数理解析研究所講究録
第 1749 巻 2011 年 157-164

Two applications of Coulomb wave functions in hydrodynamics
(流体力学におけるクーロン波動関数の応用 2 例)

Takahiro Nishiyama
(西山 高弘)

Department of Applied Science, Yamaguchi University, Ube 755-8611, Japan
e-mail: t-nishi@yamaguchi-u.ac.jp

1 Introduction

The regular Coulomb wave function $F_L(\eta,\rho)$ for $L\in\mathbb{N}\cup\{0\}$, $\eta\in\mathbb{R}$ and $\rho>0$ is defined by

$$\begin{aligned}F_L(\eta,\rho)&=C_L(\eta)\rho^{L+1}e^{-i\rho}\,{}_1F_1(L+1-i\eta\,;2L+2\,;2i\rho)\\&=(2i)^{-(L+1)}C_L(\eta)M_{i\eta,L+1/2}(2i\rho),\end{aligned}$$

where ${}_1F_1$ and M denote Kummer's and Whittaker's regular confluent hypergeometric functions, respectively, and

$$C_L(\eta)=\frac{2^L|\Gamma(L+1+i\eta)|}{e^{\pi\eta/2}(2L+1)!}=\begin{cases}\dfrac{2^L}{(2L+1)!}\sqrt{\dfrac{2\pi\prod_{k=0}^{L}(k^2+\eta^2)}{\eta(e^{2\pi\eta}-1)}} & \text{for } \eta\neq 0,\\ \dfrac{2^L L!}{(2L+1)!} & \text{for } \eta=0,\end{cases}$$

[1, Chapter 14], [3, Appendix I.A.14]. The value of $F_L(\eta,\rho)$ is real because of the Kummer transformation

$$e^{-i\rho}{}_1F_1(L+1-i\eta;2L+2;2i\rho)=e^{i\rho}{}_1F_1(L+1+i\eta;2L+2;-2i\rho)$$

[1, Eq. 13.1.27]. If η is a constant, then $w(\rho)=F_L(\eta,\rho)$ is a solution to

$$\frac{d^2w}{d\rho^2}+\left[1-\frac{2\eta}{\rho}-\frac{L(L+1)}{\rho^2}\right]w=0.$$

As another solution to this equation that is independent of $F_L(\eta,\rho)$, the irregular Coulomb wave function $G_L(\eta,\rho)$ is defined by

$$\begin{aligned}G_L(\eta,\rho)&=\frac{(\pm 2i)^{2L+1}\rho^{L+1}e^{\mp i\rho}}{C_L(\eta)(2L+1)!}\Gamma(L+1\mp i\eta)U(L+1\mp i\eta,2L+2,\pm 2i\rho)\pm iF_L(\eta,\rho)\\&=\frac{(\pm 2i)^L}{C_L(\eta)(2L+1)!}\Gamma(L+1\mp i\eta)W_{\pm i\eta,L+1/2}(\pm 2i\rho)\pm iF_L(\eta,\rho)\end{aligned}$$

so that $G_L(\eta,\rho)\frac{d}{d\rho}F_L(\eta,\rho) - F_L(\eta,\rho)\frac{d}{d\rho}G_L(\eta,\rho) = 1$. Here U and W denote Kummer's and Whittaker's irregular confluent hypergeometric functions, respectively. There are various formulas for $F_L(\eta,\rho)$ and $G_L(\eta,\rho)$ in [1]. In particular, the case $L = \eta = 0$ is easy: $F_0(0,\rho) = \sin\rho$ and $G_0(0,\rho) = \cos\rho$.

Coulomb wave functions are mainly used in quantum physics, especially in scattering theories (see [8], and references therein, e.g. [7, Chapter III]). In the field of hydrodynamics, however, there are only a few papers using them. In this article, two applications of Coulomb wave functions in hydrodynamics are introduced. One is to an orthogonal series associated with steady Euler flows (§ 2), and the other is to the stability problem for pipe Poiseuille flow (§ 3).

