Direct and inverse bifurcation problems for nonlinear Sturm-Liouville problems

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1 Introduction

We consider the nonlinear Sturm-Liouville problem

\begin{align}
-\ddot{u}(t) + f(u(t)) &= \lambda u(t), \quad t \in I := (0,1), \\
\dot{u}(t) &> 0, \quad t \in I, \\
\dot{u}(0) = u(1) &= 0,
\end{align}

where \( \lambda > 0 \) is a positive parameter. \( f(u) \) is assumed to satisfy the conditions (A.1) - (A.2):

(A.1) \( f(u) \) is \( C^1 \) for \( u \geq 0 \) satisfying \( f(u) > 0 \) for \( u > 0 \). Furthermore, \( f(0) = f'(0) = 0 \).

(A.2) \( f(u)/u \) is strictly increasing for \( u \geq 0 \). Moreover, \( f(u)/u \to \infty \) as \( u \to \infty \).

The following are the typical examples of \( f(u) \) which satisfy (A.1) and (A.2).

\begin{align}
(1.4) & \quad f(u) = u^p \quad (u \geq 0), \\
(1.5) & \quad f(u) = u^p + u^m \quad (u \geq 0), \\
(1.6) & \quad f(u) = u^p \left(1 - \frac{1}{1+u^2}\right) \quad (u \geq 0),
\end{align}

where \( p > m > 1 \) are constants.
Before stating our result, let us briefly recall some known facts (cf. [1]).

(a) For each given $\alpha > 0$, there exists a unique solution $(\lambda, u) = (\lambda_\alpha(\alpha), u_\alpha) \in \mathbb{R}_+ \times C^2(\bar{I})$ of (1.1)-(1.3) with $\|u_\alpha\|_q = \alpha$. Here, $\|u_\alpha\|_q$ is the $L^q$-norm of $u_\alpha$, and $\lambda_\alpha(\alpha)$ is called $L^q$-bifurcation curve.

(b) The set $\{(\lambda_\alpha(\alpha), u_\alpha) : \alpha > 0\}$ gives all solutions of (1.1)-(1.3) and is an unbounded curve of class $C^1$ in $\mathbb{R}_+ \times L^q(I)$ emanating from $(\pi^2, 0)$. Furthermore, $\lambda_\alpha(\alpha)$ is strictly increasing for $\alpha > 0$ and $\lambda_\alpha(\alpha) \to \infty$ as $\alpha \to \infty$.

The objective here is to discuss inverse bifurcation problems for nonlinear Sturm-Liouville problems from an asymptotic point of view.

The direct bifurcation problem, that is, for a given nonlinear term $f(u)$, the problem to investigate the local and global behavior of bifurcation curve has a long history and has been studied by many authors. We refer to [1-17] and the references therein. However, it seems that there exists a few works concerning inverse bifurcation problems. We only refer to [21].

Recently, the following basic result was obtained in [20].

**Theorem 1.0 ([20]).** Assume that $f_1(u)$ and $f_2(u)$ are unknown to satisfy (A.1) (A.2). Further, assume that the connected components of the set $V := \{u \geq 0 : f_1(u) = f_2(u)\}$ are locally finite. Let $\lambda_2(1, \alpha)$ and $\lambda_2(2, \alpha)$ be the $L^2$-bifurcation curves of (1.1)-(1.3) associated with the nonlinear term $f(u) = f_1(u)$ and $f(u) = f_2(u)$, respectively. Assume that $\lambda_2(1, \alpha) = \lambda_2(2, \alpha)$ for any $\alpha > 0$. Then $f_1(u) \equiv f_2(u)$ for $u \geq 0$.

Motivated by the result above, we here introduce an asymptotic approach to inverse bifurcation problem for (1.1)-(1.3). To be more precise, we assume that the nonlinear term $f(u)$ is unknown. Then we show that, if the asymptotic formula for the $L^q$-bifurcation curve $\lambda_q(\alpha)$ as $\alpha \to \infty$ is known precisely, then we are able to characterize the asymptotic property of $f(u)$ for $u \gg 1$. Here, $1 \leq q < \infty$ is a constant and we fix it throughout this paper. We call this idea **asymptotic approach for inverse bifurcation problems**.

