

GAUSS' VARIATIONAL PROBLEM AND THE NAVIER-STOKES EQUATIONS.

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ABSTRACT. The “two-constants” theory introduced first by Laplace in 1805 still forms the basis of current theory describing isotropic, linear elasticity, describing the capillarity. By using “two-constants” theory, the Navier-Stokes equations are formulated. These equations with the two coefficients in the ratio 1 : 3 originated from Poisson [16] in 1831. Moreover, these equations contained both a linear and a nonlinear term developed earlier in Navier’s equations [20] in 1827. Still earlier, the nonlinear term was introduced by Euler [7] in 1752-5.

We show the process of formulation of *calculus of variations* using the two functions characterized from the attraction and repulsion, and his criticism to Laplace imaging the *Gaussian function* as the rapidly decreasing function by Gauss in 1830. And we introduce a contribution to the hydromechanics, because he was a contemporary of the epoch of formulation of the Navier-Stokes equations, which are our main theme in our paper.

Particularly, from the viewpoint of mathematics, several important topics such as integral theory in §4.6 and §4.11 which are his selling points. We show his unique *RDF* and reduction of integral from sextuplex to quadruplex, in the sections §4.2, §4.5 and §4.6. In and after §4.7, we show his calculus of variations in the capillarity against the *RDF* and calculation of it by Laplace.

Finally, to the question of capillarity, to be solved by variational equation described in §4.7, we sketch his answers deduced from the previous work of theory in curved surface [7], to the height and angle in question in §4.15 and §4.16.

1. INTRODUCTION

¹ In 1805, Laplace introduced the “two-constants” theory, so-called because of the prominence of two constants in his theory, in regard to capillary action with constants denoted by H and K . ² (cf. Table 1, 2). Thereafter, contributing investigators in formulating *NS* equations, i.e. equations describing equilibrium or capillary situations, have presented various pairs of constants. The original two-constant theory is commonly accepted as describing isotropic, linear elasticity. [5, p.121]. However, the persistence of just two constants in later developments is to be particularly noted. We believe that Poisson was one of few who were aware of this aspect when he introduced Laplace’s deductions when, in 1831, he states, “they incorporate the two special constants of which I mentioned just a while ago, . . .” [27, p.4].

Next, another topic discussed in the final section is the rapidly decreasing functions [*RDFs*] which werekerneled in the “two-constants” and which provided the common, mathematical interpretation of fluid properties among the then progenitors, in particular by Gauss, a contemporary of the progenitors of the *NS* equations, who contributed to the formulation of fluid mechanics in the development of Laplace’s capillarity.

Finally, we uncover reasons for the practice in naming these fundamental equations of fluid motion “*NS* equations”. In Table 6, we present a chronology outlining this practice. The last entry from 1934 by Prandtl [27] grouped the equations containing three terms:

- 1) the nonlinear term
- 2) the Laplacian term multiplied by ν
- 3) the gradient term of divergence multiplied by $\frac{\kappa}{3}$, which takes its rise in the fluid equation by Poisson, and used the nomenclature “the Navier-Stokes equations” for this set of equations.

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¹In §4, the free translation from Latin of Gauss to English is of ours. Throughout this paper, in citation of bibliographical sources, by surrounding our own paragraph or sentences of commentaries between (↓) and (↑) ((↑) is used only when not following to next section,) and by =* or ⇒*, we detail the statement by Gauss, because we would like to discriminate and to avoid confusion from the descriptions by original authors. The mark : ⇒ mean transformation of the statements in brevity by ours. And all the frames surrounding the statements are inserted for important remark by ours.

²Of capillary action, Laplace[12, V.4, Supplement p.2] acknowledges Clairaut[4, p.22], and Clairaut cites Maupertuis[15].

These equations with the two coefficients in the ratio 1 : 3 originated from Poisson [16] in 1831. Moreover, these equations contained both a linear and a nonlinear term developed earlier in Navier's equations [20] in 1827. Still earlier, the nonlinear term was introduced by Euler [7] in 1752-5. cf. Table 2.

2. A UNIVERSAL METHOD FOR THE TWO-CONSTANTS THEORY

In this section, we propose a universal method to describe the kinetic equations that arise in isotropic, linear elasticity. This method is outlined as follows:

- The partial differential equations describing waves in elastic solids or flows in elastic fluids are expressed by using one constant or a pair of constants C_1 and C_2 such that:

$$\begin{aligned} \text{for elastic solids:} & \quad \frac{\partial^2 \mathbf{u}}{\partial t^2} - (C_1 T_1 + C_2 T_2) = \mathbf{f}, \\ \text{for elastic fluids:} & \quad \frac{\partial \mathbf{u}}{\partial t} - (C_1 T_1 + C_2 T_2) + \cdots = \mathbf{f}, \end{aligned}$$

where T_1, T_2, \dots are the terms depending on tensor quantities constituting our equations.

- The two coefficients C_1 and C_2 associated with the tensor terms are the two constants of the theory, definitions of which depend on the contributing author. For example, ε and E were introduced by Navier, R and G by Cauchy, k and K in elastic and $(K+k)\alpha$ and $\frac{(K+k)\alpha}{3}$ in fluid by Poisson, ε and $\frac{\varepsilon}{3}$ by Saint-Venant, and μ and $\frac{\mu}{3}$ by Stokes. Since Poisson, the ratio of two coefficient in fluid was fixed by 3. Moreover, C_1 and C_2 can be expressed in the following form:

$$\begin{cases} C_1 \equiv \mathcal{L}r_1g_1S_1, \\ C_2 \equiv \mathcal{L}r_2g_2S_2, \end{cases} \quad \begin{cases} S_1 = \iint g_3 \rightarrow C_3, \\ S_2 = \iint g_4 \rightarrow C_4, \end{cases} \quad \Rightarrow \quad \begin{cases} C_1 = C_3\mathcal{L}r_1g_1 = \frac{2\pi}{15}\mathcal{L}r_1g_1, \\ C_2 = C_4\mathcal{L}r_2g_2 = \frac{2\pi}{3}\mathcal{L}r_2g_2. \end{cases}$$

Here \mathcal{L} corresponds to either \sum_0^∞ as argued for by Poisson or \int_0^∞ as argued for by Navier. A heated debate had developed between the two over this point. It is a matter of personnel preference as to how the two constants should be expressed.

3. THE RAPIDLY DECREASING FUNCTIONS KERNELED IN THE "TWO-CONSTANTS"

In Table 1, we show the form of g_1 and g_2 , which are kernel functions and with which the progenitors of the fluid equation developed their formulae. Here we refer to these functions as rapidly decreasing functions (*RDFS*).³ While formulating the equilibrium equations, we obtain the competing theories of "two-constants" in capillary action between Laplace and Gauss.

In 1830, after Laplace's death, Gauss [8] started publishing his studies on capillarity following his famous paper on curved surfaces [7]. In the paper, Gauss criticized Laplace's calculations of 1805-7 in which the "two-constants" in his calculation of capillary action were introduced. At about this time, Gauss had studied what became to be called *Gaussian function* or *Gaussian curve* and using this as his *RDF* Gauss criticised Laplace's example function e^{-if} as the equivalent function of $\varphi(f)$. Here, $\varphi(f)$ is the *RDF*, which depends on distance f . In that paper, Gauss [8] pointed out various deficiencies: 1. Laplace had mentioned only attractive action without considering the repulsive action; 2. Laplace could not identify the correct example function as the equivalent function of the *RDF*; and 3. Laplace lacked any proof from say a geometrical point of view. The following are Gauss' criticisms to Laplace in the preface of [8].

- Judging from the second dissertation: *< Supplément à la théorie de l'action capillaire >*, Mr. Laplace investigated a little, not only the complete attraction, but also the partial one by $\varphi(f)$, and tacitly understood incompletely the general attraction; by the way, if we would refer the latter by him about our sensible modification, it is easy to see being conspicuous about it.⁴

³We show the then family of *RDF* by using our notation $f \in \mathcal{RFD}$, and f is a function kernelized in the two-constants belonging to the then rapidly decreasing function.

⁴N.Bowditch, the editor of the complete works of Laplace, cites only the title of Gauss' paper : [8] but siding with Laplace with the following comments :

• He considers exponential e^{-if} as an example of equivalent function with $\varphi(f)$, denoting the large quantity by i , or $\frac{1}{i}$ becomes infinitesimal.

But it is not at all necessary to limit the generality by such a large quantity, the things are more clear than words, we would see easiest, only to investigate if these integrations would be extended, not only infinite but also to an arbitrary sensible distance, or if anything, occurring wider in the finitely measurable distance in experiment. [8, p.33]

Here, we can consider these arguments on the *RDFs* as simple examples of today's distributions and hypergeometric functions of Schwarz in 1945, but which were popular in the 1830s, during the time the *NS* equations were being discussed in their microscopically-descriptive formulation.

However, Gauss' criticisms in 1830 naturally drew no rebuttal. We present a sketch of these assertions on the *RDFs* in Table 3 in their original, cross-indexed narratives, where, we show the then family of *RDF* by using our notation $f \in \mathcal{RFD}$, and f is a function kernelized in the two-constants belonging to the then rapidly decreasing function.

Gauss didn't mention the following fact, and Bowditch⁵ also didn't comment on Gauss's work in Laplace's total works[13] except for only one comment of the name "Gauss" [13, p.686].⁶

In his historical descriptions about the study of capillarity action, we would like to recognize that there is no counterattack to Gauss, but the correct valuation. Gauss [9] stated his conclusion about Laplace's paper "his calculations in the pages, p.44 and the followings it, ⁷ have non effect in vain."

4. Gauss' papers of the capillary action

Gauss states common motivations with Laplace about *MD* (the microscopically-descriptive we call it below) equations. He states the difficulties of integral $\int r^2 \varphi r . dr$, in which he confesses that he also is included in the person who feels difficulties to calculate the *MD* integral.

4.1. Criticism to Laplace in Preface of Gauss' paper.

¶ 2.

• Since Mr. Laplace, from here, presented conveniently the unique supposition about the inner, molecular activity, moreover, giving up diminution of law for the increasing distance, we have got the first result in the surface of the fluid figure based on the accurate calculation, and have established the general equation for the equilibratory figure, not only the precise capillary phenomenon as described, but also try to explain the relating problems.

• This investigation is discussed getting the consented with and confirmed in everywhere, by the exact experiment, among the first class of increasing natural philosophers, geometricians, and referred and criticized by the some authorities from all the directions to the maximum part such as a minor or nonsense.

¶ 3. (Two *RDF* functions and two-constants defined by Laplace.)

• In the calculation by Mr. Laplace, we have at least a thing, which we can give evidence about it, and for which we would not absolutely consent with him.

