Remarks on a paper of Zhang and Sun

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ABSTRACT

We are interested in the existence of positive solutions for second order boundary value problem: \((E) \quad y'' + h(t)f(y) = 0, \quad 0 < t < 1\), subject to multi-point boundary conditions. We prove an extension of a recent result by Zhang and Sun [3] and illustrate with examples.

1. Introduction

We are interested in the existence of positive solutions for the second order nonlinear differential equation

\[ y'' + h(t)f(y) = 0, \quad 0 \leq t \leq 1, \quad (1.1) \]

where \( h(t) \in L^{1}(0,1) \) and \( f(y) \in C(\mathbb{R}, \mathbb{R}_{+}) \) are non-negative functions subject to multi-point boundary condition

\[ y(0) = \langle \alpha, y(\xi) \rangle = \sum_{i=1}^{m} \alpha_{i}\xi_{i}, \quad y(1) = \langle \beta, y(\xi) \rangle = \sum_{i=1}^{m} \beta_{i}\xi_{i} \quad (1.2) \]
where $\alpha = (\alpha_1, \cdots, \alpha_m), \beta = (\beta_1, \cdots, \beta_m), \alpha_i, \beta_i$ real, and $0 < \xi_1 < \cdots < \xi_m < 1$. Denote by $(\alpha, y(\xi))$ the scalar product between $m$-vectors $\alpha$ and $y(\xi) = (y(\xi_1), \cdots, y(\xi_m))$. We assume that $\alpha_i, \beta_i \geq 0$ for $i = 1, \cdots, m$, and $h(t)$ may be singular at $t = 0$ or $t = 1$, or both.

In a recent paper [3], Zhang and Sun proved the following generalization of Krasnoselskî Cone Fixed Point Theorem:

**Theorem A** ([3], p.583, Corollary 2.1). Let $\Omega_1$ and $\Omega_2$ be two bounded open sets in a Banach space $X$ and $P \subseteq X$ be an ordered cone such that $\theta \in \overline{\Omega_1} \subseteq \Omega_2$. Suppose that $A : P \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow P$ is completely continuous and $\rho : P \rightarrow [0, \infty)$ is a uniformly continuous convex functional with $\rho(\theta) = 0$ and $\rho(x) > 0$, $x \neq \theta$. If one of the two conditions:

(a) (Expansion) For $x \in P \cap \partial \Omega_1$, $\rho(Ax) \leq \rho(x)$ and for $x \in P \cap \partial \Omega_2$, $\inf_{x \in \partial \Omega_2} \rho(x) > 0$, $\rho(Ax) \geq \rho(x)$; or

(b) (Compression) For $x \in P \cap \partial \Omega_2$, $\rho(Ax) \leq \rho(x)$ and for $x \in P \cap \partial \Omega_1$, $\inf_{x \in \partial \Omega_1} \rho(x) > 0$, $\rho(Ax) \geq \rho(x)$;

then $A$ has a fixed point $\hat{x} \in P \cap (\overline{\Omega_2} \setminus \Omega_1)$, i.e. $A\hat{x} = \hat{x}$.

Clearly if $\rho(x) = \|x\|$, where $\| \cdot \|$ denotes the norm of the Banach space $X$, then Theorem A reduces to the classical Krasnoselskî theorem, see Guo and Lakshmikantham [2]. As an application of Theorem A, Zhang and Sun proved an existence theorem for the multipoint boundary value problem (1.1), (1.2) where $\alpha_i \equiv 0$ for $i = 1, 2, \cdots, m$ in (1.2) subject to the assumptions:

$(H_1)$ \hspace{1em} $0 < \overline{\beta} = \sum_{i=1}^{m} \beta_i < 1$,

$(H_2)$ \hspace{1em} $h : (0, 1) \rightarrow [0, \infty)$ is continuous, $h \in L^1(0, 1)$ and $h(t) \not\equiv 0$ any subinterval in the open interval $(0, 1)$. 
Theorem B ([3], p.584, Theorem 3.1). Suppose that there exist positive constants $r, R$ and $\tau$ such that $0 < r < R$ and $\tau \in \left(0, \frac{1}{2}\right]$ satisfying one of the two conditions:

