $M$ such that $e(M)=0$ and the Euler characteristic of the orbifold $\chi(O_M)<0$.

Suppose $M=(g;\frac{\beta_{1,1}}{\alpha_1},\cdots,\frac{\beta_{1,m_1}}{\alpha_1},\cdots,\frac{\beta_{n,1}}{\alpha_n},\cdots,\frac{\beta_{n,m_n}}{\alpha_n})$, where all the integers $\alpha_i>1$ are different from each other, and $\sum_{i=1}^n\sum_{j=1}^{m_i}\frac{\beta_{i,j}}{\alpha_i}=0$.

For each $\alpha_i$ and each $a\in U_{\alpha_i}$, define $\theta_a(\alpha_i)=\#\{\beta_{i,j} \mid p_i(\beta_{i,j})=a\}$ (with repetition allowed), $p_i$ is the natural projection from $\{n \mid gcd(n,\alpha_i)=1\}$ to $U_{\alpha_i}$. Define $B_l(\alpha_i)=\{a \mid \theta_a(\alpha_i)=l\}$ for $l=1,2,\cdots$, there are only finitely many $l$ such that $B_l(\alpha_i)\neq\emptyset$. Let $C_l(\alpha_i)=\{b\in U_{\alpha_i} \mid ab\in B_l(\alpha_i),\ \forall a\in B_l(\alpha_i)\}$ if $B_l(\alpha_i)\neq\emptyset$ and $C_l(\alpha_i)=U_{\alpha_i}$ otherwise. Finally define $C(\alpha_i)=\bigcap_{l=1}^{\infty}C_l(\alpha_i)$, and $\bar{C}(\alpha_i)=p_i^{-1}(C(\alpha_i))$.

**Theorem 3.6.** $D(M(g;\frac{\beta_{1,1}}{\alpha_1},\cdots,\frac{\beta_{1,m_1}}{\alpha_1},\cdots,\frac{\beta_{n,1}}{\alpha_n},\cdots,\frac{\beta_{n,m_n}}{\alpha_n}))=\bigcap_{i=1}^n\bar{C}(\alpha_i)$.

3.3. **A brief comment of the topic and organization of the paper.** Theorem 3.1 was appeared in [Wa1]. The proof of the "only if" part in Theorem 3.1 is based on the results on simplicial volume developed by Gromov, Thurston and Soma (see [So]), and various classical results by others on 3-manifold topology and group theory ([He], [SW], [R]). The proof of "if" part in Theorem 3.1 is a sequence elementary constructions, which were essentially known before, for example see [HL] and [KM] for (3). That graph manifolds admit no self-maps of degrees $>1$ also follows from a recent work [De2].

The table in Theorem 3.2 is quoted from [Du], which generalizes the earlier work [HKWZ], which is presented as below.

**Proposition 3.7.** *For 3-manifold $M$ supporting $S^3$ geometry,*

$$D_{iso}(M)=\{k^2+l|\pi_1(M)|, \text{ where } k \text{ and } |\pi_1(M)| \text{ are co-prime}\}.$$

The topic of mapping degrees between (and to) 3-manifolds covered by $S^3$ has been discussed for long time and has much relation with other topics (see [Wa2] for details). We just mention several papers: in very old papers [Rh] and [Ol], the degrees of maps between any given pairs of lens spaces are obtained by using equivalent maps between spheres; in [HWZ], $D(M,L(p,q))$ can be computed for any 3-manifold $M$; and in a recent one [MP], an algorithm (or formula) is given for the degrees of maps between given pairs of 3-manifolds covered by $S^3$ in term of their Seifert invariants.

Theorem 3.4 is proved in [SWW].

Theorem 3.3, Theorem 3.5 and Theorem 3.6 are proved in [SWWZ].

3.4. **Some examples of computation.**

**Example 3.8.** Let $M_1=(P\#\bar{P})\ \#\ (L(7,1)\ \#\ L(7,2)\ \#\ 2L(7,3))$ and $M_2=(2P\#\bar{P})\#(L(7,1)\#L(7,2)\#L(7,3))$, where $P$ is the Poincare homology three sphere. Apply Theorem 3.3 we have

$$D(M_1)=\{840n+i \mid n\in\mathbb{Z},\ i=1,71,121,169,191,239,241,289,311,359,361,$$

$$409, 431, 479, 481, 529, 551, 599, 601, 649, 671, 719, 769, 839.\}$$

$$D(M_2) = \{840n + i \mid n \in \mathbb{Z},\ i = 1, 121, 169, 289, 361, 529.\}$$

**Example 3.9.** By Theorem 3.4, for the torus bundle $M_\phi$, $\phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, among the first 20 integers $> 0$, exactly $1, 4, 5, 9, 11, 16, 19, 20 \in D(M_\phi)$.

