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Oscillation theorems for second-order nonlinear difference equations of Euler type

大阪府立大学・大学院工学研究科 山岡直人 (Naoto Yamaoka)
Department of Mathematical Sciences,
Osaka Prefecture University

We consider the second-order nonlinear difference equation

$$\Delta^2 x(n) + \frac{1}{n(n+1)} f(x(n)) = 0, \qquad n \in \mathbb{N} \tag{1}$$

where $f(x)$ is a real valued continuous function satisfying

$$xf(x) > 0 \qquad \text{if } x \neq 0. \tag{2}$$

Here the forward difference operator Δ is defined as $\Delta x(n) = x(n+1) - x(n)$ and $\Delta^2 x(n) = \Delta(\Delta x(n))$.

A nontrivial solution $x(n)$ is said to be *oscillatory* if for every positive integer N there exists $n \geq N$ such that $x(n)x(n+1) \leq 0$. Otherwise it is said to be *non-oscillatory*. In other words, a solution $x(n)$ is non-oscillatory if it is either eventually positive or eventually negative.

Since equation (1) is one of the discrete equation of the differential equation

$$x'' + \frac{1}{t^2} f(x) = 0, \qquad ' = \frac{d}{dt}, \tag{3}$$

the oscillation problem for equation (3) plays an important role in the oscillation of solutions of equation (1). Over the past a decade, a great deal of effort has been devoted to the study of oscillation of solutions of equation (3). For example, those results can be found in [1–7]. In particular, Sugie and Kita [3] gave the following pair of an oscillation theorem and a non-oscillation theorem for equation (3).

Theorem A. *Assume* (2) *and suppose that there exists* λ *with* $\lambda > 1/4$ *such that*

$$\frac{f(x)}{x} \geq \frac{1}{4} + \frac{\lambda}{(\log x^2)^2} \tag{4}$$

for $|x|$ *sufficiently large. Then all non-trivial solutions of equation* (3) *are oscillatory.*

Theorem B. *Assume* (2) *and suppose that*

$$\frac{f(x)}{x} \leq \frac{1}{4} + \frac{1}{4(\log x^2)^2} \tag{5}$$

for $x > 0$ *or* $x < 0$, $|x|$ *sufficiently large. Then all non-trivial solutions of equation* (3) *are non-oscillatory.*

Remark 1. To discuss the oscillation problem for equation (3), Sugie and Kita assumed that $f(x)$ satisfies a suitable smoothness condition for the uniqueness of solutions of (3) to the initial value problem.

The purpose of this paper is to give an oscillation theorem for equation (1) corresponding to Theorem A. Our main result is stated as follows.

Theorem 1. *Assume* (2) *and suppose that there exists* λ *with* $\lambda > 1/4$ *such that* (4) *holds for* $|x|$ *sufficiently large. Then all non-trivial solutions of equation* (1) *are oscillatory.*

Judging from Theorem B, it seems reasonable to expect as follows.

Conjecture 1. *Assume* (2) *and suppose that* (5) *holds for* $x > 0$ *or* $x < 0$, $|x|$ *sufficiently large. Then all non-trivial solutions of equation* (1) *are non-oscillatory.*

To prove Theorem 1, we prepare some lemmas.

Lemma 1. *Assume* (2) *and suppose that equation* (1) *has a positive solution. Then the solution is increasing for* n *sufficiently large and it tends to* ∞ *as* $n \to \infty$.

Proof. Let $x(n)$ be a positive solution of equation (1). Then there exists $n_0 \in \mathbb{N}$ such that $x(n) > 0$ for $n \geq n_0$. Hence, by (2) we have

$$\Delta^2 x(n) = -\frac{1}{n(n+1)} f(x(n)) < 0 \tag{6}$$

for $n \geq n_0$.

We first show that $\Delta x(t) > 0$ for $n \geq n_0$. By way of contradiction, we suppose that there exists $n_1 \geq n_0$ such that $\Delta x(n_1) \leq 0$. Then, using (6), we have

$$\Delta x(n) < \Delta x(n_1) \leq 0$$

for $n > n_1$, and therefore, we can find $n_2 > n_1$ such that $\Delta x(n_2) < 0$. Using (6) again, we get

$$\Delta x(n) \leq \Delta x(n_2) < 0$$

for $n \geq n_2$. Hence we obtain

$$x(n) \leq \Delta x(n_2)(n - n_2) + x(n_2) \to -\infty$$

as $n \to \infty$, which is a contradiction to the assumption that $x(n)$ is positive for $n \geq n_0$. Thus, $x(n)$ is increasing for $n \geq n_0$.