(a) (Expansion) $\quad R \geq \tau^{-2}(1-\tau)^{-2}r; \quad f(u) \leq \sigma_1^{-1}h_0^{-1}r$ for $0 \leq u \leq \tau^{-1}(1-\tau)^{-1}r$ and $f(y) \geq \sigma_2 h^{-1}_\tau R$ for $R \geq u \geq \tau(1-\tau)R$; or

(b) (Compression) $\quad R \geq \sigma_1 \sigma_2 h_0 h^{-1}_\tau r; \quad f(u) \leq \sigma_1^{-1}h_0^{-1}R$ for $0 \leq u \leq \tau^{-1}(1-\tau)^{-1}R$ and $f(u) \geq \sigma_2 h^{-1}_\tau$ for $R \geq u \geq \tau(1-\tau)r$, where

$$\sigma_1 = \frac{1}{4} + \frac{\beta}{1-\beta}, \quad \sigma_2 = \tau^{-2}(1-\tau)^{-1}, \quad h_0 = \int_0^1 h(t)dt \quad \text{and} \quad h_\tau = \int_\tau^{1-\tau} h(t)dt;$$

then the boundary value problem (1.1) with multipoint boundary condition $\bar{\alpha} = 0$ in (1.2), i.e.

$$y(0) = 0, \quad y(1) = \sum_{i=1}^m \beta_i y(\xi_i) = \langle\beta, y(\xi)\rangle$$  \hspace{1cm} (1.3)

has at least one positive solution.

The purpose of this note is to generalize Theorem B to cover the more general boundary condition (1.2). We obtain bounds on the nonlinear function $f(y)$ sharper than those given in Theorem B, and illustrate our results by examples.

2. Main Result

It is easy to verify that a solution to the boundary value problem (1.1), (1.2) is equivalent to the existence of a fixed point of the operator $A : P \to P$ defined by

$$Ay(t) = \int_0^1 K(t,s)h(s)f(y(s))ds$$  \hspace{1cm} (2.1)

where $P$ is the cone of non-negative functions in $C[0,1]$ and

$$K(t,s) = g(t,s) + \frac{t}{A} \left\{ (1-\bar{\alpha}\langle\alpha,\xi\rangle)\langle\beta, g(\xi,s)\rangle + (\bar{\beta} - \langle\beta,\xi\rangle)\langle\alpha, g(\xi,s)\rangle \right\}$$  \hspace{1cm} (2.2)
with $\overline{\alpha} = \sum_{i=1}^{m} \alpha_{i}, g(\xi, s) = (g(\xi_{1}, s), \cdots, g(\xi_{m}, s))$ and $\Lambda = (1 - \overline{\alpha})(1 - \langle \beta, \xi \rangle) + (1 - \overline{\beta})\langle \alpha, \beta \rangle$.

When $\overline{\alpha} = 0$ in (2.2), $K(t, s)$ reduces to that given in [3] for the simpler BVP (1.1), (1.3), whilst $g(t, s)$ is the usual Green’s function for the two point boundary value problem, i.e. (1.1), (1.2) in absence of all interior boundary points $\xi_{i}, i = 1, 2, \cdots, m$, and is given by

$$g(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1. \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases} \tag{2.3}$$

Note that $g(t, s) \leq s(1-s)$ for all $t, s \in [0, 1]$ and given $\tau \in (0, 1/2]$, $g(t, s) \geq \tau g(s, s)$ for $t \in [\tau, 1-\tau]$ and $s \in [0, 1]$. Since $\alpha_{i}, \beta_{i} \geq 0$ for $i = 1, \cdots, m$, it is easy to deduce the following estimates:

$$K(t, s) \geq g(t, s) \geq \tau g(s, s) \quad \tau \leq t \leq 1-\tau, \ 0 \leq s \leq 1 \tag{2.4}$$

$$K(t, s) \leq \nu g(s, s), \ \nu = \frac{1}{\Lambda} \{1 - \langle \beta, \xi \rangle + \langle \alpha, \xi \rangle + \max(\overline{\beta} - \overline{\alpha}, 0)\} \tag{2.5}$$

where $\Lambda = (1 - \overline{\alpha})(1 - \langle \beta, \xi \rangle) + \langle \alpha, \xi \rangle(1 - \overline{\beta})$.