2 An orthogonal series associated with steady Euler flows

When an Euler flow is two-dimensional and in a steady state, then it is described by a stream function $\psi(x,y)$ as

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -g(\psi)$$

with an arbitrary differentiable function g [2, Section 7.4]. It is clear that each basis function of the two-dimensional Fourier series satisfies this equation with g linear. Therefore, the two-dimensional Fourier series can be regarded as a superposition of steady planar Euler flows.

Similarly, a steady axisymmetric Euler flow is described by a Stokes stream function $\phi(r,x)$ in the cylindrical coordinate system (r,θ,x) as

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial r}\right) + \frac{\partial^2\phi}{\partial x^2} = -r^2 h(\phi)$$

if the θ-component of velocity is equal to zero [2, Section 7.5]. Here h is an arbitrary differentiable function. If h is linear, then

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial r}\right) + \frac{\partial^2\phi}{\partial x^2} = -\lambda r^2\phi \tag{1}$$

with a constant λ. What is an orthogonal series whose basis functions mean steady axisymmetric Euler flows?

Set $\phi = \Phi(r)\, e^{2in\pi x/b}$ ($n \in \mathbb{Z}$, $b > 0$) in (1). Then $\Phi(r)$ should satisfy

$$r\frac{d}{dr}\left(\frac{1}{r}\frac{d\Phi}{dr}\right) - 4\left(\frac{n\pi}{b}\right)^2\Phi + \lambda r^2\Phi = 0. \tag{2}$$

As mentioned by Herrnegger [5] and Maschke [6], it has a solution

$$\Phi(r) = R_0^n(\sqrt{\lambda}; r) := F_0\left(\frac{1}{\sqrt{\lambda}}\left(\frac{n\pi}{b}\right)^2, \frac{\sqrt{\lambda}\,r^2}{2}\right)$$

when $\Phi(0) = 0$ is imposed. The author [9], [10] pointed out that there exists a set $\{\lambda_{m,n}\}$ $(m \in \mathbb{N})$ for each fixed $n \in \mathbb{Z}$ and a constant $a > 0$ such that $R_0^n(\sqrt{\lambda_{m,n}}; a) = 0$ and

$$\left(\frac{2n\pi}{ab}\right)^2 < \lambda_{1,n} < \lambda_{2,n} < \lambda_{3,n} < \cdots .$$

Furthermore, using the Hilbert–Schmidt theory, he deduced that $\{R_0^n(\sqrt{\lambda_{m,n}}; r)\}$ $(m \in \mathbb{N})$ for each fixed n is a complete orthogonal system on $(0, a)$ with the weight function r. In other words, every function $f(r)$ that satisfies $\int_0^a [f(r)]^2 r\,dr < \infty$ can be represented in the form

$$f(r) \sim \sum_{m=1}^{\infty} R_0^n(\sqrt{\lambda_{m,n}}; r) \frac{\int_0^a R_0^n(\sqrt{\lambda_{m,n}}; t) f(t)\, t\, dt}{\int_0^a [R_0^n(\sqrt{\lambda_{m,n}}; t)]^2\, t\, dt} \tag{3}$$

in the square integrable space with the weight r.

In consequence, the set $\{\phi_{m,n}(r, x)\} := \{R_0^n(\sqrt{\lambda_{m,n}}; r) e^{2in\pi x/b}\}$ $(m \in \mathbb{N},\ n \in \mathbb{Z})$ is a complete orthogonal system with the weight r on $(0, a) \times (-b/2, b/2)$ such that each basis expresses a steady axisymmetric Euler flow. It should be noted that the author [10] derived an integral transform whose kernel is a Stokes stream function of a steady axisymmetric Euler flow by letting $a \to \infty$ and $b \to \infty$.