We next suppose that $x(n)$ is bounded from above. Then there exists $L > 0$ such that $\lim_{n\to\infty} x(n) = L$. Since $f(x)$ is continuous on $\mathbb{R}$, we have $\lim_{n\to\infty} f(x(n)) = f(L)$, and therefore, there exists $n_3 \geq n_0$ such that

$$0 < \frac{f(L)}{2} < f(x(n))$$

for $n \geq n_3$. Hence, we have

$$\begin{aligned}\Delta x(m) =& \Delta x(n) + \sum_{j=m}^{n-1} \frac{1}{j(j+1)} f(x(j)) \\ >& \frac{f(L)}{2} \sum_{j=m}^{n-1} \frac{1}{j(j+1)} = \frac{f(L)}{2} \left(\frac{1}{m} - \frac{1}{n} \right)\end{aligned}$$

for $n > m \geq n_3$. Taking the limit of this inequality as $n \to \infty$, we get

$$\Delta x(m) \geq \frac{f(L)}{2m}$$

for $m \geq n_3$, and therefore, we obtain

$$x(m+1) \geq x(n_3) + \frac{f(L)}{2} \sum_{k=n_3}^{m} \frac{1}{k} \to \infty$$

as $m \to \infty$. This contradicts the assumption that $x(n)$ is bounded from above. Thus, we have $\lim_{n\to\infty} x(n) = \infty$. The proof is now complete. □

Lemma 2. *Suppose that the difference inequality*

$$\Delta w(n) + \frac{1}{n + w(n)} \left(w(n) - \frac{1}{2} \right)^2 \leq 0 \tag{7}$$

has a positive solution. Then the solution is nonincreasing and tends to $1/2$ *as* $n \to \infty$.

Proof. Let $w(n)$ be a positive solution of (7). Then there exists $n_0 \in \mathbb{N}$ such that $w(n) > 0$ for $n \geq n_0$. Hence, we see that $w(n)$ is nonincreasing because $w(n)$ satisfies

$$\Delta w(n) \leq -\frac{1}{n + w(n)} \left(w(n) - \frac{1}{2} \right)^2 \leq 0$$

for $n \geq n_0$. Thus, we can find $\alpha \geq 0$ such that $w(n) \searrow \alpha$ as $n \to \infty$. If $\alpha \neq 1/2$, then there exists $n_1 \geq n_0$ such that $|w(n) - 1/2| > |\alpha - 1/2|/2$ for $n \geq n_1$. Since $w(n)$ is nonincreasing, there exists $n_2 \geq n_1$ such that $w(n) < n$ for $n \geq n_2$. Hence, we have

$$\Delta w(n) \leq -\frac{1}{n + w(n)} \left(w(n) - \frac{1}{2} \right)^2 \leq -\frac{1}{2n} \left(\frac{\alpha - 1/2}{2} \right)^2$$

for $n \geq n_2$, and therefore, we get

$$w(n+1) - w(n_2) \leq -\frac{1}{2}\left(\frac{\alpha - 1/2}{2}\right)^2 \sum_{j=n_2}^{n} \frac{1}{j} \to -\infty$$

as $n \to \infty$. This is a contradiction to the assumption that $w(n)$ is positive for $n \geq n_0$. □

Lemma 3. *Suppose that $w(n)$ and $v(n)$ satisfy $w(n_0) = v(n_0)$,*

$$w(n+1) \leq F(n, w(n)) \quad \textit{and} \quad v(n+1) = F(n, v(n))$$

for $n \geq n_0$ where $F(n,x)$ is nondecreasing with respect to $x \in \mathbb{R}$ for each fixed n. Then

$$w(n) \leq v(n) \tag{8}$$

for $n \geq n_0$.

Proof. We use mathematical induction on n. Let $n = n_0$. Then it is clear that (8) holds. Assume that (8) holds for $n = n_1$. Since F is nondecreasing with respect to x for each fixed n, we have

$$w(n_1+1) \leq F(n_1, w(n_1)) \leq F(n_1, v(n_1)) = v(n_1+1).$$

Thus, we see that (8) holds for $n = n_1 + 1$. This completes the proof. □

We next consider the second-order linear difference equation

$$\Delta^2 x(n) + \frac{1}{n(n+1)}\left\{\frac{1}{4} + \frac{\lambda}{l(n)l(n+1)}\right\} x(n) = 0, \tag{9}$$

where $l(n)$ satisfies $\Delta l(n) = 2/(2n+1)$.