• In the previous commentary : < *Théorie de l'action capillaire* >, denoting with φf intensity of the

This theory of capillary attraction was first published by La Place in 1806, and in 1807 he gave a supplement. In neither of these works is the repulsive force of the heat of fluid taken into consideration, because he supposed it to be unnecessary. But in 1819, he observed that this action could be taken into account, by supposing the force $\varphi(f)$ to represent the difference between the attractive force of the particles of the fluid $A(f)$, and the repulsive force of the heat $R(f)$ so that the combined action would be expressed by, $\varphi(f) = A(f) - R(f)$; ... [13, p.685]

Maybe this was stated under the covering fire from Gauss' criticisms of Laplace. Gauss may not have read Laplace's works after 1819 in which he had changing his thoughts. As yet we have not been able to investigate this fact.

⁵The present work is a reprint, in four volumes, of Nathaniel Bowditch's English translation of volumes I, II, III and IV of the French-language treatise *Traité de Mécanique Céleste* by P.S.Laplace. The translation was originally published in Boston in 1829, 1832, 1834, and 1839, under the French title, "*Mécanique Céleste*", which has now been changed to its English-language form, "*Celestial Mechanics*."

⁶We refer to Bowditch's comment number : [9173g] in [13].

⁷There are 35 pages of calculation between p.44 and p.78 in his *Supplement*.

TABLE 1. The expression of the total momentum of molecular actions by Laplace, Navier, Cauchy, Poisson, Saint-Venant & Stokes. (Remark. 6-8 : capillarity, 9-10 : equilibrium, else : kinetic equation)

| no | name | problem | C_1 | C_2 | C_3 | C_4 | \mathcal{L} | r_1 | r_2 | g_1 | g_2 | remark |
|-----|-----------------------------------|---------------------------|------------|----------------------|------------------------|-------|---------------------------------|---------------|-------|--------------------------------------|---|---|
| 1 | Navier 1827 [17] | elastic solid | ϵ | | $\frac{2\pi}{15}$ | | $\int_0^\infty d\rho \rho^4$ | ρ^4 | | $f\rho$ | | ρ : radius |
| 2 | Navier fluid 1827 [18] | motion of fluid | ϵ | | $\frac{2\pi}{15}$ | | $\int_0^\infty d\rho \rho^4$ | | | $f(\rho)$ | | ρ : radius |
| | | | | | | | $\int_0^\infty d\rho$ | ρ^2 | | $F(\rho)$ | | |
| 3 | Cauchy 1828 [3] | system of particles | R | | $\frac{2\pi}{15}$ | | $\int_0^\infty dr r^3$ | | | $f(r)$ | | $f(r) \equiv \pm[r f'(r) - f(r)]$ |
| | | | | | | | $\int_0^\infty dr$ | r^3 | | $f(x)$ | $f(r) \neq f(r)$ | |
| 4 | Poisson 1829 [25] | elastic solid | k | | $\frac{2\pi}{15}$ | | $\sum \frac{1}{\alpha^3} r^5$ | r^5 | | $\frac{d \cdot \frac{1}{2} f r}{dr}$ | | |
| | | | | | | | $\sum \frac{1}{\alpha^3}$ | r^3 | | $f r$ | | |
| 5 | Poisson 1831 [26] | motion of fluid | k | | $\frac{1}{30}$ | | $\sum \frac{1}{\epsilon^3} r^3$ | r^3 | | $\frac{d \cdot \frac{1}{2} f r}{dr}$ | | $C_3 = \frac{1}{4\pi} \frac{2\pi}{15} = \frac{1}{30}$ |
| | | | | | | | $\sum \frac{1}{\epsilon^3}$ | r | | $f r$ | $C_4 = \frac{1}{4\pi} \frac{2\pi}{3} = \frac{1}{6}$ | |
| 6 | Laplace 1806,7 [13] | capillary action | H | | 2π | | $\int_0^\infty dz z$ | | | $\Psi(z)$ | | z : distance |
| | | | | | | | $\int_0^\infty dz$ | | | $\Psi(z)$ | | |
| 6-2 | Rewritten by Poisson 1831 [27] | | H | | $\frac{\pi}{4} \rho^2$ | | $\int_0^\infty dr r^4$ | | | φr | | [27, pp.14-15] |
| | | | | | | | $\int_0^\infty dr$ | r^3 | | φr | | |
| 7 | Gauss 1830 [8] | capillary action | | | | | | | | | | attraction : $-f x \cdot dx = d\varphi x$, $\int f x \cdot dx = -\varphi x$, repulsion : $-F x \cdot dx = d\Phi x$, $\int F x \cdot dx = -\Phi x$ |
| 8 | Poisson 1831 [27] | capillary action | H | | $\frac{\pi}{4} \rho^2$ | | $\int_0^\infty dr r^4$ | | | φr | | [27, p.14] |
| | | | | | | | $\int_0^\infty dr$ | r^3 | | φr | [27, p.12] | |
| 9 | Navier fluid 1827 [18] | equilibrium of fluid | p | | $\frac{4\pi}{3}$ | | $\int_0^\infty d\rho \rho^3$ | ρ^3 | | $f(\rho)$ | | ρ : radius |
| 10 | Poisson 1831 [26] | equilibrium of fluid | q | | $\frac{1}{4}$ | | $\sum \frac{1}{\epsilon^3} r^4$ | $\frac{1}{r}$ | | $r_i^2 z' R$ | | $C_3 = \frac{1}{4\pi} \pi = \frac{1}{4}$ $C_4 = \frac{1}{4\pi} \frac{2\pi}{3} = \frac{1}{6}$ |
| | | | p | | $\frac{1}{6}$ | | $\sum \frac{1}{\epsilon^3}$ | r | | R | | |
| 11 | Saint-Venant 1843 [31] | fluid | ϵ | $\frac{\epsilon}{3}$ | | | | | | | | |
| 12 | Stokes 1849 [32] | fluid | μ | $\frac{\mu}{3}$ | | | | | | | | |
| 13 | Stokes 1849 [32] | elastic solid | A | B | | | | | | | | $A = 5B$ |

attraction in the distance f , the integrals ⁸

$$\int_x^\infty \varphi f \cdot df = \Pi x, \quad \int_x^\infty \Pi f \cdot f df = \Psi x,$$

⁸cf. Laplace states the two-constants (1) in his original paper. Poisson cites these equations with the same H and K . cf. the entry no.8 in Table 1.

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TABLE 2. The kinetic equations of the hydrodynamics until the "Navier-Stokes equations" was fixed. (Rem. HD : hydro-dynamics, N under entry-no : non-linear, gr.dv : grad.div, E : $\frac{\Delta}{gr.dv}$ of elastic, F : $\frac{\Delta}{gr.dv}$ and the group of entry 6-14 show F = 3 in fluid.)