Noting that

$$\int_0^a [R_0^n(\sqrt{\lambda_{m,n}}; r)]^2\, r\, dr = \frac{A_{m,n} B_{m,n}}{2a\sqrt{\lambda_{m,n}}}$$

is valid for

$$A_{m,n} = \frac{d}{dr} R_0^n(\sqrt{\lambda_{m,n}}; r)\bigg|_{r=a}, \qquad B_{m,n} = \frac{\partial}{\partial u} R_0^n(u; a)\bigg|_{u=\sqrt{\lambda_{m,n}}}$$

[10, Eq. (4.1)], we can prove the following theorem, which is a more specific result than (3):

Theorem 1 ([11]). *If $\int_0^a |f(t)| t dt < \infty$ and the total variation of f is bounded on $[\alpha_1, \alpha_2] \subset (0, a)$, then*

$$\frac{f(r-0) + f(r+0)}{2} = 2a \sum_{m=1}^{\infty} \frac{\sqrt{\lambda_{m,n}}}{A_{m,n} B_{m,n}} R_0^n(\sqrt{\lambda_{m,n}}; r) \int_0^a R_0^n(\sqrt{\lambda_{m,n}}; t) f(t)\, t\, dt$$

for all fixed $r \in (\alpha_1, \alpha_2)$ and $n \in \mathbb{Z}$.

The proof is done by extending $F_L(\eta, \rho)$ and $G_L(\eta, \rho)$ $(L = 0$ or $1)$ to complex η and ρ. It is similar to the proof of Watson [20, Sections 18.21–18.24] on the Fourier–Bessel series, the best-known orthogonal series with the weight function r. Because of the gamma function, however, Coulomb wave functions with complex arguments are more delicate to treat than Bessel functions.

3 Stability problem for pipe Poiseuille flow

The stability problem for pipe Poiseuille flow has a long history. The analytical study of its dependence on the Reynolds number R was started by Sexl [17]. After that, many researchers investigated behavior of small disturbances to the pipe flow from various theoretical viewpoints and deduced the linear stability at every R (see [4], and references therein). It was Pekeris [13] who first applied a confluent hypergeometric function (i.e. a Coulomb wave function with complex arguments) to the stability analysis of pipe Poiseuille flow. Sexl & Spielberg [18] followed. In this section, by using the result of asymptotic analysis of Skovgaard [19], we consider the distribution of complex phase velocities for small axisymmetric torsional disturbances to pipe Poiseuille flow.

Let $\Omega(r)$ be a function such that $\Omega(r)e^{i\alpha(x-ct)}/r$ is a normal mode for axisymmetric torsional disturbances to the pipe flow which has the velocity $1-r^2$ $(0<r<1)$ in the x-direction in the cylindrical coordinate system (r,θ,x). Here the wave-number $\alpha>0$ and the complex phase velocity $c\in\mathbb{C}$ are constants. Pekeris [13] derived the linearized equation of the same type as (2):

$$r\frac{d}{dr}\left(\frac{1}{r}\frac{d\Omega}{dr}\right)-\alpha^2\Omega-i\alpha R(1-r^2-c)\Omega=0$$

with the boundary conditions $\Omega(1)=0$ and $|\lim_{r\to+0}\Omega(r)/r|<\infty$. Setting

$$\kappa=\frac{1}{4}\left[\frac{\sqrt{\alpha R}\,(1-c)}{e^{i\pi/4}}-\frac{\alpha^2e^{i\pi/4}}{\sqrt{\alpha R}}\right],$$

we solve it as

$$\begin{aligned}\Omega(r)&\propto F_0\left(i\kappa,\ \frac{1}{2}\sqrt{\alpha R}\,e^{i\pi/4}r^2\right)\propto F_0\left(-i\kappa,\ -\frac{1}{2}\sqrt{\alpha R}\,e^{i\pi/4}r^2\right)\\&\propto \mu(\alpha,R,c;r):=M_{\kappa,1/2}(\sqrt{\alpha R}\,e^{-i\pi/4}r^2)\end{aligned}$$