As for the asymptotic behavior of $\lambda_q(\alpha)$ and $u_\alpha$ as $\alpha \to \infty$, it is known from [1] that

\[
\frac{u_\alpha(t)}{\|u_\alpha\|_\infty} \to 1
\]
locally uniformly on I as $\alpha \to \infty$. We set $g(u) := f(u)/u$. Then as $\alpha \to \infty$,

(1.8) \[ \lambda_q(\alpha) = g(\|u_\alpha\|_\infty) + \xi_\alpha, \]

where $\xi_\alpha = O(1)$ is the remainder term. By (1.7), we see that $\|u_\alpha\|_\infty = \alpha(1 + o(1))$ for $\alpha \gg 1$. By this and (1.8), for $\alpha \gg 1$,

(1.9) \[ \lambda_q(\alpha) = g(\alpha) + o(g(\alpha)). \]

Motivated by (1.9), as a direct problem, more precise asymptotic formula for $\lambda_q(\alpha)$ as $\alpha \to \infty$ has been given in [18].

**Theorem 1.1 ([18]).** Let $f(u) = u^p$, where $p > 1$ is a given constant. Then as $\alpha \to \infty$,

(1.10) \[ \lambda_q(\alpha) = \alpha^{p-1} + C_0 \alpha^{(p-1)/2} + C_1 + o(1), \]

where

\[
C_0 = \frac{2(p-1)}{q} C_2, \quad C_1 = \frac{2(p-1)}{q} C_2^2, \quad C_2 = \int_0^1 \frac{1 - s^q}{\sqrt{1 - s^2 - 2(1 - s^{p+1})/(p+1)}} ds.
\]

The formula (1.10) has been obtained first for $q = 2$ in [15] by using the relationship between $\lambda_2(\alpha)$ and the critical value associated with $\lambda_2(\alpha)$.

From a viewpoint of Theorems 1.1, we consider the following inverse problem.

**Problem 1.** Let $f(u)$ be unknown to satisfy (A.1) and (A.2). Assume that as $\alpha \to \infty$,

(1.11) \[ \lambda_q(\alpha) = g(\alpha) + Ag(\alpha)^{1/2} + O(1), \]

where $A > 0$ is a constant. Then can you conclude that $f(u) = u^p$ for some constant $p > 1$?

To state our results, we assume additional conditions (A.3) and (A.4). We put $f(u) := u^p h(u)$. 
(A.3) $h(u)$ is a $C^1$ function for $u > 0$, and there exists a constant $\delta_0 > 0$ such that $h(u) \geq \delta_0$ for $u > 0$. Furthermore, for an arbitrary fixed constant $0 < \epsilon \ll 1$, as $u \rightarrow \infty$,

\begin{align}
\max_{c \leq s \leq 1} \frac{|uh'(us)|}{h(u)} &= O((u^{p-1}h(u))^{-1/2}), \\
\max_{0 \leq s \leq \epsilon} s^p \frac{|uh'(us)|}{h(u)} &= O((u^{p-1}h(u))^{-1/2}).
\end{align}

(A.4) There exists a constant $0 < \delta_1 \ll 1$ such that for $(1 + \delta_1)v > u > v \gg 1$,

\begin{align}
f(u) &= f(v) + f'(v)(u-v) + O(f(v)/v^2)(u-v)^2.
\end{align}

The typical examples of $h(u)$ (i.e. $f(u)$) satisfying (A.3) and (A.4) are:

$h(u) = 1$ \quad ($f(u) = u^p$), \quad $h(u) = 1 + u^{m-p}$ \quad ($f(u) = u^p + u^m$, \quad $1 < m \leq \frac{p+1}{2}$).

The answer to Problem 1 is as follows.

**Theorem 1.2.** Assume that all conditions in Problem 1, (A.3) and (A.4) are satisfied. Then $f(u) = u^p h(u)$ with $p = 1 + (qA)/(2C_2)$ and $h(u) = D + d(u)$, where $C_2$ is a constant in Theorem 1.1, $A$ is a constant in (1.11), $D > 0$ is an arbitrary positive constant and $d(u) = O(u^{(1-p)/2})$ for $u \gg 1$.

**Remark 1.3.** (i) The next inverse bifurcation problem we consider in a near future should be to establish the asymptotic uniqueness of unknown $f(u)$ from the asymptotic behavior of $\lambda_\alpha(\alpha)$ as $\alpha \rightarrow \infty$.