| no | name/prob | the kinetic equations | Δ | gr.dv | E | F |
|----|--|--|-----------------|-------------------------|--------------------|---------------|
| 1 | Euler (1752-55) N [6, p.127] fluid | $\begin{cases} X - \frac{1}{h} \frac{dp}{dx} = \frac{du}{dt} + u \frac{du}{dx} + v \frac{du}{dy} + w \frac{du}{dz}, \\ Y - \frac{1}{h} \frac{dp}{dy} = \frac{dv}{dt} + u \frac{dv}{dx} + v \frac{dv}{dy} + w \frac{dv}{dz}, \\ Z - \frac{1}{h} \frac{dp}{dz} = \frac{dw}{dt} + u \frac{dw}{dx} + v \frac{dw}{dy} + w \frac{dw}{dz}, \end{cases} \quad \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$ | | | | |
| 2 | Navier (1827)[17] elastic solid | $(6-1)N^e \quad \begin{cases} \frac{\Pi}{g} \frac{d^2x}{dt^2} = \epsilon \left(3 \frac{d^2x}{dx^2} + \frac{d^2x}{dy^2} + \frac{d^2x}{dz^2} + 2 \frac{d^2y}{dxdx} + 2 \frac{d^2z}{dcdx} \right), \\ \frac{\Pi}{g} \frac{d^2y}{dt^2} = \epsilon \left(\frac{d^2y}{dx^2} + 3 \frac{d^2y}{dy^2} + \frac{d^2y}{dz^2} + 2 \frac{d^2x}{dadb} + 2 \frac{d^2z}{dadb} \right), \\ \frac{\Pi}{g} \frac{d^2z}{dt^2} = \epsilon \left(\frac{d^2z}{dx^2} + \frac{d^2z}{dy^2} + 3 \frac{d^2z}{dz^2} + 2 \frac{d^2x}{dadc} + 2 \frac{d^2y}{dbdc} \right) \end{cases}$ where Π is density of the solid, g is acceleration of gravity. | ϵ | 2ϵ | $\frac{1}{2}$ | |
| 3 | Navier (1827)[18] N fluid | $\begin{cases} \frac{1}{\rho} \frac{dp}{dx} = X + \epsilon \left(3 \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2 \frac{d^2v}{dxdx} + 2 \frac{d^2w}{dxdz} \right) - \frac{du}{dt} - \frac{du}{dx} \cdot u - \frac{du}{dy} \cdot v - \frac{du}{dz} \cdot w; \\ \frac{1}{\rho} \frac{dp}{dy} = Y + \epsilon \left(\frac{d^2v}{dx^2} + 3 \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 2 \frac{d^2u}{dxdy} + 2 \frac{d^2w}{dxdz} \right) - \frac{dv}{dt} - \frac{dv}{dx} \cdot u - \frac{dv}{dy} \cdot v - \frac{dv}{dz} \cdot w; \\ \frac{1}{\rho} \frac{dp}{dz} = Z + \epsilon \left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3 \frac{d^2w}{dz^2} + 2 \frac{d^2u}{dxdz} + 2 \frac{d^2v}{dydz} \right) - \frac{dw}{dt} - \frac{dw}{dx} \cdot u - \frac{dw}{dy} \cdot v - \frac{dw}{dz} \cdot w; \end{cases}$ | ϵ | 2ϵ | | $\frac{1}{2}$ |
| 4 | Cauchy (1828)[3] system of particles in elastic and fluid | $\begin{cases} (L+G) \frac{\partial^2 \xi}{\partial x^2} + (R+H) \frac{\partial^2 \xi}{\partial y^2} + (Q+I) \frac{\partial^2 \xi}{\partial z^2} + 2R \frac{\partial^2 \eta}{\partial x \partial y} + 2Q \frac{\partial^2 \zeta}{\partial x \partial z} + X = \frac{\partial^2 \xi}{\partial t^2}, \\ (R+G) \frac{\partial^2 \eta}{\partial x^2} + (M+H) \frac{\partial^2 \eta}{\partial y^2} + (P+I) \frac{\partial^2 \eta}{\partial z^2} + 2P \frac{\partial^2 \xi}{\partial y \partial z} + 2R \frac{\partial^2 \xi}{\partial x \partial y} + Y = \frac{\partial^2 \eta}{\partial t^2}, \\ (Q+G) \frac{\partial^2 \zeta}{\partial x^2} + (P+H) \frac{\partial^2 \zeta}{\partial y^2} + (N+I) \frac{\partial^2 \zeta}{\partial z^2} + 2Q \frac{\partial^2 \xi}{\partial x \partial z} + 2P \frac{\partial^2 \eta}{\partial y \partial z} + Z = \frac{\partial^2 \zeta}{\partial t^2}, \\ G = H = I, \quad L = M = N, \quad P = Q = R, \quad L = 3R \end{cases}$ | $R+$ G | $2R$ | if G $= 0$ | $\frac{1}{2}$ |
| 5 | Poisson (1831)[26] elastic solid in general equations | $\begin{cases} X - \frac{d^2u}{dt^2} + a^2 \left(\frac{d^2u}{dx^2} + \frac{2}{3} \frac{d^2v}{dydx} + \frac{2}{3} \frac{d^2w}{dzdx} + \frac{1}{3} \frac{d^2u}{dy^2} + \frac{1}{3} \frac{d^2u}{dz^2} \right) = \frac{\Pi}{\rho} \frac{d^2u}{dx^2}, \\ Y - \frac{d^2v}{dt^2} + a^2 \left(\frac{d^2v}{dy^2} + \frac{2}{3} \frac{d^2u}{dxdy} + \frac{2}{3} \frac{d^2w}{dxdz} + \frac{1}{3} \frac{d^2u}{dx^2} + \frac{1}{3} \frac{d^2v}{dz^2} \right) = \frac{\Pi}{\rho} \frac{d^2v}{dy^2}, \\ Z - \frac{d^2w}{dt^2} + a^2 \left(\frac{d^2w}{dz^2} + \frac{2}{3} \frac{d^2u}{dxdz} + \frac{2}{3} \frac{d^2v}{dydz} + \frac{1}{3} \frac{d^2u}{dx^2} + \frac{1}{3} \frac{d^2v}{dy^2} \right) = \frac{\Pi}{\rho} \frac{d^2w}{dz^2}, \end{cases}$ | $\frac{a^2}{3}$ | $\frac{2a^2}{3}$ | $\frac{1}{2}$ | |
| 6 | Poisson (1831)[26] fluid in general equations | $\begin{cases} \rho \left(\frac{Du}{Dt} - X \right) + \frac{dp}{dx} + \alpha(K+k) \left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} \right) + \frac{\alpha}{3}(K+k) \frac{d}{dx} \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right) = 0, \\ \rho \left(\frac{Dv}{Dt} - Y \right) + \frac{dp}{dy} + \alpha(K+k) \left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} \right) + \frac{\alpha}{3}(K+k) \frac{d}{dy} \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right) = 0, \\ \rho \left(\frac{Dw}{Dt} - Z \right) + \frac{dp}{dz} + \alpha(K+k) \left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2} \right) + \frac{\alpha}{3}(K+k) \frac{d}{dz} \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right) = 0, \\ \rho \left(X - \frac{d^2x}{dt^2} \right) = \frac{d\omega}{dx} + \beta \left(\frac{d^2u}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2w}{dz^2} \right), \\ \rho \left(Y - \frac{d^2y}{dt^2} \right) = \frac{d\omega}{dy} + \beta \left(\frac{d^2u}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2w}{dz^2} \right), \\ \rho \left(Z - \frac{d^2z}{dt^2} \right) = \frac{d\omega}{dz} + \beta \left(\frac{d^2u}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2w}{dz^2} \right) \end{cases}$ WHERE $\omega \equiv p - \alpha \frac{d\psi t}{dt} - \frac{\beta + \beta'}{\chi t} \frac{dx t}{dt}$, $\beta \equiv \alpha(K+k)$ | β | $\frac{\beta}{3}$ | | 3 |
| 7 | Saint-Venant (1843)[31] fluid | His equations are not in his paper [31], however we are available for it by his tensor. | ϵ | $\frac{\epsilon}{3}$ | | 3 |
| 8 | Stokes (1849)[32] fluid | $(12)_S \quad \begin{cases} \rho \left(\frac{Du}{Dt} - X \right) + \frac{dp}{dx} - \mu \left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} \right) - \frac{\mu}{3} \frac{d}{dx} \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right) = 0, \\ \rho \left(\frac{Dv}{Dt} - Y \right) + \frac{dp}{dy} - \mu \left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} \right) - \frac{\mu}{3} \frac{d}{dy} \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right) = 0, \\ \rho \left(\frac{Dw}{Dt} - Z \right) + \frac{dp}{dz} - \mu \left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2} \right) - \frac{\mu}{3} \frac{d}{dz} \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right) = 0. \end{cases}$ | μ | $\frac{\mu}{3}$ | | 3 |
| 9 | Maxwell (1865-66) [16] HD | $\begin{cases} \rho \frac{\partial u}{\partial t} + \frac{dp}{dx} - C_M \left[\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + \frac{1}{3} \frac{d}{dx} \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right) \right] = \rho X, \\ \rho \frac{\partial v}{\partial t} + \frac{dp}{dy} - C_M \left[\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + \frac{1}{3} \frac{d}{dy} \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right) \right] = \rho Y, \\ \rho \frac{\partial w}{\partial t} + \frac{dp}{dz} - C_M \left[\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2} + \frac{1}{3} \frac{d}{dz} \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right) \right] = \rho Z \end{cases}$ where, $C_M \equiv \frac{\rho M}{6k\rho\Theta_2}$ | C_M | $\frac{C}{3}$ | | 3 |
| 10 | Kirchhoff (1876)[10] HD | $\begin{cases} \mu \frac{du}{dt} + \frac{\partial p}{\partial x} - C_K \left[\Delta u + \frac{1}{3} \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right] = \mu X, \\ \mu \frac{dv}{dt} + \frac{\partial p}{\partial y} - C_K \left[\Delta v + \frac{1}{3} \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right] = \mu Y, \\ \mu \frac{dw}{dt} + \frac{\partial p}{\partial z} - C_K \left[\Delta w + \frac{1}{3} \frac{\partial}{\partial z} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right] = \mu Z, \end{cases} \quad \begin{cases} \frac{1}{\mu} \frac{d\mu}{dt} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \\ \text{where, } C_K \equiv \frac{1}{3\kappa} \frac{\rho}{\mu} \end{cases}$ | C_K | $\frac{\Delta}{3}$ | | 3 |
| 11 | Rayleigh (1883)[30] N HD | $\begin{cases} \frac{1}{\rho} \frac{dp}{dx} = -\frac{du}{dt} + \nu \nabla^2 u - u \frac{du}{dx} - v \frac{du}{dy}, \\ \frac{1}{\rho} \frac{dp}{dy} = -\frac{dv}{dt} + \nu \nabla^2 v - u \frac{dv}{dx} - v \frac{dv}{dy}, \end{cases} \quad \frac{du}{dx} + \frac{dv}{dy} = 0$ | ν | | | |
| 12 | Boltzmann (1895)[1] HD | $(221)_B \quad \begin{cases} \rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} - \mathcal{R} \left[\Delta u + \frac{1}{3} \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right] = \rho X, \\ \rho \frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} - \mathcal{R} \left[\Delta v + \frac{1}{3} \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right] = \rho Y, \\ \rho \frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} - \mathcal{R} \left[\Delta w + \frac{1}{3} \frac{\partial}{\partial z} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right] = \rho Z \end{cases}$ | \mathcal{R} | $\frac{\mathcal{R}}{3}$ | | 3 |
| 13 | Prandtl (1905)[28] N HD | $\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) + \nabla(V+p) = k \nabla^2 \mathbf{v}, \quad \text{DIV } \mathbf{v} = 0$ | k | | | |
| 14 | Prandtl (1934)[29] N HD | $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = X - \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\nu}{3} \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right),$ FOR INCOMPRESSIBLE, IT IS SIMPLIFIED $\text{DIV } \mathbf{w} = 0$, $\frac{D\mathbf{w}}{dt} = g - \frac{1}{\rho} \text{GRAD } p + \nu \Delta \mathbf{w}$ | ν | $\frac{\nu}{3}$ | | 3 |

TABLE 3. Cross-indexed differences on the RDFs $f \in \mathcal{RFD}$ (Remark. 1,5,6 : on capillarity)

| no | Name/Problem/ Bibl. (Year read) - Year published/ | 1 Laplace | 2 Poisson | 3 Navie | 4 $f(r)$ at $r = 0$ | 5 $f(r)$ at $r = \infty$ |
|----|--|---|--|---|------------------------------|-----------------------------------|
| 1 | Laplace capillary action : [13] 1806-07 | $L_1 : K, H$ L_2 :force attractive only and $f \simeq c^{-if}, f \in \mathcal{RDF}$ | | | 0 | 0 |
| 2 | Poisson elastic : [22],(1828)-28; [25],1829;[26],(1829)-31 fluid : [26],(1829)-31 disputing origin: [22],1828 (with Navier : [23],1828;[24],1828) | Refer to Laplace's $f \in \mathcal{RFD}$ | k, K | $P_1 \rightarrow N_1 :$ $f \simeq ab^{-\left(\frac{r}{\pi a}\right)^m}$ $P_2 \rightarrow N_2$: not by integral but by sum because $k = -K = 0$ at once. $P_3 \rightarrow N_3 : k = \varepsilon$ of Navier $P_4 \rightarrow N_4 : f \in \mathcal{RFD}$ | 0 | 0 |
| 3 | Navier elastic:[17],(1821)-27 fluid:[18],(1822)-27 (with Poisson : [?],1828; [19],1829; [20],1829; [21],1829 with Arago[21],1829) | Refer to Laplace's integral | $N_1 \rightarrow P_1 : f \simeq e^{-kp}$ $N_2 \rightarrow P_2$: not by sum but by integral as Laplace does $N_3 \rightarrow P_3 : [r^4 f(r)]_0^\infty \neq 0,$ $\varepsilon \neq k$ $N_4 \rightarrow P_4 : r^4 f(r)$ for $r = 0,$ $f \in \mathcal{RFD}$ but only in $r = \infty,$ $f(r) \not\rightarrow 0$ as $r \rightarrow 0$ | ε in elastic ε, E in fluid | $\neq 0$ | 0 |
| 4 | Cauchy elastic & fluid :[3] | | | | 0 | 0 |
| 5 | Gauss capillary action : [8] (to Laplace [8],1830 to Bessel[9],1830) | $G_1 \rightarrow L_1$:Laplace's deduction is conspicuous. $G_2 \rightarrow L_2$:no necessary to limit i of c^{-if} to be very large. | | | - | |
| 6 | Poisson capillary action : [27],1831, (to Gauss[27]) | Same K and H with Laplace | | | 0 | 0 |

; The integral of two values : ⁹

$$2\pi \int_0^\infty \Psi f . df = K, \quad 2\pi \int_0^\infty \Psi f . f . df = H, \quad (1)$$

where denoting by π the $\frac{1}{2}$ of the circumference of the circle with radius = 1.

• In a word, the $\langle \text{indoles} \rangle$ of the function φf reserves ineffective, as long as this f were insensible for all sensible value.

¶ 4. (Criticism to Laplace by Gauss.)

• However, something similar to simple carelessness form the basis, such that he discusses about the form than about the relating action with it.