with

$$\mu(\alpha,R,c;1)=0. \tag{4}$$

This (4) determines the value of c for given α and R. If $R|1-c|\to\infty$ with α fixed, then κ is asymptotically equal to k defined by

$$k=\frac{\sqrt{\alpha R}\,(1-c)}{4e^{i\pi/4}}=\frac{\sqrt{\alpha R}}{4|z|}\,e^{-i(\arg z+\pi/4)},$$

where $z=1/(1-c)$. Therefore, in the limit

$$\sqrt{R}\,|1-c|\to\infty\quad\text{and}\quad R|1-c|\to\infty\quad\text{with }\alpha\text{ fixed,} \tag{5}$$

we have $|k|\to\infty$ and $\mu(\alpha,R,c;1)\sim M_{k,1/2}(4kz)$, to which the result of asymptotic analysis of Skovgaard [19] is applicable.

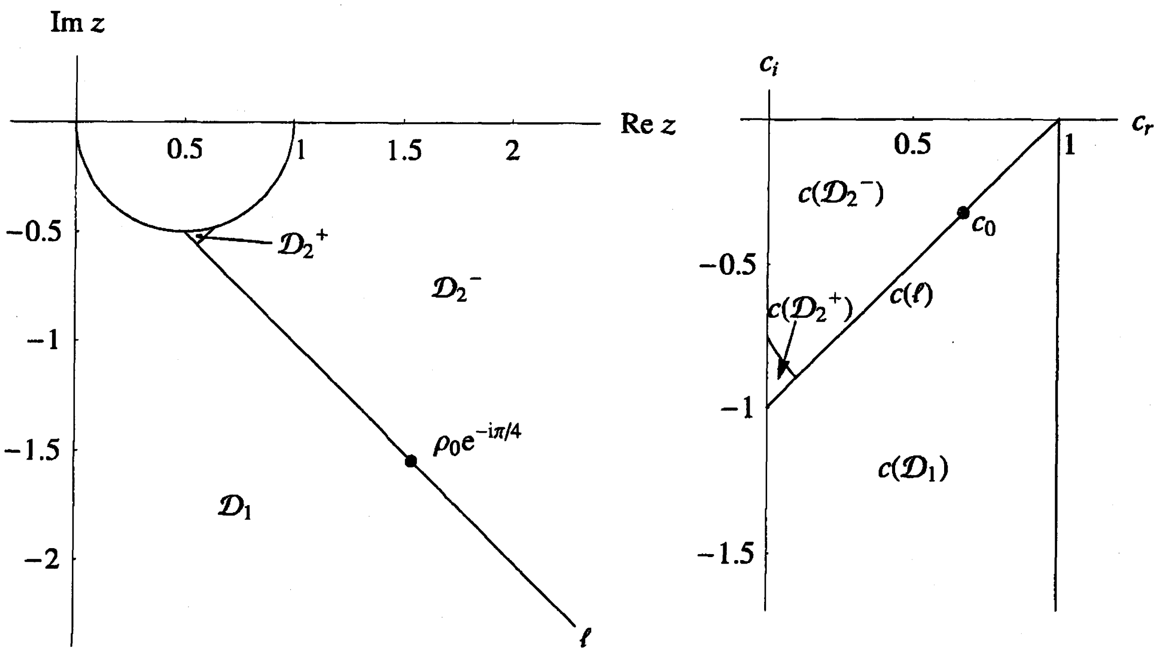

Figure 1: The sets D_1, $D_2^{\pm}$ and ℓ of z, and the corresponding sets of $c = 1 - 1/z$.