(ii) The condition (A.3) is not technical one. Indeed, if we consider $f(u) = u^5 e^u$ and $q = 2$, then $g(u) = u^4 e^u$ does not satisfy (A.3), and we know from [16] that as $\alpha \rightarrow \infty$

\begin{align}
\lambda_2(\alpha) &= \alpha^4 e^{\alpha} + \frac{\pi}{4} \alpha^3 e^{\alpha/2} + \frac{\pi}{4} u^2 e^{\alpha/2}(1 + o(1)),
\end{align}

which is different from (1.11). Therefore, (1.11) does not hold without (A.3).

## 2 Sketch of the Proof of Theorem 1.2

In what follows, $C$ denotes various positive constants independent of $\alpha \gg 1$. We write $(\lambda, u_\alpha)$ for a unique solution pair of (1.1)--(1.3) with $\|u_\alpha\|_q = \alpha$. We begin with the fundamental
tools which play important roles in what follows. It is well known that

\begin{equation}
(2.1) \quad u_\alpha(t) = u_\alpha(1-t), \quad t \in I, \quad \|u_\alpha\|_{\infty} = u_\alpha \left( \frac{1}{2} \right),
\end{equation}

\begin{equation}
(2.2) \quad u'_\alpha(t) > 0, \quad 0 \leq t < \frac{1}{2}.
\end{equation}

Multiply (1.1) by \( u'_\alpha(t) \). Then

\[(u''_\alpha(t) + \lambda u_\alpha(t) - f(u_\alpha(t)))u'_\alpha(t) = 0.\]

This along with (2.1) implies that

\begin{equation}
(2.3) \quad \frac{1}{2}u'_\alpha(t)^2 + \frac{1}{2}\lambda u_\alpha(t)^2 - F(u_\alpha(t)) = \text{constant}
= \frac{1}{2}\lambda \|u_\alpha\|_{\infty}^2 - F(\|u_\alpha\|_{\infty}), \quad \text{(put \( t = 1/2 \))}
\end{equation}

where \( F(u) := \int_0^u f(s)ds \). We set

\begin{equation}
(2.4) \quad L_\alpha(\theta) = \lambda(\|u_\alpha\|_{\infty}^2 - \theta^2) - 2(F(\|u_\alpha\|_{\infty}) - F(\theta)).
\end{equation}

This along with (2.2) and (2.3) implies that for \( 0 \leq t \leq 1/2 \)

\begin{equation}
(2.5) \quad u'_\alpha(t) = \sqrt{L_\alpha(u_\alpha(t))}.
\end{equation}

By this and (2.1), we obtain

\begin{equation}
(2.6) \quad \|u_\alpha\|^q_{\infty} - \alpha^q = 2 \int_0^{1/2} \left( \|u_\alpha\|^q_{\infty} - u^q_\alpha(t)u'_\alpha(t) \right)dt = 2 \int_0^{\|u_\alpha\|_{\infty}} \left( \|u_\alpha\|^q_{\infty} - \theta^q \right) \frac{d\theta}{\sqrt{L_\alpha(\theta)}}
= \frac{2\|u_\alpha\|^q_{\infty}}{\sqrt{\lambda}} \int_0^1 \frac{1 - s^q}{\sqrt{B_\alpha(s)}}ds
= \frac{2\|u_\alpha\|^q_{\infty}}{\sqrt{\lambda}} \left\{ \int_0^1 \frac{1 - s^q}{\sqrt{J(s)}}ds + \int_0^1 \left( \frac{1 - s^q}{\sqrt{B_\alpha(s)}} - \frac{1 - s^q}{\sqrt{J(s)}} \right)ds \right\}
= \frac{2\|u_\alpha\|^q_{\infty}}{\sqrt{\lambda}} \left( C_2 + M_\alpha \right),
\end{equation}

where

\begin{equation}
(2.7) \quad J(s) := 1 - s^2 - \frac{2}{p+1}(1 - s^{p+1}),
\end{equation}

\begin{equation}
(2.8) \quad B_\alpha(s) := 1 - s^2 - \frac{2}{\lambda \|u_\alpha\|^2_{\infty}}(F(\|u_\alpha\|_{\infty}) - F(\|u_\alpha\|_{\infty}s)),
\end{equation}

\begin{equation}
(2.9) \quad M_\alpha := \int_0^1 \left( \frac{1 - s^q}{\sqrt{B_\alpha(s)}} \right)ds.
\end{equation}
Lemma 2.1. \( f'(\alpha) \leq C\alpha^{p-1} \) for \( \alpha \gg 1 \).