• Judging from the second dissertation : $\langle \text{Supplément à la théorie de l'action capillaire} \rangle$, Mr. Laplace investigated a little, not only the complete attraction, but also the partial one by φf , and tacitly understood incompletely the general attraction ; by the way, if we would refer the latter by him about our sensible modification, it is easy to see being conspicuous about it.

• He considers exponential e^{-if} as an example of equivalent function with φf , denoting the large quantity by i , namely $\frac{1}{i}$ becomes infinitesimal. But it is not at all necessary to limit the generality by such a large quantity, the things are more clear than words, we would see easiest, only to investigate if these integrations would be extended, not only infinite but also to an arbitrary sensible distance, or if anything, occurring wider in the finitely measurable distance in experiment. ¹⁰

⁹Poisson rewrite these equations to the equivalent with Laplace. cf. the entry no.6-2 in Table 1.

¹⁰(↓) We show his Latin original as follows :

4.2. Three capitals of force and two RFDs : φ and Φ .

We consider the force reducing to three capitals.

- I. Gravity.
- II. The attractive force, which itself corresponds to the points m, m', m'', \dots . The intensity of attraction of function is propotional with the distance if this function, the *< characteristic >* denoted by f in mass and supposed that the attraction is uniformly concentrated in the point.
- III. The forces, m, m', m'', \dots are attractive to the infinitesimal fixed points. For these forces, with the similar way, we will designate the *< characteristic F >* such that the inverse-directional distance is used, and with M, M', M'', \dots , which are treated as a fixed point in one case, or a mass in the other case, which are supposed in these concentrate.

We get $\sum Pdp$ of the previous article as follows :

$$\begin{aligned} & -gdz \\ & - m'f(m, m')d(m, m') - m''f(m, m'')d(m, m'') - m'''f(m, m''')d(m, m''') - \dots \\ & - MF(m, M)d(m, M) - M'F(m, M')d(m, M') - M''F(m, M'')d(m, M'') - \dots \end{aligned} \quad (2)$$

where, the difference $d(m, m')$, $d(m, m'')$ etc. are partial, relative to the only motion of the force of m . We denote :

$$\varphi \text{ such that : } -fx.dx = d\varphi x, \quad \int fx.dx = -\varphi x, \quad (3)$$

$$\Phi \text{ such that : } -Fx.dx = d\Phi x, \quad \int Fx.dx \equiv -\Phi x \quad (4)$$

where, $\varphi\infty = 0$, and in case of $\varphi t \Rightarrow \int_t^\infty fx.dx = -\varphi t$.

(Ψ)Gauss didn't describe explicitly about $\varphi 0$. By the way, this method without taking of "two-constants" by Gauss corresponds to other's style by such as Laplace, Poisson, Navier and so on. Poisson [27, p.8] considers this method as one of Gauss' characteristic, however Poisson chose his own method like Laplace. cf. the entry no.8 in Table 1.(\uparrow)

The function Ω is expressed by the following sequence :

$$\begin{aligned} \Omega = \sum m \{ & -gz + \frac{1}{2}m'\varphi(m, m') + \frac{1}{2}m''\varphi(m, m'') + \frac{1}{2}m'''\varphi(m, m''') + \dots \\ & + M\Phi(m, M) + M'\Phi(m, M') + M''\Phi(m, M'') + \dots \end{aligned}$$

where, *< characteristic Σ >* represents the expression of sum, in which m', m'', m''', \dots follow permuting after m .

4.3. The sum of force : Ω .

For brevity, we express :

$$\Omega = -gc \int zds + \frac{1}{2}c^2 \iint ds.ds'.\varphi(ds, ds') + cC \iint ds.dS.\Phi(ds, dS) \quad (5)$$

where, s, s' are specially denoted spaces (satisfied with the mobile material), however with the duplex integration¹¹, integrate twice with the element to resolve it.

Sed ne opus quidem est, generalitatem tantopere limitare, quum is, qui rem potius quam verba intuetur, facillime videat, sufficere, si intergrationes illae non in infinitum, sed tantummodo usque ad distantiam sensibilem arbitrariam, aut si mavis ad distantiam finitam dimensionibus in experimentis occurrentibus maiorem extendantur. [8, p.33]

¹¹In below, Gauss uses "duplex" not only as both P and U , but also as two triangles.

(↓) Here the integral (5) contains sextuplex integral when using both (3) and (4). Moreover, Poisson comments Ω consisted of three terms. ¹²

4.4. Transformation of the expression and the definition of s, S, φ, Φ .

We take the transformation as follows :

- of the second and third terms to two cases of the particular problem, where, proposition of the dual spaces whatever, single element of the first space with second element, we combine and product from the third factor, put from the element volume of the first space and the volume element of the second space, and the function data of the mutual distance, and then we can sum up to the last,
- the second term to the same way, where the both space is the same,
- the third to it, where all of a side of space is from the other side of space,

then, the problem is solved. The two different cases are completed, namely

- when one side of space is part of the other side of space,
- or when each side has the common part with the other part.

Although, moreover, the first case is sufficient to institute us, or we can easy return the rest to the other side, when the work evaluate, the problem in itself complete by accepting the general sign.

In this problem, we denote the spaces by s and S , the function on distance denoted with the < characteristic φ >, as the same as in the application to the second located term S and s of (5), and to the third located term, we may replace Φ with φ . The integration is given as follows :

$$\iint ds.dS.\varphi(ds, dS) \quad (6)$$

We would like to show that the spacial elements, depending on the three variables, which imply that the sextuplex integral are to be reduced to the quadruplex integral. (↓) Here the integral (6) contains triple integral when using either (3) or (4), then (5) contains sextuplex integral. ¹⁵

4.5. Reduced integral from sextuplex to quadruplex.

Our integral (I) neglecting the insensible factors : $= - \int \pi\theta'\rho.d\tau + \int \pi\theta\rho.d\tau'$. Clearly this is not important, either the parts τ and τ' or to the surface T to t is rather important. The value of the sextuplex integral : (6) becomes

$$\iint ds.dS.\varphi(ds, dS) = 4\pi\sigma\psi\theta - \pi T\theta\theta + \pi T'\theta\theta - \pi \int d\tau.\theta'\rho + \pi \int d\tau'.\theta\rho \quad (7)$$

(↓) Just this transformation is boastful reductional method of integral from the sextuplex to quadruplex.

4.6. Method of reduction of $\iint ds.dS.\varphi(ds, dS)$ from sextuplex to quadruplex.

- Therefore, we can assume the primitive function θ' , i.e.,

$$2r^2 \int \frac{\theta r.dr}{r^3} = -\theta'r \Rightarrow \frac{\theta'r}{r^2} = \int \frac{2\theta x.dx}{x^3} \quad (8)$$

- We consider the integral from $x = r$ to an arbitrary, sensible and constant value, denoted by R . Namely we integrate as follows : ¹⁶

$$\int_R^r \frac{2\theta x.dx}{x^3} = \frac{\theta r}{r^2} - \frac{\theta R}{R^2} \quad (9)$$

¹²Poisson cites Gauss' *minimum* denoted by Ω in (5) in his preface of [27] and states :

Dans le cas d'un liquide homogène et incompressible, il réduit d'abord cette quantité à une intégrale quadruple ; et en considérant spécialement le cas où les forces appliquées au liquide sont la pesanteur et l'attraction mutuel de ses molécules, dont la sphère d'activité est insensible, il réduit de nouveau la quantité dont il s'agit, qui est ensuit composée de trois termes, savoir,

- (1) le produit du poids du liquide et de l'ordonnée verticale de son centre du gravité,
- (2) l'aire de sa surface libre multipliée par une constante¹³ qui ne dépend que de la matière du liquide,
- (3) et l'aire des parois fixes contre lesquelles il s'appuie, multipliée par une seconde constante¹⁴ de la matière du liquide et de celle de la partie solide du système.

[27, pp.7-8]

¹⁵Poisson recognizes this Gauss' achievement in [27].

¹⁶(↓) This function is rapidly decreasing function. Here, θr , θR mean $\theta(r)$, $\theta(R)$ and are assumed as $\theta(r) > \theta(R)$.

Clearly this integral is smaller than this $\int \frac{2\theta x \cdot dx}{x^3}$ with the interval, this is $= \frac{\theta r}{r^2} - \frac{\theta R}{R^2}$. Moreover, it is smaller than $\frac{\theta r}{r^2}$. Otherwise, by infinite integral, it become as follows :

$$\int \frac{2\theta x \cdot dx}{x^3} = -\frac{\theta x}{x^2} + \int \frac{d\theta x}{x^2} = -\frac{\theta x}{x^2} - \int \frac{\psi x \cdot dx}{x^2} \tag{10}$$

Moreover, from (8), (9) and (10),

$$\frac{\theta' r}{r^2} = \int \frac{2\theta x \cdot dx}{x^3} =^* \left[-\frac{\theta x}{x^2} - \int \frac{\psi x \cdot dx}{x^2} \right]_{x=r} = \left(\frac{\theta r}{r^2} - \frac{\theta R}{R^2} \right) - \int \frac{\psi r \cdot dx}{r^2} =^* \left(\frac{\theta r}{r^2} - \frac{\theta R}{R^2} \right) - \frac{\psi r}{r} \tag{11}$$

• Integrating with the smaller interval than the integral $\int \frac{\psi x \cdot dx}{x^2}$. Moreover, from (11), this is smaller than $\frac{\psi r}{r}$; therefore, the value of $\frac{\psi' r}{r^2}$ is greater than the right-side expression of (12) ¹⁷

$$\frac{\theta' r}{r^2} = \left(\frac{\theta r}{r^2} - \frac{\theta R}{R^2} \right) - \frac{\psi r}{r} \Rightarrow \theta' r = \theta r - \frac{r^2 \cdot \theta R}{R^2} - r\psi r \tag{12}$$

From (12), the interval of $\theta' r : \theta r$ and $\theta r - r^2 \cdot \frac{\theta R}{R^2} - r\psi r =^* \theta' r$

• If we differentiate this expression, by r decreasing infinitely, then we see clearly that we can evaluate this quantity to be infinitesimal, for example, when ψ_0 in (7) is the finite quantity. Thus we have concluded that it is due to $\theta'_0 = \theta_0$. It is clearly considerable that, the formula (7) of previous art.16 (§4.5) turns into

- $-\pi T \theta_0$ and for instance, under the interval $-\pi \int d\tau \cdot \theta' \rho$
- $\pi T' \theta_0$ and for instance, under the interval $\pi \int d\tau' \cdot \theta' \rho$,

if the difference or the distance is insensible or considerable as null, to count respectively the part of T, T' or τ, τ' .