We are now ready to state and prove our main result:

**Theorem 1** Suppose that there exist positive constants $r, R$ and $\tau$ such that $0 < r < R$ and $\tau \in (0, 1/2]$ satisfying one of the two conditions:

(a) *(Expansion)* $R \geq \tau^{-2}(1 - \tau)^{-2}r; f(u) \leq \nu^{-1}h_{1}^{-1}r$ for $0 \leq u \leq \tau^{-1}(1-\tau)^{-1}r$ and $f(u) \geq mR$ for $R \geq u \geq \tau(1-\tau)R$, where $h_{1} = \frac{1}{\tau} \int_{0}^{1} s(1-s)h(s)ds$; or

(b) *(Compression)* $R \geq m\nu h_{1}r; f(u) \leq \nu^{-1}h_{1}^{-1}R$ for $0 \leq u \leq \tau^{-1}(1-\tau)^{-1}R$ and $f(u) \geq mr$ for $r \geq u \geq \tau(1-\tau)r$;

where

$$m = \left(\tau \int_{\tau}^{1-\tau} s(1-s)h(s)ds \right)^{-1},$$

then the boundary value problem (1.1) (1.2) has at least one positive solution.

**Proof of Theorem 1** Let $P_{1} = \{y \in P : y(t) \text{ concave}, y(t) \geq t(1-t)\|y\|, 0 \leq t \leq 1\}$. We first prove that $A : P_{1} \rightarrow P_{1}$. Let $y \in P_{1}$. Note that $g(t, s) \geq t(1-t)s(1-s)$ for all
$t, s \in [0, 1]$. Now use (2.4), (2.5) in (2.1), we observe

$$Ay(t) = \int_{0}^{1} K(t, s) h(s) f(y(s)) ds$$

$$\geq t(1 - t) \int_{0}^{1} K(t_0, s) h(s) f(y(s)) ds$$

$$= t(1 - t) \nu^{-1} Ay(t_0) \quad \text{for any } t_0 \in [0, 1].$$

This shows $Ay(t) \geq t(1 - t) \|Ay\|$ proving $A : P_1 \rightarrow P_1$.

We only prove part (b) as part (a) is similar. Let $y \in P_1 \cap \partial B_R$ where $B_R = \{y \in P_1 : \rho(y) < R\}$ and $\partial B_R = \{y \in P_1 \cap \overline{B_R} : \rho(y) = R\}$. Observe by (2.5)

$$\rho(Ay) \leq \nu \int_{0}^{1} s(1 - s) h(s) f(y(s)) ds.$$  \hspace{1cm} (2.6)

Since $\rho(y) = R$ and $y \in P_1$ imply

$$R \geq y(\bar{t}) = \max_{\tau \leq t \leq 1 - \tau} y(t) \geq \nu_1^{-1} \bar{t}(1 - \bar{t}) \|y\|,$$

which in turn implies for all $s \in [0, 1]$

$$0 \leq y(s) \leq \|y\| \leq \nu_1 [\bar{t}(1 - \bar{t})]^{-1} R \leq \nu_1 \tau^{-1}(1 - \tau)^{-1} R.$$  \hspace{1cm} (2.7)

Now (2.7) implies by assumption (a) that $f(y(s)) \leq M, s \in [0, 1]$, which upon using this in (2.6), we find

$$\rho(Ay) \leq \nu \left( \int_{0}^{1} s(1 - s) h(s) ds \right) = \nu h_1 MR = R = \rho(y).$$

since $M = \nu^{-1} h_1^{-1}$.