Let us make ready for stating asymptotic forms of $\mu(\alpha, R, c; 1)$. As proved in [13], every c ($= c_r + ic_i$) of (4) satisfies $0 < c_r < 1$ and $c_i < 0$. Consequently, z should belong to one of the three sets

$$\begin{aligned} D_1 &= \left\{s \,:\, -\pi/2 < \arg s < -\pi/4,\ \left|s - \tfrac{1}{2}\right| > \tfrac{1}{2},\ |s| < \infty\right\}, \\ D_2 &= \left\{s \,:\, -\pi/4 < \arg s < 0,\ \left|s - \tfrac{1}{2}\right| > \tfrac{1}{2},\ |s| < \infty\right\}, \\ \ell &= \left\{s \,:\, \arg s = -\pi/4,\ \left|s - \tfrac{1}{2}\right| > \tfrac{1}{2},\ |s| < \infty\right\}. \end{aligned}$$

We define the function ξ by

$$\xi(z) = \begin{cases} \frac{1}{2} z^{1/2}(z-1)^{1/2} - \frac{1}{2}\ln[z^{1/2} + (z-1)^{1/2}] - i\pi/4 & \text{for } z \in D_1, \\ \frac{1}{2} z^{1/2}(z-1)^{1/2} - \frac{1}{2}\ln[z^{1/2} + (z-1)^{1/2}] & \text{for } z \in D_2 \cup \ell. \end{cases}$$

Here, and from now on, multivalued functions should be understood to take their principal values. Using this ξ, we divide D_2 into the two sets

$$D_2^+ = \{s \in D_2 \,:\, \operatorname{Im}\xi(s) \geq 0\}, \qquad D_2^- = \{s \in D_2 \,:\, \operatorname{Im}\xi(s) < 0\}.$$

Figure 1 shows the locations of D_1, $D_2^{\pm}$ and ℓ on the z-plane and the corresponding sets on the c-plane. It also shows the point $z = \rho_0 e^{-i\pi/4} \in \ell$ with $\rho_0 \approx 2.1844$, at which $\arg \xi(z) = -\pi/2$ holds, and the corresponding point

$$c = c_0 = 1 - \rho_0^{-1} e^{i\pi/4} \approx 0.67629 - 0.32371\, i. \tag{6}$$

We now express asymptotic forms of $\mu(\alpha, R, c; 1)$ in the limit (5) as follows:

$$\mu(\alpha, R, c; 1) \sim 2(-\xi)^{1/2} \left(\frac{z}{z-1}\right)^{1/4} I_1(4k\xi) \qquad \text{for } z \in D_1, \tag{7}$$

$$\mu(\alpha, R, c; 1) \sim \frac{2^{2/3}3^{1/6}\xi^{1/6}e^{i\pi(k-2/3)}}{k^{1/3}} \left(\frac{z}{z-1}\right)^{1/4} \mathrm{Ai}((6k)^{2/3}\xi^{2/3}e^{-2i\pi/3}) \qquad \text{for } z \in D_2^-, \tag{8}$$

$$\mu(\alpha, R, c; 1) \sim \frac{2^{5/3}3^{1/6}\xi^{1/6}e^{i\pi/6}\sin \pi k}{k^{1/3}} \left(\frac{z}{z-1}\right)^{1/4} \mathrm{Ai}((6k)^{2/3}\xi^{2/3}e^{2i\pi/3}) \qquad \text{for } z \in \ell \text{ with } |z| > \rho_0. \tag{9}$$

Here I_1 denotes the first-kind modified Bessel function of the first order, and Ai denotes the Airy function. The case $z \in D_2^+$ or $z \in \ell$ with $|z| \leq \rho_0$ is omitted because the asymptotic form has no zero. For details of the derivation of (7)–(9), see [19], and also [12].