4.7. Variation problem to be solved.

In the application of previous survey to the evolution the second term of the expression Ω in the art. 3, in the art. 6 denote by S in the art.16 σ, T, T' will be use as $s, t, 0$, if t is the total surface of the space s , in which the fluid is filled. Therefore whenever this space extensional sensible part however insensible concentration is kept, this sort of gap (crevice), the part of the second part of the expression Ω of (5) in the art. 4.3 becomes $= \frac{1}{2} \pi c^2 (s\phi_0 - t\theta_0)$. In static equilibrium it is due to the maximum value, this turns into $-gc \int z ds + \frac{1}{2} c^2 s \psi_0 - \frac{1}{2} \pi c^2 t \theta_0 + \pi c C T \Theta_0$. In an arbitrary fluid, of which the figure is yield oneself to the space s meaning invariant, of which the expression becomes as follows : $\int z ds + \frac{\pi c \theta_0}{2g} \cdot t - \frac{\pi C T \Theta_0}{g} \cdot T$, and in an equilibrium state which is due to *minimum*. Here, we denote

$$\frac{\pi c \theta_0}{2g} \equiv \alpha^2, \quad \frac{\pi C T \Theta_0}{2g} \equiv \beta^2, \quad t \equiv T + U, \tag{13}$$

and by W , then

$$W \equiv \int z ds + (\alpha^2 - 2\beta^2)T + \alpha^2 U \tag{14}$$

4.8. Geometric structure for analysis.

Here, we consider : • the surface, denoted by s , • a part U , on which all the points is determined by the coordinate x, y, z , these three values are the distances to an arbitrary horizontal plane. It is capable to recognize z is, for example, as the indeterminated function by x, y , for these secondary partial differential with our conventional method, by omitting a bracket, we show it by $\frac{dz}{dx} \cdot dx, \frac{dz}{dy} \cdot dy$. ¹⁸ The structure we are considering is as follows :

- (1) We define the points consisted of an arbitrary and every points on the surface, denoting s with respect to the rectangular surface, normal to the exterior direction of s , and in addition, we set

¹⁷(ψ) Multiplying by r^2 , which is infinitesimal value. Today's description of (12) is $\theta'(r) = \theta(r) - \frac{r^2 \cdot \theta(R)}{R^2} - r\psi(r)$.
¹⁸(ψ) These descriptions by Gauss mean as follows :

$$\frac{dz}{dx} \cdot dx \equiv \left(\frac{dz}{dx} \right)_x = \frac{d^2 z}{dx^2}, \quad \frac{dz}{dy} \cdot dy \equiv \left(\frac{dz}{dy} \right)_y = \frac{d^2 z}{dy^2}, \quad \frac{dz}{dx} \cdot dx' \equiv \left(\frac{dz}{dx} \right)_{x'} = \frac{d^2 z}{dx x'}$$

an angle by cosine between this normal direction to the axis of rectangular coordinate x, y and z with parallel, which we denote by ξ, η and ζ . Thereby it will be :

$$\xi^2 + \eta^2 + \zeta^2 = 1, \quad \frac{dz}{dx} = -\frac{\xi}{\zeta}, \quad \frac{dz}{dy} = -\frac{\eta}{\zeta}, \quad \Rightarrow^* \quad 1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2 = \frac{1}{\zeta^2} \quad (15)$$

(2) The boundary of surface U become linear in itself, as the same as denoted by P , and while the motion is supposed necessarily, this element dP (as the same way of dU as the surface) is treated as positive only.

(3) The angle by cosine, that directions of the element dP are expressed with the axis of coordinate of x, y, z , denoted by X, Y, Z : since we would avoid giving ambiguous sense about the direction, we define these angles as follows :

- at first, we assume that the normal direction in the element dP to the surface U , and draw a tangent
- next, looking this line innerward, we draw the second side.
- finally, in the normal direction with respect to the surface, we put the third side in the space s to the exterior,

and constituting similarly the next system of three rectangles and the coordinate axis x, y, z .

Thus, we see easily the following expressions (cf. *Disquisitiones generales circa superficies curvas*), using the angle by cosine with the direction to the axis of the coordinates x, y, z are respectively

$$\eta^0 Z - \zeta^0 Y, \quad \zeta^0 X - \xi^0 Z, \quad \xi^0 Y - \zeta^0 X \quad \Rightarrow^* \quad \begin{bmatrix} \alpha & \beta & \gamma \\ X & Y & Z \\ \xi^0 & \eta^0 & \zeta^0 \end{bmatrix}, \quad (16)$$

here, we suppose that ξ^0, η^0, ζ^0 are the values of ξ, η, ζ for the points of the element dP .

(↓) where, α, β, γ are temporarily used values of ours to correspond to (35). By the way, we see (16) is the same with the determinant to be mentioned again below (35).

4.9. Variation of a triangle dU of the surface U .

Here we would like to supplement the preliminary. We assume the surface U is the part by an arbitrary infinitesimal perturbation.

- If we consider sufficiently all the perturbation, for this boundary P always invariant, at any rate, it maintains, in this vertical surface, we can induce clearly the variation of only the third coordinate z , this problem is far easy to evaluate it ;
- moreover, the maximum problem in general, in the following investigating method, considering the variable boundary, in which ambiguity and difficulty combine elegantly, bring up perturbation ; how we can show, always from the start of all, three coordinates handle the variation.

We the force as we image it, and anywhere on the surface, in which the coordinates, which are x, y, z , had substituted in another, these coordinates are $x + \delta x, y + \delta y, z + \delta z$, where $\delta x, \delta y, \delta z$ are able to regard as if these were the indeterminate functions of x, y , if these values stay infinitesimal. Now we would like to inquire into the variation of singular (individual) element, expressed with W and surely the initial are made of variation of these elements dU .

Now, we assume a triangle consisted of three points : P_1, P_2, P_3 .¹⁹ We put the element of U by a triangle dU consisted of these points, of which the coordinates are :

$$\begin{cases} P_1 : & x & y & z \\ P_2 : & x + dx & y + dy & z + \frac{dz}{dx}.dx + \frac{dz}{dy}.dy \\ P_3 : & x + d'x & y + d'y & z + \frac{dz}{dx}.d'x + \frac{dz}{dy}.d'y \end{cases}$$

If we assume $dx.d'y - dy.d'x > 0$, then the twice area of this triangle is gained by our principle as follows :

$$(dx.d'y - dy.d'x) \sqrt{\left[1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right]} \quad (17)$$

↓(17) becomes $\frac{(dx.d'y - dy.d'x)}{\zeta}$ from (15). (↑)

- location value by perturbation of P_1 : $x + \delta x, y + \delta y, z + \delta z$.

¹⁹(↓) The symbols : P_1, P_2, P_3 are of ours insted of "the first point", etc.

- Location value by perturbation of P_2 :

$$\begin{bmatrix} x + dx \\ y + dy \\ z + \frac{dz}{dx} dx + \frac{dz}{dy} dy \end{bmatrix}, \begin{bmatrix} \delta x + \frac{d\delta x}{dx} dx + \frac{d\delta x}{dy} dy \\ \delta y + \frac{d\delta y}{dx} dx + \frac{d\delta y}{dy} dy \\ \delta z + \frac{d\delta z}{dx} dx + \frac{d\delta z}{dy} dy \end{bmatrix}, \begin{bmatrix} (x + \delta x) + (1 + \frac{d\delta x}{dx}) dx + \frac{d\delta x}{dy} dy \\ (y + \delta y) + \frac{d\delta y}{dx} dx + (1 + \frac{d\delta y}{dy}) dy \\ (z + \delta z) + (\frac{dz}{dx} + \frac{d\delta z}{dx}) dx + (\frac{dz}{dy} + \frac{d\delta z}{dy}) dy \end{bmatrix}$$

- Location value by perturbation of P_3 :

$$\begin{bmatrix} x + d'x \\ y + d'y \\ z + \frac{dz}{dx} d'x + \frac{dz}{dy} d'y \end{bmatrix}, \begin{bmatrix} \delta x + \frac{d\delta x}{dx} d'x + \frac{d\delta x}{dy} d'y \\ \delta y + \frac{d\delta y}{dx} d'x + \frac{d\delta y}{dy} d'y \\ \delta z + \frac{d\delta z}{dx} d'x + \frac{d\delta z}{dy} d'y \end{bmatrix}, \begin{bmatrix} (x + \delta x) + (1 + \frac{d\delta x}{dx}) d'x + \frac{d\delta x}{dy} d'y, \\ (y + \delta y) + \frac{d\delta y}{dx} d'x + (1 + \frac{d\delta y}{dy}) d'y, \\ (z + \delta z) + (\frac{dz}{dx} + \frac{d\delta z}{dx}) d'x + (\frac{dz}{dy} + \frac{d\delta z}{dy}) d'y \end{bmatrix}$$

(\Downarrow) We can also show the matrix only with variation as follows :

$$\begin{bmatrix} \delta x & \delta y & \delta z \\ (1 + \frac{d\delta x}{dx}) dx + \frac{d\delta x}{dy} dy & \frac{d\delta y}{dx} dx + (1 + \frac{d\delta y}{dy}) dy & E dx + D dy \\ (1 + \frac{d\delta x}{dx}) d'x + \frac{d\delta x}{dy} d'y & \frac{d\delta y}{dx} d'x + (1 + \frac{d\delta y}{dy}) d'y & E d'x + D d'y \end{bmatrix}$$

where, $E \equiv \frac{dz}{dx} + \frac{d\delta z}{dx}$, $D \equiv \frac{dz}{dy} + \frac{d\delta z}{dy}$ (18)

By the way, these principle comes from Lagrange [11, pp.189-236],²⁰ in which Lagrange states his *méthode des variations*²¹ in hydrostatics. (\Uparrow)

The duplex triangles²² including these points, by the same method, for brevity, by denoting the sum by N , (17) is expressed as follows :

$$(dx.d'y - dy.d'x)\sqrt{N}$$

(\Downarrow) These values : $dx.d'y - dy.d'x$, $dz.d'x - dx.d'z$ and $dy.d'z - dz.d'y$ are calculated in permutation by Jacobian $|J|$ of the three determinants extracted from (18) :

$$(x, y) : \begin{vmatrix} 1 + \frac{d\delta x}{dx} & \frac{d\delta x}{dy} \\ \frac{d\delta y}{dx} & 1 + \frac{d\delta y}{dy} \end{vmatrix}, \quad (x, z) : \begin{vmatrix} 1 + \frac{d\delta x}{dx} & \frac{d\delta x}{dy} \\ E & D \end{vmatrix}, \quad (y, z) : \begin{vmatrix} 1 + \frac{d\delta y}{dy} & \frac{d\delta y}{dx} \\ D & E \end{vmatrix}$$

(\Uparrow)

We denote temporarily the following sum by N , then

$$\begin{aligned} N &= \left[\left(1 + \frac{d\delta x}{dx}\right) \left(1 + \frac{d\delta y}{dy}\right) - \frac{d\delta x}{dy} \frac{d\delta y}{dx} \right]^2 + \left[\left(1 + \frac{d\delta x}{dx}\right) \left(\frac{dz}{dy} + \frac{d\delta z}{dy}\right) - \frac{d\delta x}{dy} \left(\frac{dz}{dx} + \frac{d\delta z}{dx}\right) \right]^2 \\ &+ \left[\left(1 + \frac{d\delta y}{dy}\right) \left(\frac{dz}{dx} + \frac{d\delta z}{dx}\right) - \frac{d\delta y}{dx} \left(\frac{dz}{dy} + \frac{d\delta z}{dy}\right) \right]^2 \\ &= C^2 + [D_1^2 + D_2^2] D^2 + [E_1^2 + E_2^2] E^2 - 2[D_1 E_2 + E_1 D_2], \end{aligned} \quad (19)$$

$$\text{where, } C \equiv \left(1 + \frac{d\delta x}{dx}\right) \left(1 + \frac{d\delta y}{dy}\right) - \frac{d\delta x}{dy} \frac{d\delta y}{dx} = 1 + \frac{d\delta x}{dx} + \frac{d\delta y}{dy}, \quad D \equiv \frac{dz}{dy} + \frac{d\delta z}{dy}, \quad E \equiv \frac{dz}{dx} + \frac{d\delta z}{dx}$$

and D_1, D_2, E_1, E_2 are the two terms consisting of D and E respectively, and these coefficients are correspond to the variables of the equation (20) showed in our footnote on the theory of curved surface

²⁰Section 7. *De l'équilibre des fluides incompressibles*, §2. *Où l'on déduit les lois générales de l'équilibre des fluides incompressibles de la nature des particules qui les composent.* [11, pp.204-236]

²¹Lagrange[11, p.201]. Today's mathematical nomenclature is *calculus of variations* or *calcul des variations* by *The mathematical dictionary* (4th edition in 2007) edited by MSJ, 1954, p.432, (Japanese).