Proposition 1. *Equation* (9) *has the general solution*

$$x(n) = \begin{cases} K_1 \prod_{j=n_0}^{n-1}\left(1 + \frac{1}{2j} + \frac{z}{jl(j)}\right) + K_2 \prod_{j=n_0}^{n-1}\left(1 + \frac{1}{2j} + \frac{1-z}{jl(j)}\right) & \textit{if } \lambda \neq \frac{1}{4}, \\ K_3 \prod_{j=n_0}^{n-1}\left(1 + \frac{1}{2j} + \frac{1}{2jl(j)}\right) + K_4 \sum_{r=n_0}^{n-1} \prod_{j=r+1}^{n-1}\left(1 + \frac{1}{2j} + \frac{1}{2jl(j)}\right) & \\ \qquad \times \prod_{k=n_0}^{r}\left\{1 - \frac{1}{2k}\left(1 + \frac{1}{l(k)}\right)\right\} & \textit{if } \lambda = \frac{1}{4}, \end{cases}$$

where K_1, K_2, K_3, K_4 are arbitrary constants and z is the root of the characteristic equation

$$z^2 - z + \lambda = 0. \tag{10}$$

Proof. Let $x(n)$ be a solution of equation (9) satisfying

$$\Delta x(n) = \frac{1}{n}\left(\frac{1}{2} + \frac{z}{l(n)}\right) x(n).$$

Then we have

$$\begin{aligned}
\Delta^2 x(n) =& \Delta\left\{\frac{1}{n}\left(\frac{1}{2} + \frac{z}{l(n)}\right)\right\} x(n) + \frac{1}{n+1}\left(\frac{1}{2} + \frac{z}{l(n+1)}\right)\Delta x(n) \\
=& \left\{\Delta\left(\frac{1}{n}\right)\left(\frac{1}{2} + \frac{z}{l(n)}\right) + \frac{1}{n+1}\Delta\left(\frac{z}{l(n)}\right)\right\} x(n) \\
& + \frac{1}{n+1}\left(\frac{1}{2} + \frac{z}{l(n+1)}\right)\frac{1}{n}\left(\frac{1}{2} + \frac{z}{l(n)}\right) x(n) \\
=& \left\{-\frac{1}{n(n+1)}\left(\frac{1}{2} + \frac{z}{l(n)}\right) - \frac{z\Delta\, l(n)}{(n+1)l(n)l(n+1)}\right\} x(n) \\
& + \frac{1}{n(n+1)}\left(\frac{1}{4} + \frac{z}{2l(n)} + \frac{z}{2l(n+1)} + \frac{z^2}{l(n)l(n+1)}\right) x(n) \\
=& -\frac{1}{n(n+1)}\left\{\frac{1}{2} + \frac{z}{l(n)} + \frac{zn\Delta l(n)}{l(n)l(n+1)}\right. \\
& \left. - \left(\frac{1}{4} + \frac{z}{2l(n)} + \frac{z}{2l(n+1)} + \frac{z^2}{l(n)l(n+1)}\right)\right\} x(n) \\
=& -\frac{1}{n(n+1)}\left\{\frac{1}{4} + \frac{z}{2l(n)} - \frac{z}{2l(n+1)} + \frac{zn\Delta l(n) - z^2}{l(n)l(n+1)}\right\} x(n) \\
=& -\frac{1}{n(n+1)}\left\{\frac{1}{4} + \frac{\frac{z}{2}\Delta l(n)}{l(n)l(n+1)} + \frac{zn\Delta l(n) - z^2}{l(n)l(n+1)}\right\} x(n) \\
=& -\frac{1}{n(n+1)}\left\{\frac{1}{4} + \frac{\left(n+\frac{1}{2}\right)\Delta l(n) z - z^2}{l(n)l(n+1)}\right\} x(n) \\
=& -\frac{1}{n(n+1)}\left\{\frac{1}{4} + \frac{z - z^2}{l(n)l(n+1)}\right\} x(n),
\end{aligned}$$

and therefore, we obtain the characteristic equation (10). Hence, we see that

$$\phi(n) = \prod_{j=n_0}^{n-1}\left(1 + \frac{1}{2j} + \frac{z}{jl(j)}\right) \quad \text{and} \quad \psi(n) = \prod_{j=n_0}^{n-1}\left(1 + \frac{1}{2j} + \frac{1-z}{jl(j)}\right)$$

are solutions of equation (9). we also see that $\phi(n)$ and $\psi(n)$ are linearly independent if $\lambda \neq 1/4$.