**Example 3.10.** For Nil 3-manifold $M = M(0; \frac{\beta_1}{2}, \frac{\beta_2}{3}, \frac{\beta_3}{6})$,

$$D(M) = \{l^2 | l = m^2 + mn + n^2, l \equiv 1 \mod 6, m, n \in \mathbb{Z}\}.$$

The numbers in $D(M)$ smaller than 10000 are exactly 1,49,169,361,625,961,1369, 1849, 2401,3721, 4489, 5329, 6241, 8291, 9409.

**Example 3.11.** For $H^2 \times E^1$ manifold $M = M(2; \frac{1}{5}, \frac{1}{5}, -\frac{2}{5}, \frac{1}{7}, \frac{2}{7}, -\frac{3}{7})$, apply Theorem 3.6 we have $D(M) = \{5n + 1 \mid n \in \mathbb{Z}\} \bigcap \{7n + i \mid n \in \mathbb{Z},\ i = 1, 2, 4\} = \{35n + i \mid n \in \mathbb{Z},\ i = 1, 11, 16\}$.

## 4. Realization of self-map of degree $\pm 1$ by a homeomorphisms

This section is based on [Sun].

Given a closed orientable $n$-manifold $M$, it is natural to ask, whether all the degree $\pm 1$ self-maps on $M$ can be homotopic to homeomorphisms. Without specific description, all the manifolds below are closed and orientable.

If the property stated above holds for $M$, we say $M$ has property H. In particular, if all the degree 1 $(-1)$ self-maps on $M$ can be homotopic to homeomorphisms, we say $M$ has property 1H $(-1\text{H})$. $M$ has property $H$ if and only if $M$ has both property 1H and property $-1$H. We can observe that, if $M$ admits an orientation-reversing self-homeomorphism, then $M$ has property 1H if and only if $M$ has property $-1$H. So we mostly only concern property 1H.

Below we would like to determine which prime 3-manifolds, which are the basic part of 3-manifolds, has property H.

It is known that each degree $\pm 1$ self-map map $f$ on $M$ induces an isomorphism $f_*$ : $\pi_1(M) \to \pi_1(M)$.

Hyperbolic 3-manifolds and Haken manifolds have property H by the celebrated Mostow rigidity theorem [M] and Waldhausen's theorem on Haken manifolds(see 13.6 of [He]).

This two theorems cover most cases of irreducible 3-manifolds, including: the manifolds with nontrivial JSJ decomposition, hyperbolic manifolds, Seifert manifolds $M$ with incompressible surface. So the remaining cases are:

Class 1. manifolds supporting $S^3$-geometry;

Class 2. Seifert manifolds supporting Nil or $\widetilde{PSL(2, R)}$ geometries with orbifold $S^2(p, q, r)$;

4.1. **Main Results.** Class 1. According to [Or] or [Sc], the fundamental group of a 3-manifold supporting $S^3$-geometry is among the following eight types: $\mathbb{Z}_p$ , $D^*_{4n}$ , $T^*_{24}$ , $O^*_{48}$ , $I^*_{120}$ , $T'_{8\cdot 3^q}$ ,$D'_{n'\cdot 2^q}$ and $\mathbb{Z}_m \times \pi_1(N)$, where $N$ is a $S^3$ 3-manifold, $\pi_1(N)$ belongs to the previous seven ones, and $|\pi_1(N)|$ is coprime to $m$. The cyclic group $Z_p$ is realized by lens space $L(p,q)$, each group in the remaining types is realized by a unique $S^3$-manifold.

**Theorem 4.1.** *For $M$ supporting $S^3$-geometry, $M$ has property 1H if and only if $M$ belongs to one of the following classes:*
*i)* $S^3$;
*ii)* $L(p,q)$ *satisfies one of the following:*
*a)* $p = 2, 4, p_1^{e_1}, 2p_1^{e_1}$;
*b)* $p = 2^s (s > 2), 4p_1^{e_1}, p_1^{e_1}p_2^{e_2}, 2p_1^{e_1}p_2^{e_2}$, $q^2 \equiv 1 \mod p$ *and* $q \neq \pm 1$;
*iii)* $\pi_1(M) = \mathbb{Z}_m \times D^*_{4k}$, $(m,k) = (1,2^k), (p_1^{e_1}, 2), (1, p_2^{e_2})$ *or* $(p_1^{e_1}, p_2^{e_2})$;
*iv)* $\pi_1(M) = D'_{2^{k+2}p_1^{e_1}}$;
*v)* $\pi_1(M) = T^*_{24}$ *or* $\mathbb{Z}_{p_1^{e_1}} \times T^*_{24}$;
*vi)* $\pi_1(M) = T'_{8\cdot 3^{k+1}}$;
*vii)* $\pi_1(M) = O^*_{48}$ *or* $\mathbb{Z}_{p_1^{e_1}} \times O^*_{48}$;
*viii)* $\pi_1(M) = I^*_{120}$ *or* $\mathbb{Z}_{p_1^{e_1}} \times I^*_{120}$.
*Where all the $p_1, p_2$ are odd prime numbers, $e_1, e_2, k, m$ are positive integers.*