²²(\Downarrow) The duplex triangles construct a rectangle made of arbitrary two adjoining triangles.

by Gauss [7].²³

Extending (19) with neglecting the second order of δ , for example, $\frac{d\delta x}{dy} \cdot \frac{d\delta y}{dx}$ or $(\frac{d\delta y}{dy})^2$, etc., and for brevity, denoting the sum by L , then

(\Downarrow)

$$\begin{aligned} \bullet \quad C^2 &= \left(1 + \frac{d\delta x}{dx} + \frac{d\delta y}{dy}\right)^2 \cong 1 + 2\left(\frac{d\delta x}{dx} + \frac{d\delta y}{dy}\right) \\ \bullet \quad \left[\left(1 + \frac{d\delta x}{dx}\right)^2 + \left(\frac{d\delta y}{dy}\right)^2\right] D^2 &\cong \left(\frac{dz}{dy}\right)^2 + 2\frac{d\delta x}{dx} \left(\frac{dz}{dy}\right)^2 + 2\frac{dz}{dy} \frac{d\delta z}{dy} \end{aligned}$$

Samely changing x with y in corresponding expression,

$$\begin{aligned} \bullet \quad \left[\left(\frac{d\delta x}{dy}\right)^2 + \left(1 + \frac{d\delta y}{dy}\right)^2\right] E^2 &\cong \left(\frac{dz}{dx}\right)^2 + 2\frac{d\delta y}{dy} \left(\frac{dz}{dx}\right)^2 + 2\frac{dz}{dx} \frac{d\delta z}{dx} \\ \bullet \quad -2\left[\left(1 + \frac{d\delta x}{dx}\right) \frac{d\delta x}{dy} + \left(1 + \frac{d\delta y}{dy}\right) \frac{d\delta y}{dx}\right] DE &\cong -2\frac{dz}{dx} \frac{dz}{dy} \left(\frac{d\delta x}{dy} + \frac{d\delta y}{dx}\right) \end{aligned}$$

(\Uparrow)

$$\sqrt{N} = \left[\left[1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right] \cdot \left[1 + \frac{L}{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2}\right] \right]^{\frac{1}{2}} =^* \left(L + 1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right)^{\frac{1}{2}}$$

where, L is gained by extracting only one order terms in the expanded terms from (19) :

(\Downarrow) Here, we see the coefficient 2 included in L in (22) come from two triangles.

N

$$\begin{aligned} &=^* C^2 + (\bullet)D^2 + (\bullet)E^2 + (\bullet)DE \\ &=^* \underbrace{1 + 2\left(\frac{d\delta x}{dx} + \frac{d\delta y}{dy}\right)}_{\bullet C^2} + \underbrace{\left(\frac{dz}{dy}\right)^2 + 2\frac{d\delta x}{dx} \left(\frac{dz}{dy}\right)^2 + 2\frac{dz}{dy} \frac{d\delta z}{dy}}_{\bullet D^2} + \underbrace{\left(\frac{dz}{dx}\right)^2 + 2\frac{d\delta y}{dy} \left(\frac{dz}{dx}\right)^2 + 2\frac{dz}{dx} \frac{d\delta z}{dx}}_{\bullet E^2} - \underbrace{2\frac{dz}{dx} \frac{dz}{dy} \left(\frac{d\delta x}{dy} + \frac{d\delta y}{dx}\right)}_{\bullet DE} \\ &=^* 2\left[\frac{d\delta x}{dx} \left\{1 + \left(\frac{dz}{dy}\right)^2\right\} - \frac{dz}{dx} \frac{dz}{dy} \left(\frac{d\delta x}{dy} + \frac{d\delta y}{dx}\right) + \frac{d\delta y}{dy} \left\{1 + \left(\frac{dz}{dx}\right)^2\right\} + \left(\frac{dz}{dy} \frac{d\delta z}{dy} + \frac{dz}{dx} \frac{d\delta z}{dx}\right)\right] + \left[1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right] \\ &=^* 2L + \left[1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right] \end{aligned} \quad (22)$$

(\Uparrow) We continue from Gauss. From (22)

$$\begin{aligned} L &= \left[\frac{d\delta x}{dx} \left\{1 + \left(\frac{dz}{dy}\right)^2\right\} - \frac{dz}{dx} \frac{dz}{dy} \left(\frac{d\delta x}{dy} + \frac{d\delta y}{dx}\right) + \frac{d\delta y}{dy} \left\{1 + \left(\frac{dz}{dx}\right)^2\right\} + \left(\frac{dz}{dy} \frac{d\delta z}{dy} + \frac{dz}{dx} \frac{d\delta z}{dx}\right)\right] \\ &=^* \frac{1}{2} \left[N - \left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}\right] \end{aligned} \quad (23)$$

(\Downarrow) Gauss' expression is without 2 of the top in the last right-hand side of (23), for L is a triangle. (\Uparrow)

Here we may recall (15), then the followings hold : the ratio of the first triangle to the second and plus 1 becomes,

$$1 + \frac{L}{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2} =^* 1 + \frac{\text{1st triangle}}{\text{2nd triangle}} =^* 1 + \zeta^2 L$$

Moreover, this is independent of the figure of a triangle dU , then, it turns out,

$$\delta dU = \frac{LdU}{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2} =^* \zeta^2 LdU \quad (24)$$

²³In *Disquisitiones generales circa superficies curvas*, Gauss deduces the following concluding equation (cf. [7]) :

$$EG - F^2 = E\left(\frac{dr}{dq}\right)^2 - 2F\frac{dr}{dp}\frac{dr}{dq} + G\left(\frac{dr}{dp}\right)^2 \quad (20)$$

We see (19) resembles one in [7].

$$N = C^2 + G'D^2 + E'E^2 - 2F'DE \quad (21)$$

If we assume that $\frac{dr}{dp} \equiv D$, $\frac{dr}{dq} \equiv E$, $E' = E_2^2 + E_1^2$, $F' = D_1E_2 + E_1D_2$ and $G' = D_1^2 + D_2^2$, then E' , F' and G' correspond to E , F and G in [7].

Expanding L in (24) using (15) and (23), then

$$\delta dU = dU \left[\frac{d\delta x}{dx} (\eta^2 + \zeta^2) - \left(\frac{d\delta x}{dy} + \frac{d\delta y}{dx} \right) \xi \eta + \frac{d\delta y}{dy} (\xi^2 + \zeta^2) - \frac{d\delta z}{dx} \xi \zeta - \frac{d\delta z}{dy} \eta \zeta \right], \quad (25)$$

(↓) where, we used the followings : $\zeta^2 \left(1 + \frac{dz}{dx} \right) = \zeta^2 + \zeta^2 \frac{\xi^2}{\zeta^2} = \xi^2 + \zeta^2$, $\zeta^2 \left(1 + \frac{dz}{dy} \right) = \zeta^2 + \zeta^2 \frac{\eta^2}{\zeta^2} = \zeta^2 + \eta^2$. Here, the coefficient of 2 in (23) is unnecessary, since dU is a triangle according to Gauss' description.

4.10. Integral expression by decomposing dU into dQ and dU .

From (25), all variation of the surface U is obtained by the following two integrals

$$\int dU \left[(\eta^2 + \zeta^2) \frac{d\delta x}{dx} - \xi \eta \left(\frac{d\delta y}{dx} \right) - \xi \zeta \frac{d\delta z}{dx} \right] \equiv A, \quad (x\text{-differential part}) \quad (26)$$

$$\int dU \left[-\xi \eta \frac{d\delta x}{dy} + (\xi^2 + \zeta^2) \frac{d\delta y}{dy} - \eta \zeta \frac{d\delta z}{dy} \right] \equiv B, \quad (y\text{-differential part}) \quad (27)$$

and these are separately treated. We consider as follows :

- at first, we take a plane, normal to the coordinate axis y , and such as, for the value of this y to be determinated suitably taking the exterior value to the peripheral, and for the last value of y to be in the surface U ;
- next, for this plane, on the peripheral P , we split into two part, or four, or six, etc., for the points of which by the first coordinate, to be followed by x^0, x', x'', \dots ; namely, as if the indeces are different each other, we should number suitably by the indicies to these points ;
- then, by the same way, we split the surface with other plane, for this infinite neighbourhood to be parallel, and to encounter with the point of the second coordinate $y + dy$;
- finally, between these planes, we could get the elements of peripheral dP^0, dP', dP'', \dots ,

then we could see easily the expressed as follows :

$$dy = -Y^0 dP^0 = +Y' dP' = -Y'' dP'' = +Y''' dP''' \text{ etc.} \quad (28)$$

(↓) where dP^* means the various P , not the derivative, and the sign changes superior or inferior, according to that the line μP^* from the center μ takes interior or exterior of the space S . (↑) If, in addition to, we consider the infinitely many planes, rectangles to the coordinate axis x , of which the element dx between x^0 and x' , or between x'' and x''' , or etc., it corresponds to the element : ²⁴

$$dU = \frac{dx \cdot dy}{\zeta}, \quad (29)$$

(↓) Namely, this correspondence comes from (25)

$$\begin{aligned} \int \delta dU &= \int \left[dU (\eta^2 + \zeta^2) \frac{d\delta x}{dx} - \frac{d\delta y}{dx} \xi \eta - \frac{d\delta z}{dx} \xi \zeta \right] + \int dU \left[(\xi^2 + \zeta^2) \frac{d\delta y}{dy} - \frac{d\delta x}{dy} \xi \eta - \frac{d\delta z}{dy} \eta \zeta \right] \\ &= dy \int dx \frac{1}{\zeta} \left[(\eta^2 + \zeta^2) \cdot \frac{d\delta x}{dx} - \frac{d\delta y}{dx} \xi \eta - \frac{d\delta z}{dx} \xi \zeta \right] + dx \int dy \frac{1}{\zeta} \left[(\xi^2 + \zeta^2) \cdot \frac{d\delta y}{dy} - \frac{d\delta x}{dy} \xi \eta - \frac{d\delta z}{dy} \eta \zeta \right] \end{aligned}$$