The zeros of the right hand sides of (7)–(9) approximately determine c of (4) in the limit (5). Since all zeros of I_1 and Ai are located on the imaginary axis and the negative real axis, respectively, the equality $\arg(k\xi) = -\pi/2$ is necessarily satisfied by all z that make the right hand side of (7) or (8) vanish. It leads to

$$\arg\left\{ z^{1/2}(z-1)^{1/2} - \ln[z^{1/2} + (z-1)^{1/2}] - \frac{i\pi}{2} \right\} - \arg z + \frac{\pi}{4} = 0 \quad \text{for } z \in D_1, \tag{10}$$

$$\arg\left\{ z^{1/2}(z-1)^{1/2} - \ln[z^{1/2} + (z-1)^{1/2}] \right\} - \arg z + \frac{\pi}{4} = 0 \qquad \text{for } z \in D_2^-. \tag{11}$$

Moreover, as another necessary condition for $\mu(\alpha, R, c; 1)$ to vanish approximately, we add

$$|z| > \rho_0 \quad \text{for } z \in \ell, \tag{12}$$

under which the factor $\sin \pi k$ in (9) has zeros (while Ai in (9) has no zero). By numerically solving (10) and (11) with respect to $c = 1 - 1/z$ and adding the straight line segment given by (12), we obtain the Y-shaped contour shown in figure 2, on which zeros of $\mu(\alpha, R, c; 1)$ are approximately located. The three branches of this contour meet at a point $c = c_0$, already appeared in (6) and figure 1. Figure 2 shows the locations of c computed by Schmid & Henningson [15, Table 1, $n = 0$], [16, p.506, $n = 0$], too. Most of them are on or near the Y-shaped contour. Although some are off the leftward branch of the contour, they are not of torsional disturbances but of meridional disturbances (see [14, Figure 2]). It should be noted that the Y-shaped structure in figure 2 is independent of α and R. Of course, the location of each individual c of (4) depends on α and R, as investigated in detail in [12]. In particular, about $c_r \approx 2/3$ on the downward branch of the contour, the following theorem can be analytically proved:

Theorem 2 ([12]). *If α and R are fixed at arbitrary positive numbers, then there exist sequences of c of* (4) *such that $c_i \to -\infty$. For all these sequences, $c_r \to 2/3$ holds.*