By [HKWZ] and elementary number theory, among all the $S^3$-manifolds, only $S^3$ and lens spaces admit degree $-1$ self-maps. When considering about property $-1$H, it is reasonable to restrict the manifold to be $L(p,q)$.

**Proposition 4.2.** *$L(p,q)$ has property $-1H$ if and only if $L(p,q)$ belongs to one of the following classes:*
*i)* $4|p$ *or some odd prime factor of* $p$ *is in* $4k+3$ *type;*
*ii)* $q^2 \equiv -1 \mod p$ *and* $p = 2, p_1^{e_1}, 2p_1^{e_1}$, *where $p_1$ is $4k+1$ type prime number.*

Essentially, it is known that the manifolds in Class 2 have property H. However, the author can't find a proper reference and he can just copy the proof of Theorem 3.9 of [Sc] to prove this result.

**Theorem 4.3.** *For Seifert manifolds $M$ supporting Nil or $\widetilde{PSL}(2,R)$ geometries with orbifold $S^2(p,q,r)$, $M$ has property H.*

Synthesize from Mostow and Waldhausen's theorem and Theorem 4.1, 4.3, Proposition 4.2, we get the following consequence:

**Theorem 4.4.** *Suppose $M$ is a prime geometrizable 3-manifold.*
*1) $M$ has property 1H if and only if $M$ belongs to one of the following classes:*
*i) $M$ does not support $S^3$-geometry;*
*ii) $M$ is in one of the classes stated in Theorem 4.1*
*2)$M$ has property -1H if and only if $M$ belongs to one of the following classes:*
*i) $M$ does not support $S^3$-geometry;*
*ii) $M$ is in one of the classes stated in Proposition 4.2.*
*3)$M$ has property H if and only if $M$ belongs to one of the following classes:*
*i) $M$ does not support $S^3$-geometry;*
*ii) $M$ is in one of the classes except ii) stated in Theorem 4.1;*
*iii) $L(p,q)$ satisfies one of the following:*
*a)* $p = 2, 4$;
*b)* $p = p_1^{e_1}, 2p_1^{e_1}$, *where $p_1$ is $4k+3$ type prime number;*

*c)* $p = p_1^{e_1}, 2p_1^{e_1}$, *where* $p_1$ *is* $4k+1$ *type prime number and* $q^2 \equiv -1$ *mod p;*
*d)* $p = 2^s (s > 2), 4p_1^{e_1}$, $q^2 \equiv 1$ *mod p*, $q \neq \pm 1$;
*e)* $p = p_1^{e_1} p_2^{e_2}, 2p_1^{e_1} p_2^{e_2}$, *where one of* $p_1, p_2$ *is* $4k+3$ *type prime number,* $q^2 \equiv 1$ *mod p*, $q \neq \pm 1$.

Indeed the proof of above theorems in [Sun] give much stronger results. For simplicity, we only explain the situation for 1H.

Let $K(M) = \{\phi \in Out(\pi_1(M)) | \ \exists f : \ M \to M, \ f_* \in \phi, \ deg(f) = 1\}$. It is known $K(M)$ is $1-1$ corresponds with {degree 1 self-maps $f$ on $M$}/homotopy.

Let $K'(M) = \{\phi \in Out(\pi_1(M)) | \ \phi$ is realized by orientation preserving homeomorphism $\}$, which is a subgroup of $K(M)$. $K'(M)$ is $1-1$ corresponds with $\mathcal{MCG}^+(M)$, the orientation preserving subgroup of mapping class group of $M$.

To determine whether $M$ has property 1H, we need only determine whether $K(M) = K'(M)$, or whether $|K(M)| = |\mathcal{MCG}^+(M)|$. Define the realization coefficient of $M$ to be

$$RC(M) = \frac{|K(M)|}{|K'(M)|}.$$

So $M$ has property 1H if and only if $RC(M) = 1$. The $RC(M)$ is completely determined for each 3-manifold support $S^3$-geometry in [Sun].

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