(↑)

Therefore, from here, it is clear for a part of integral by part : A , that corresponds to the part of the surface depending on between the interval : $y, y + dy$, to have by the following integral, i.e., substituting the right hand-side of (29) into A of (26), then $A = dy \int dx \left(\frac{\eta^2 + \zeta^2}{\zeta} \cdot \frac{d\delta x}{dx} - \frac{\xi \eta}{\zeta} \cdot \frac{d\delta y}{dx} - \xi \delta z \right)$, extending from $x = x^0$ to $x = x'$, next, from $x = x''$ to $x = x'''$ etc. In fact, the limit of this integral by part is expressed as follows :

$$A = \left(\frac{\eta^2 + \zeta^2}{\zeta} \delta x - \frac{\xi \zeta}{\zeta} \delta y - \xi \delta z \right) dy - dy \int \left(\delta x \frac{\eta^2 + \zeta^2}{\zeta} - \delta y \frac{d\xi \eta}{\zeta} - \delta z \frac{d\xi}{dx} \right) dx \quad (30)$$

Here, we construct A using (28) and (29), then

$$\begin{aligned} &\left(\frac{\eta^{02} + \zeta^{02}}{\zeta^0} \delta x^0 - \frac{\xi^0 \eta^0}{\zeta^0} \delta y^0 - \xi^0 \delta z^0 \right) Y^0 dP^0 + \left(\frac{\eta'^2 + \zeta'^2}{\zeta'} \delta x' - \frac{\xi' \eta'}{\zeta'} \delta y' - \xi' \delta z' \right) Y' dP' \\ &+ \left(\frac{\eta''^2 + \zeta''^2}{\zeta''} \delta x'' - \frac{\xi'' \eta''}{\zeta''} \delta y'' - \xi'' \delta z'' \right) Y'' dP'' + \text{etc.} - \int \zeta dU \left(\delta x \frac{\eta^2 + \zeta^2}{\zeta} - \delta y \frac{d\xi \eta}{\zeta} - \delta z \frac{d\xi}{dx} \right) \end{aligned}$$

²⁴(↓) In fact, comparerig the two expressions : (26) with (30) and (27) with (30), then this correspondence deduced.

or in sum,

$$\sum \left(\frac{\eta^2 + \zeta^2}{\zeta} \delta x - \frac{\xi \eta}{\zeta} \delta y - \xi \delta z \right) Y dP - \int \zeta dU \left(\delta x \frac{\eta^2 + \zeta^2}{\zeta} - \delta y \frac{d\xi \eta}{\zeta} - \delta z \frac{d\xi}{\zeta} \right)$$

This total quantity A is expressed by

$$A = \int \left(\frac{\eta^2 + \zeta^2}{\zeta} \delta x - \frac{\xi \eta}{\zeta} \delta y - \xi \delta z \right) Y dP - \int \zeta dU \left(\delta x \frac{\eta^2 + \zeta^2}{\zeta} - \delta y \frac{d\xi \eta}{\zeta} - \delta z \frac{d\xi}{\zeta} \right)$$

where, the first integral is extended to all the circumference of P , and the second is extended to all the surface of U .

4.11. Analytic reduction of δU to Q and V via A and B .

By calculation from (26) as the same as (27), we get B similarly and immediately

$$A = \int \left(\frac{\eta^2 + \zeta^2}{\zeta} \delta x - \frac{\xi \eta}{\zeta} \delta y - \xi \delta z \right) Y dP - \int \zeta dU \left(\delta x \frac{\eta^2 + \zeta^2}{\zeta} - \delta y \frac{d\xi \eta}{\zeta} - \delta z \frac{d\xi}{\zeta} \right) \quad (31)$$

$$B = \int \left(\frac{\xi \eta}{\zeta} \delta x - \frac{\xi^2 + \zeta^2}{\zeta} \delta y - \eta \delta z \right) X dP + \int \zeta dU \left(\delta x \frac{\xi \eta}{\zeta} - \delta y \frac{d\xi^2 + \zeta^2}{\zeta} + \delta z \frac{d\eta}{\zeta} \right) \quad (32)$$

Here we determine for all the circumference P , we get ζQ from the first terms of both (31) and (32),

$$\left(\frac{\xi \eta}{\zeta} \delta x - \frac{\xi^2 + \zeta^2}{\zeta} \delta y - \eta \delta z \right) X + \left(\frac{\eta^2 + \zeta^2}{\zeta} \delta x - \frac{\xi \eta}{\zeta} \delta y - \xi \delta z \right) Y \equiv Q,$$

$$\left[X \xi \eta + Y (\eta^2 + \zeta^2) \right] \delta x - \left[X (\xi^2 + \zeta^2) + Y \xi \eta \right] \delta y + (X \eta \zeta - Y \xi \zeta) \delta z = \zeta Q$$

Moreover, for every point of the surface U , we get V from the second terms of both (31) and (32),

$$\left(\frac{d\xi \eta}{\zeta} - \frac{d\eta^2 + \zeta^2}{\zeta} \right) \zeta \delta x + \left(\frac{d\xi \eta}{\zeta} - \frac{d\xi^2 + \zeta^2}{\zeta} \right) \zeta \delta y + \left(\frac{d\xi}{\zeta} + \frac{d\eta}{\zeta} \right) \zeta \delta z \equiv V \quad (33)$$

That is, we can put

$$\delta U = \int Q dP + \int V dU \quad (34)$$

The first integral is to be extended along all the circumference P , and the second is on all surface U .

(\Downarrow) This is the what is called *Gaussian integral formula* in two dimensions.

4.12. Geometric reduction of Q and V .

Formula for Q and V notably contradict $X\xi + Y\eta + Z\zeta = 0$, Q has always the symmetric form as follows :

$$Q = (Y\zeta - Z\eta)\delta x + (Z\xi - X\zeta)\delta y + (X\eta - Y\xi)\delta z \quad \Rightarrow^* \quad Q = \begin{vmatrix} \delta x & \delta y & \delta z \\ X & Y & Z \\ \xi & \eta & \zeta \end{vmatrix} \quad (35)$$

When we see the form of V , we can reduce from the formulae (15)

$$\frac{dz}{dx} = -\frac{\xi}{\zeta}, \quad \frac{dz}{dy} = -\frac{\eta}{\zeta}, \quad \Rightarrow \quad \frac{d\xi}{\zeta} = \frac{d\eta}{\zeta} \quad (36)$$

therefore, $\frac{d\xi \eta}{\zeta} = \frac{\xi}{\zeta} \cdot \frac{d\eta}{\zeta} + \eta \frac{d\xi}{\zeta} = \frac{\xi}{\zeta} \cdot \frac{d\eta}{\zeta} + \eta \frac{d\xi}{\zeta}$. Moreover, for $\xi^2 + \eta^2 + \zeta^2 = 1$, we can deduce

$$\xi \frac{d\xi}{\zeta} + \eta \frac{d\eta}{\zeta} + \zeta \frac{d\zeta}{\zeta} = 0 \quad (37)$$

by dividing the both side of hand of (37) with ζ ,

$$\frac{\xi}{\zeta} \frac{d\xi}{\zeta} = -\left(\frac{\eta}{\zeta} \frac{d\eta}{\zeta} + \frac{d\zeta}{\zeta} \right), \quad \frac{d\eta^2 + \zeta^2}{\zeta} = \eta \frac{d\eta}{\zeta} + \left(\frac{\eta}{\zeta} \cdot \frac{d\eta}{\zeta} + \frac{d\zeta}{\zeta} \right) = \eta \frac{d\eta}{\zeta} - \frac{\xi}{\zeta} \frac{d\xi}{\zeta} \quad (38)$$

We may replace the coefficient of $\zeta\delta x$ in V of (33), using (36) and (38),

$$\frac{d\frac{\xi\eta}{\zeta}}{dy} - \frac{d\frac{\eta^2+\zeta^2}{\zeta}}{dx} = \frac{d\frac{\xi\eta}{\zeta}}{dy} - \eta\frac{d\eta}{dx} + \frac{\xi}{\zeta}\frac{d\xi}{dx} = \left(\frac{\xi}{\zeta}\frac{d\eta}{dy} + \eta\frac{d\xi}{dy}\right) - \eta\frac{d\xi}{dy} + \frac{\xi}{\zeta}\frac{d\xi}{dx} = \frac{\xi}{\zeta}\left(\frac{d\xi}{dx} + \frac{d\eta}{dy}\right)$$

Samely for $\zeta\delta y$, $\frac{d\frac{\xi\eta}{\zeta}}{dx} - \frac{d\frac{\xi^2+\zeta^2}{\zeta}}{dy} = \frac{\eta}{\zeta}\left(\frac{d\xi}{dx} + \frac{d\eta}{dy}\right)$. Then V of (33) is reduced as follows :

$$V = (\xi\delta x + \eta\delta y + \zeta\delta z)\left(\frac{d\xi}{dx} + \frac{d\eta}{dy}\right).$$

4.13. Geometric meaning of $\frac{d\xi}{dx} + \frac{d\eta}{dy}$ in V .

Before going forward, we must illustrate conveniently the important geometrical expression. Here we restrict the various direction, we would like to present the following its intuitionally facile method, which we introduced in *Disquisitiones generales circa superficies curvas*. We consider the following layout of structure.

- At first, we put the sphere, of which the radius = 1 at the center of an arbitrary surface, we denote the axis of the coordinates x, y and z by the points (1), (2) and (3),
- next, taking exterior domain denoted by s , we number a point denoting by the point (4) toward the normal direction on surface ;
- then, at an arbitrary point on surface, drawing various rectangle direction toward point of itself, which we denote by the point (5),
- finally, the variation of itself, we suppose that the quantity $\sqrt{\delta x^2 + \delta y^2 + \delta z^2}$ is always positive, and we denote the quantity by δe for brevity, then $\delta x = \delta e \cdot \cos(1, 5)$, $\delta y = \delta e \cdot \cos(2, 5)$, $\delta z = \delta e \cdot \cos(3, 5)$,

(↓) By the way, for understanding Gauss' method of description of angle, we can show the same method by Lagrange in 1788.(↑↑)

Here, we consider the every point on the surface. In this boundary, if we call the periphery P , we can consider the two directions. (↓) (Remark. (•) is a unique point naming, and (•, •) means the angle between two points taking an intermediate of the origin.) (↑)

- At first, we denote the corresponding point to dP by the point (6),
- next, we draw the rectangle direction to the surface, which is the inner normally-directed tangential to the surface, then we denote the point by (7),
- then, by the hypothesis, these points (6), (7) and (4) look toward the same direction ,²⁵ using above-mentioned (1), (2) and (3) then (4, 6), (4, 7) and (6, 7) make a cube,²⁶ when we consider each angle as the rectangle.