Next let $y \in P_1 \cap \partial \Omega_r$ and $\rho(y) = \max_{\tau \leq t \leq 1 - \tau} y(t) = y(\bar{t})$ for some $\bar{t} \in [\tau, 1 - \tau]$. Since $y \in P_1$ so

$$r = \rho(y) \geq y(\bar{t}) \geq \bar{t}(1 - \bar{t}) \|y\| \geq \tau(1 - \tau) \|y\| \geq \tau(1 - \tau) \rho(y) = \tau(1 - \tau)r.$$
For $r \geq y(s) \geq \tau(1 - \tau)r$ we have by assumption (a) $f(y(s)) \geq mr$ for $s \in [\tau, 1 - \tau]$. Now by (2.4) and $g(s, s) = s(1 - s)$, we obtain
\[
\rho(Ay) \geq \int_0^1 K(t, s) h(s) f(y(s)) ds \geq \tau \left( \int_{\tau}^{1-\tau} h(s) ds \right) m r = r = \rho(y).
\]
since $m = \left( \tau \int_{\tau}^{1-\tau} h(s) ds \right)^{-1}$. This completes the proof.

**Remark 1.** When $\overline{\alpha} = 0$, $\nu = 1 + \overline{\beta}(1 - \langle \beta, \xi \rangle)^{-1}$ by (2.5). Note that $h_1 \leq \frac{1}{4} h_0$, so
\[
\nu^{-1} h_1^{-1} \geq \left( 1 + \frac{\overline{\beta}}{1 - \overline{\beta}} \right)^{-1} 4 h_0^{-1} \geq \sigma_1 h_0^{-1}.
\]
Also
\[
m = \left( \tau \int_{\tau}^{1-\tau} s(1 - s) h(s) ds \right)^{-1} \leq \tau^{-2} (1 - \tau)^{-1} h_\tau^{-1} = \sigma_2 h_\tau^{-1}.
\]
This shows that when Theorem 1 is applied to the boundary value problem (1.1), (1.3) studied in Zhang and Sun [3], we in fact can obtain sharper bounds on the nonlinear function $f(y)$.

3. Discussion

We discuss two examples given in [3; p.585, Example 3.1] for a special case of boundary value problem (1.1), (1.3):
\[
\begin{array}{l}
y'' + h(t)f(y(t)) = 0, \quad 0 < t < 1 \\
y(0) = 0, \quad y(1) = \frac{1}{2} y(\eta), \quad 0 < \eta < 1,
\end{array}
\]
where $h(t) = [t(1-t)]^{1/2} \in L_1(0,1)$. Two nonlinear functions are exhibited to illustrate Theorem B as follows:

(Expansion) $f_1(u) = \begin{cases} (3/20\pi) u & u \leq 16/3 \\ (4/35\pi)(57576u - 307065) & u > 16/3 \end{cases}$

with $r = 1, R = 30$;

(Compression) $f_2(u) = \begin{cases} (1024/3\pi) u & u \leq 3/16 \\ (8/7\pi)(16u + 61413) & u > 3/16 \end{cases}$
with \( r = 1, R = 90 \).

Using improved upper and lower bounds on \( f(u) \) as given in Theorem 1, we give the following alternative examples for the boundary value problem (3.1):

\[
\text{(Expansion)} \quad \hat{f}_1(u) = \begin{cases} 
(3/5\pi)u & u \leq 16/3 \\
(4/35\pi)(47376u - 252644) & u > 16/3
\end{cases}
\]

with \( r = 1, R = 30 \);

\[
\text{(Compression)} \quad \hat{f}_2(u) = \frac{7}{80}u + \frac{53}{\pi}, \quad u \geq 0
\]

with \( r = 1, R = 15 \).

**Remark 2.** We note that the example \( \hat{f}_1(u) \) differs only by small margin with \( f_1(u) \) but for the Compression part of Theorem 1, \( \hat{f}_2(u) \) is considerably simpler than \( f_2(u) \).

**Remark 3.** Both Theorem B and Theorem 1 impose a significant distance between constants \( r \) and \( R \). In another recent paper by Avery, Henderson and O’Regan [1], there is an example of a two point boundary value problem where it is only required that \( 0 < r < R \). In other words, \( R \) can be as close to \( r \) as one pleases. No example is known for multipoint boundary value problem when only \( 0 < r < R \) is assumed even for the three point problem.

**REFERENCES**