Next, we consider the case that $\lambda = 1/4$. Then the characteristic equation (9) has the double root $1/2$. Let

$$u(n) = \Delta x(n) - \frac{1}{2n}\left(1 + \frac{1}{l(n)}\right) x(n). \tag{11}$$

Then $u(n)$ satisfies

$$\Delta u(n) = -\frac{1}{2(n+1)}\left(1 + \frac{1}{l(n+1)}\right) u(n),$$

and therefore, we have

$$
\begin{aligned}
u(n+1) &= \left\{1 - \frac{1}{2(n+1)}\left(1 + \frac{1}{l(n+1)}\right)\right\} u(n) \\
&= \prod_{k=n_0-1}^{n} \left\{1 - \frac{1}{2(k+1)}\left(1 + \frac{1}{l(k+1)}\right)\right\} u(n_0-1) \\
&= \prod_{k=n_0}^{n+1} \left\{1 - \frac{1}{2k}\left(1 + \frac{1}{l(k)}\right)\right\} u(n_0-1).
\end{aligned}
$$

Substituting $u(n)$ into (11), we obtain the first-order linear difference equation

$$
\Delta x(n) = \frac{1}{2n}\left(1 + \frac{1}{l(n)}\right) x(n) + \prod_{k=n_0}^{n} \left\{1 - \frac{1}{2k}\left(1 + \frac{1}{l(k)}\right)\right\} u(n_0-1),
$$

and therefore, we get

$$
\begin{aligned}
x(n) &= \prod_{j=n_0}^{n-1} \left(1 + \frac{1}{2j} + \frac{1}{2jl(j)}\right) x(n_0) \\
&\quad + \sum_{r=n_0}^{n-1} \prod_{j=r+1}^{n-1} \left(1 + \frac{1}{2j} + \frac{1}{2jl(j)}\right) \prod_{k=n_0}^{r} \left\{1 - \frac{1}{2k}\left(1 + \frac{1}{l(k)}\right)\right\} u(n_0-1).
\end{aligned}
$$

Thus, we conclude that

$$
\phi(n) = \prod_{j=n_0}^{n-1} \left(1 + \frac{1}{2j} + \frac{1}{2jl(j)}\right)
$$

and

$$
\psi(n) = \sum_{r=n_0}^{n-1} \prod_{j=r+1}^{n-1} \left(1 + \frac{1}{2j} + \frac{1}{2jl(j)}\right) \prod_{k=n_0}^{r} \left\{1 - \frac{1}{2k}\left(1 + \frac{1}{l(k)}\right)\right\}
$$

are solutions of (9). Moreover we see that $\phi(n)$ and $\psi(n)$ are linearly independent. $\square$

In case $\lambda > 1/4$, the characteristic equation (10) has conjugate roots

$$
z = \frac{1 \pm i\sqrt{4\lambda - 1}}{2}.
$$

Hence, by Euler's formula, the real solution of equation (9) can be written as

$$
x(n) = K_5 \left(\prod_{j=n_0}^{n-1} r(j)\right) \cos\left(\sum_{j=n_0}^{n-1} \theta(j)\right) + K_6 \left(\prod_{j=n_0}^{n-1} r(j)\right) \sin\left(\sum_{j=n_0}^{n-1} \theta(j)\right)
$$

where $r(j)$ and $\theta(j)$ satisfy $0 < \theta(j) < \pi/2$,

$$
r(n)\cos\theta(n) = 1 + \frac{1}{2n} + \frac{1}{2nl(n)} \quad \text{and} \quad r(n)\sin\theta(n) = \frac{\sqrt{4\lambda - 1}}{2nl(n)}
$$

for $n_0 \le j \le n-1$. Hence, together with Proposition 1, we have the following lemma.

Lemma 4. *Equation* (9) *can be classified into two types as follows.*

(i) *If* $\lambda > 1/4$, *then all non-trivial solutions of equation* (9) *are oscillatory.*

(ii) *If* $\lambda \leq 1/4$, *then all non-trivial solutions of equation* (9) *are non-oscillatory.*

We are now ready to prove our main theorem.

Proof of theorem 1. By way of contradiction, we suppose that equation (1) has a non-oscillatory solution $x(n)$. Then we may assume without loss of generality that $x(n)$ is eventually positive. Let R be a large number satisfying the assumption (4) for $|x| \geq R$. From Lemma 1, $x(n)$ is increasing and $\lim_{n\to\infty} x(n) = \infty$, and therefore, there exists $n_0 \in \mathbb{N}$ such that $x(n) \geq R$ and $\Delta x(n) > 0$ for $n \geq n_0$.