Thus, the equations (16) in the above-mentioned (§4.8) transform into

$$\eta Z - \zeta Y = \cos(1, 7), \quad \zeta X - \xi Z = \cos(2, 7), \quad \xi Y - \eta X = \cos(3, 7)$$

The formulae in the previous article take forms as follows :

$$Q = -\delta e \cdot \cos(5, 7), \quad V = \delta e \cdot \cos(4, 5) \cdot \left(\frac{d\xi}{dx} + \frac{d\eta}{dy}\right) \tag{39}$$

Here

- Q expresses the translation of this point along the periphery P , to which a plane of tangential surface U , taking as normal in the domain, positive to the opposite direction ;
- the factor V is, like $\cos(4, 5)$ clearly indicates, the translation of this point on the surface U , taking as positive in the domain of the exterior space s .

Here we may summarize Q and V in $\delta U = \int QdP + \int VdU$ by the two methods between analytic and geometric in Table 4. We may explain by replacing $\frac{d\xi}{dx} + \frac{d\eta}{dy}$ in V of (39), from the point of view in geometric meaning. In such case, it turns namely as follows : from (15), taking derivative in both side of hand of (15)

$$-2\zeta^{-3} = 2\frac{dz}{dx} \cdot \frac{d\frac{dz}{dx}}{d\zeta} + 2\frac{dz}{dy} \cdot \frac{d\frac{dz}{dy}}{d\zeta} \Rightarrow 1 = -\zeta\frac{dz}{dx} \cdot \zeta^2\frac{d\frac{dz}{dx}}{d\zeta} - \zeta\frac{dz}{dy} \cdot \zeta^2\frac{d\frac{dz}{dy}}{d\zeta} \tag{40}$$

²⁵This image is considered that there are three directions emitting from a common point and making a certain angle with two directions (i.e. points.)

²⁶(4, 6), (4, 7) and (6, 7) make a plane consisting of a cube respectively.

TABLE 4. Comparison of Q and V in $\delta U = \int QdP + \int VdU$ between analytic and geometric method

| no | value | analytic method | geometric method |
|----|-------|---|--|
| 1 | Q | $Q = \left(\frac{\xi\eta}{\zeta}\delta x - \frac{\xi^2+\zeta^2}{\zeta}\delta y - \eta\delta z\right)X + \left(\frac{\eta^2+\zeta^2}{\zeta}\delta x - \frac{\xi\eta}{\zeta}\delta y - \xi\delta z\right)Y$ | $Q = -\delta e \cdot \cos(5, 7)$ |
| 2 | V | $V = \left(\frac{d\xi\eta}{dy} - \frac{d(\eta^2+\zeta^2)}{dx}\right)\zeta\delta x + \left(\frac{d\xi\eta}{dx} - \frac{d(\xi^2+\zeta^2)}{dy}\right)\zeta\delta y + \left(\frac{d\xi}{dx} + \frac{d\eta}{dy}\right)\zeta\delta z$ | $V = \delta e \cdot \cos(4, 5) \cdot \left(\frac{d\xi}{dx} + \frac{d\eta}{dy}\right)$ $= \delta e \cdot \cos(4, 5) \cdot \left(\frac{1}{R} + \frac{1}{R'}\right)$ |

and finally we get the following expression after replacing (40) with ξ and η

$$d\zeta = \xi\zeta^2 d\frac{dz}{dx} + \eta\zeta^2 d\frac{dz}{dy} \tag{41}$$

Using (41), $\frac{d\xi}{dx} = -\zeta(\eta^2 + \zeta^2)\frac{d^2z}{dx^2} + \xi\eta\zeta\frac{d^2z}{dx \cdot dy} = \frac{1}{R}$, $\frac{d\eta}{dy} = -\zeta(\xi^2 + \zeta^2)\frac{d^2z}{dy^2} + \xi\eta\zeta\frac{d^2z}{dx \cdot dy} = \frac{1}{R'}$

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} = \frac{1}{R} + \frac{1}{R'} = -\zeta^3 \left[\frac{d^2z}{dx^2} \left\{ 1 + \left(\frac{dz}{dy}\right)^2 \right\} - \frac{2d^2z}{dx \cdot dy} \cdot \frac{dz}{dx} \cdot \frac{dz}{dy} + \frac{d^2z}{dy^2} \left\{ 1 + \left(\frac{dz}{dx}\right)^2 \right\} \right],$$

where, $\zeta^3 = \left[1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2 \right]^{-\frac{3}{2}}$ and R and R' are the radii of curvature respectively. (42)

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4.14. Geometrical method. Deducing the parts of Q .

From (34), (39) and (42)

$$(I) \delta U = \int QdP + \int VdU = - \int \delta e \cdot \cos(5, 7) \cdot dP + \int \delta e \cdot \cos(4, 5) \cdot \left(\frac{1}{R} + \frac{1}{R'}\right) dU. \tag{43}$$

Evolving further the variation, for the expression W is explained by the variation of figure of the space s , we would like to start to argue at first, from the variation of the space s . Recalling that we consider in §4.9, the prism with the equal sides and oriented to the solid body, then, on this point, we can see that this prism has the followings : ●(1) the size of basement : dU , ●(2) the height : $\xi\delta x + \eta\delta y + \zeta\delta z = \delta e \cdot \cos(4, 5)$, where $\delta e = \sqrt{\delta x^2 + \delta y^2 + \delta z^2}$, ●(3) the sign (\pm) of height depends on transposing triangle, according to the location of whole solid lying whether in terior or exterior of the space s . Hence, we can get (II) $\delta s = \int dU \cdot \delta e \cdot \cos(4, 5)$. Next, from (II), the variation of $\int zds$ (III) follows : (III) $\delta \int zds = \int zdU \cdot \delta e \cdot \cos(4, 5)$.

As long as the variational quantity T , we can see that P is the limit point having commonly the surface T and U , the transpositional point of the circumference P satisfies owing to these condition, and newly keeps in the surface space S . By the transpositional element dP , as the partial displacement of the surface T , we get easily $\pm dP \cdot \delta e \cdot \sin(5, 6)$. In general, the choice of positive or negative sign depends on the sign of $\cos(4, 5)$. We would like to explain it by introducing the new directions such that : ●(1) the space S tangential in the surface plane, ●(2) the normal-directional line P , and ●(3) the exterior space s , respectively. If denoting the responding direction with the point (8), then by the transpositioning element dP , we get the surface variation of T , from the definition, as $dP \cdot \delta e \cdot \cos(5, 8)$, namely (IV) $\delta T = \int dP \cdot \delta e \cdot \cos(5, 8)$, where, the sign of factor $\cos(5, 8)$ depends on the conditions of whether increment or decrement. When we assume that :

- at first, the point (6) were the pole of the maximum circle passing through the two points : (7) and (8), then the point (5) is the highest point in the circle made by the two points (6) and (8) ;
- next, the points (5). (7) and (8) make a rectangular triangle, having the rectangle at the point (8) ;
- then, we can get the expression : $\cos(5, 7) = \cos(5, 8) \cdot \cos(7, 8)$, where, the arc (7, 8) is the measure of angle between planes of the two surface spaces : s and S , which are tangential intersecting with the point P and the plane domain, including null space ;
- finally, we denote the angle making with (7, 8) by i , i.e. $i = (7, 8)$ and by $2\pi - i$, the angle between plane domain, in which the space s is continue.

²⁷(4) cf. Laplace [13, 14] had deduced his same expression with Gauss' (42). cf. Poisson [26], p.105.

Then we can formulate (V) as follows : (V) $\cos(5, 7) = \cos(5, 8) \cdot \cos i$.

4.15. Result.1 : deduction of height from the first fundamental theorem.

By the combination of above formulae I, \dots , IV, we get the variational expression of W , where, W is the value of (14).

$$\delta W = \int dU \cdot \delta e \cdot \cos(4, 5) \cdot \left[z + \alpha^2 \left(\frac{1}{R} + \frac{1}{R'} \right) \right] - \int dP \cdot \delta e \cdot \cos(5, 8) \cdot (\alpha^2 \cos i - \alpha^2 + 2\beta^2)$$

$$\text{where, } z + \alpha^2 \left(\frac{1}{R} + \frac{1}{R'} \right) = \text{Const.} \quad \text{If we set Const} = 0, \text{ then } z = -\alpha^2 \left(\frac{1}{R} + \frac{1}{R'} \right).$$

and, z is the height of capillary action, α and β are the values defined in (13).

4.16. Result.2 : deduction of angle from the second fundamental theorem.

$$\delta W = - \int dP \cdot \delta e \cdot \cos(5, 8) \cdot (\alpha^2 \cos i - \alpha^2 + 2\beta^2) = \alpha^2 \int dP \cdot \delta e \cdot \cos(5, 8) \cdot \left(1 - 2 \left(\frac{\beta}{\alpha} \right)^2 - \cos i \right)$$

Here, we assume A such that $\cos A = 1 - 2 \sin^2 \left(\frac{A}{2} \right) = 1 - 2 \frac{\beta^2}{\alpha^2}$. If $\sin \frac{A}{2} = \frac{\beta}{\alpha}$, then

$$\delta W = \alpha^2 \int dP \cdot \delta e \cdot \cos(5, 8) \cdot (\cos A - \cos i),$$

where, the integral is to be extended along the total line P . Remember that the factor $\cos(5, 8)$ is equivalent with $\sin(5, 6)$,²⁸ and the sign becomes plus or minus, according to fluid in motion in the neighbourhood of element dP or moreover, it reaches to the end point of P , or it comes to disappear.

5. Conclusions

- (1) The "two-constants" were defined in terms of kernel functions of *RDFs*, describing the characteristics of dissipation or diffusion within isotropic and homogeneous fluids that were necessary for the interpretation of the nature of fluid or the formulation of the equations of the fluid mechanics including kinetics, equilibrium and capillarity. With their origin perhaps arising in the work of Laplace in 1805, these sorts of functions are simple examples of today's distributions and hypergeometric function of Schwarz proposed in 1945.
- (2) Gauss [8] also contributed to develop fundamental conception of *RDF* or *MDNS* equations for fluid mechanics including capillary action, because he formulated the equations with two-functions instead of two-constants and these were the superior method from other contemporaries with the progenitors of *NS* equations.
- (3) According to Bolza [2], Gauss [8] had broken one of the neck of fundamental problems, such as *multiple integral* and *calculus of variations*, however, we must recognize that even he owed the latter to its progenitor Lagrange, and calculation of capillarity to its progenitor Laplace.

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²⁸i.e. $\cos(5, 8) = \sin(5, 6)$, where the point (8) is the point of rectangle, the points (6), (8) and (5) make a straight line in the direction from left to right.

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