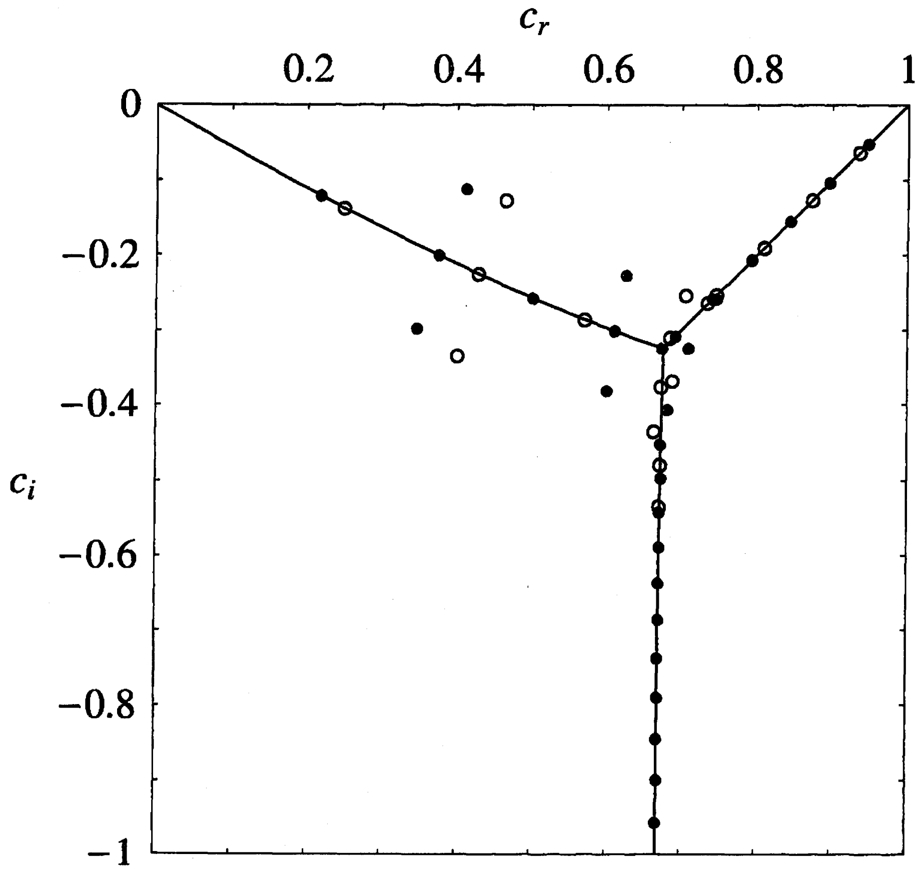

Figure 2: The Y-shaped contour obtained from (10)–(12), with c computed by Schmid & Henningson [15], [16] for $R = 3000$ (•) and $R = 2000$ (∘) when $\alpha = 1$.

Acknowledgment

The author thanks Professor Yuji Hattori for his information about [5] and [6].

References

[1] M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables*, U. S. Government Printing Office, 1964.

[2] G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press, 1967.

[3] H. Buchholz, *The Confluent Hypergeometric Function*, Springer, 1969.

[4] P. G. Drazin, W. H. Reid, *Hydrodynamic Stability*, Cambridge University Press, 1981.

[5] F. Herrnegger, On the equilibrium and stability of the belt pinch, *Proceedings of the Fifth European Conference on Controlled Fusion and Plasma Physics, Grenoble, France, August 21–25, 1972*, vol. 1, p. 26.

[6] E. K. Maschke, Exact solutions of the MHD equilibrium equation for a toroidal plasma, *Plasma Phys.* **15** (1973) 535–541.

[7] N. F. Mott, H. S. W. Massey, *The Theory of Atomic Collisions*, Clarendon, 1933.

[8] National Bureau of Standards, *Tables of Coulomb Wave Functions*, U. S. Government Printing Office, 1952.

[9] T. Nishiyama, Construction of axisymmetric steady states of an inviscid incompressible fluid by spatially discretized equations for pseudo-advected vorticity, *Int. J. Math. Math. Sci.* **2005** (2005) 3319–3346.

[10] T. Nishiyama, An integral transform whose kernel is a stream function of a steady Euler flow with axisymmetry, *Z. Angew. Math. Phys.* **58** (2007) 68–80.

[11] T. Nishiyama, Application of Coulomb wave functions to an orthogonal series associated with steady axisymmetric Euler flows, *J. Approx. Theory* **151** (2008) 42–59.

[12] T. Nishiyama, Distribution of complex phase velocities for small disturbances to pipe Poiseuille flow, *J. Fluid Mech.* **620** (2009) 299–312.

[13] C. L. Pekeris, Stability of the laminar flow through a straight pipe of circular cross-section to infinitesimal disturbances which are symmetrical about the axis of the pipe, *Proc. Nat. Acad. Sci. USA* **34** (1948) 285–295.

[14] W. H. Reid, B. S. Ng, On the spectral problem for Poiseuille flow in a circular pipe, *Fluid Dyn. Res.* **33** (2003) 5–16.

[15] P. J. Schmid, D. S. Henningson, Optimal energy density growth in Hagen–Poiseuille flow, *J. Fluid Mech.* **277** (1994) 197–225.

[16] P. J. Schmid, D. S. Henningson, *Stability and Transition in Shear Flows*, Springer, 2001.

[17] T. Sexl, Zur Stabilitätsfrage der Poiseuilleschen und Couetteschen Strömung, *Ann. Phys.* **83** (1927) 835–848.

[18] T. Sexl, K. Spielberg, Zum Stabilitätsproblem der Poiseuille-Strömung, *Acta Phys. Austriaca* **12** (1958) 9–28.

[19] H. Skovgaard, *Uniform Asymptotic Expansions of Confluent Hypergeometric Functions and Whittaker Functions*, Jul. Gjellerups, 1966.

[20] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd edition, Cambridge University Press, 1962.