We define

$$w(n) = \frac{n\Delta x(n)}{x(n)}.$$

Then, using (4), we have

$$\begin{aligned}
\Delta w(n) =& \frac{\Delta(n\Delta x(n))x(n) - n(\Delta x(n))^2}{x(n)x(n+1)} \\
=& \frac{\Delta x(n) + (n+1)\Delta^2 x(n)}{x(n+1)} - n\frac{(\Delta x(n))^2}{x(n)x(n+1)} \\
=& \frac{\Delta x(n) - f(x(n))/n}{x(n)}\frac{x(n)}{x(n+1)} - \frac{1}{n}\left(n\frac{\Delta x(n)}{x(n)}\right)^2 \frac{x(n)}{x(n+1)} \\
=& \frac{1}{n}\left\{n\frac{\Delta x(n)}{x(n)} - \frac{f(x(n))}{x(n)} - \left(n\frac{\Delta x(n)}{x(n)}\right)^2\right\}\frac{x(n)}{x(n+1)} \\
\leq& \frac{1}{n}\left\{w(n) - \left(\frac{1}{4} + \frac{\lambda}{(\log x(n)^2)^2}\right) - w(n)^2\right\}\frac{x(n)}{x(n+1)} \\
=& -\frac{1}{n}\left\{\left(w(n) - \frac{1}{2}\right)^2 + \frac{\lambda}{(\log x(n)^2)^2}\right\}\frac{x(n)}{x(n+1)} \\
=& -\frac{1}{n+w(n)}\left\{\left(w(n) - \frac{1}{2}\right)^2 + \frac{\lambda}{(\log x(n)^2)^2}\right\}
\end{aligned}$$

for $n \geq n_0$. From Lemma 2, we see that $w(n) \searrow 1/2$ as $n \to \infty$, because $w(n)$ is positive and satisfies (7) for $n \geq n_0$.

Since $\lambda > 1/4$, we can find $\varepsilon_0 > 0$ such that

$$\frac{1}{4} < \frac{1}{4}(1 + 4\varepsilon_0)^2 < \lambda. \tag{12}$$

Then we see that there exists $n_1 > n_0$ such that

$$w(n) \leq \frac{1}{2} + \varepsilon_0$$

for $n \ge n_1$, and therefore, we have

$$x(n+1) \le \left\{1 + \left(\frac{1}{2} + \varepsilon_0\right)\frac{1}{n}\right\} x(n)$$

for $n \ge n_1$. Thus, we get

$$x(n) \le \prod_{j=n_1}^{n-1} \left\{1 + \left(\frac{1}{2} + \varepsilon_0\right)\frac{1}{j}\right\} x(n_1)$$

for $n > n_1$, and therefore, there exists $n_2 \ge n_1$ such that

$$\begin{aligned}
\log x(n) &\le \sum_{j=n_1}^{n-1} \log\left\{1 + \left(\frac{1}{2} + \varepsilon_0\right)\frac{1}{j}\right\} + \log x(n_1) \\
&\le \sum_{j=n_1}^{n-1} \left(\frac{1}{2} + \varepsilon_0\right)\frac{1}{j} + \log x(n_1) \\
&\le \frac{1+4\varepsilon_0}{2} l(n)
\end{aligned}$$

for $n \ge n_2$. Hence, we obtain

$$\Delta w(n) \le -\frac{1}{n+w(n)}\left\{\left(w(n) - \frac{1}{2}\right)^2 + \frac{\lambda}{(1+4\varepsilon_0)^2 l(n) l(n+1)}\right\}$$

for $n \ge n_2$, because $l(n) < l(n+1)$ for $n \ge n_2$.

Let $v(n)$ be a solution of the difference equation

$$\Delta v(n) = -\frac{1}{n+v(n)}\left\{\left(v(n) - \frac{1}{2}\right)^2 + \frac{\lambda}{(1+4\varepsilon_0)^2 l(n) l(n+1)}\right\}$$

satisfying the initial condition $w(n_2) = v(n_2)$. Then from Lemma 3, we obtain $0 < w(n) < v(n)$ for $n \ge n_2$. Letting

$$y(n) = \prod_{j=n_0}^{n-1}\left(1 + \frac{v(j)}{j}\right),$$

we can easily see that $y(n)$ is a positive solution of the difference equation

$$\Delta^2 y(n) + \frac{1}{n(n+1)}\left\{\frac{1}{4} + \frac{\lambda}{(1+4\varepsilon_0)^2 l(n) l(n+1)}\right\} y(n) = 0.$$

Hence, from Lemma 4, we have

$$\frac{\lambda}{(1+4\varepsilon_0)^2} \le \frac{1}{4},$$

which is a contradiction to (12). $\square$

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