Lemma 2.1 is proved by direct calculation. So we omit the proof. By (A.3) and Lemma 2.1, for \( \alpha \gg 1 \),

\[
C^{-1}\alpha^{p-1} \leq \lambda \leq C\alpha^{p-1},
\]
\[
C^{-1}\alpha^{p} \leq f(\alpha) \leq C\alpha^{p},
\]
\[
C^{-1}\alpha^{p-1} \leq g(\alpha) \leq C\alpha^{p-1}.
\]

The following Lemma 2.2 plays essential roles to prove Theorem 1.2.

**Lemma 2.2.** \( M_{\alpha} = O(g(\alpha)^{-1/2}) \) as \( \alpha \rightarrow \infty \).

We tentatively accept this lemma and prove Theorem 1.2. Lemma 2.2 will be proved in Section 3.

**Proof of Theorem 1.2.** By Lemma 2.2 and Taylor expansion, for \( \alpha \gg 1 \),

\[
\|u_{\alpha}\|_{\infty} = \alpha \left(1 - \frac{2}{q\sqrt{\lambda}}(C_{2} + M_{\alpha})\right)^{-1/q}
\]
\[
= \alpha \left(1 + \frac{2}{q\sqrt{\lambda}}(C_{2} + M_{\alpha}) + \frac{2(q+1)}{q^{2}\lambda}(C_{2} + M_{\alpha})^{2}(1 + o(1))\right).
\]

By this, Lemmas 2.1 and 2.2,

\[
\lambda = \frac{f(\|u_{\alpha}\|_{\infty})}{\|u_{\alpha}\|_{\infty}} + \xi_{\alpha}
\]
\[
= \frac{1}{\alpha} \left(1 - \frac{2}{q\sqrt{\lambda}}(C_{2} + M_{\alpha}) + O(\alpha^{1-p})\right) \left(f(\alpha) + \frac{2\alpha}{q\sqrt{\lambda}} f'(\alpha)(C_{2} + M_{\alpha}) + O(\alpha)\right) + \xi_{\alpha}
\]
\[
= \frac{f(\alpha)}{\alpha} + \frac{2C_{2}}{q\sqrt{\lambda}} \left(f'(\alpha) - \frac{f(\alpha)}{\alpha}\right) + M_{\alpha} \frac{2C_{2}}{q\sqrt{\lambda}} \left(f'(\alpha) - \frac{f(\alpha)}{\alpha}\right) + O(1)
\]
\[
= \frac{f(\alpha)}{\alpha} + \frac{2C_{2}}{q} \left(f'(\alpha) - \frac{f(\alpha)}{\alpha}\right) \left(g(\alpha) + A g(\alpha)^{1/2} + O(1)\right)^{-1/2} + O(1)
\]
\[
= \frac{f(\alpha)}{\alpha} + \frac{2C_{2}}{q\sqrt{g(\alpha)}} \left(f'(\alpha) - \frac{f(\alpha)}{\alpha}\right) + O(1).
\]

This implies that for \( \alpha \gg 1 \)

\[
f'(\alpha) - r \frac{f(\alpha)}{\alpha} = O(\sqrt{g(\alpha)}),
\]
where \( r := 1 + (qA)/(2C_2) \). Then we solve (2.15) directly, and easily obtain that \( r = p \), and for \( \alpha \gg 1 \)

\[
(2.16) \quad f(\alpha) = D\alpha^p + O(\alpha^{(p+1)/2}),
\]

where \( D > 0 \) is an arbitrary constant. Thus the proof is complete. \( \blacksquare \)

3 Proof of Lemma 2.2.

In this section, we prove Lemma 2.2. Let an arbitrary \( 0 < \epsilon \ll 1 \) be fixed. For \( 0 \leq s \leq 1 \), we put

\[
K_\alpha(s) := J(s) - B_\alpha(s) = \frac{2}{\lambda\|u_\alpha\|_\infty^2} \{F(\|u_\alpha\|_\infty) - F(\|u_\alpha\|_\infty s)\} - \frac{2}{p+1} (1 - s^{p+1}).
\]

Then

\[
M_\alpha = \int_0^1 \frac{(1 - s^q)K_\alpha(s)}{\sqrt{J(s)}\sqrt{B_\alpha(s)}(\sqrt{J(s)} + \sqrt{B_\alpha(s)})} ds
\]

\[
= \int_{1-\epsilon}^1 \frac{(1 - s^q)K_\alpha(s)}{\sqrt{J(s)}\sqrt{B_\alpha(s)}(\sqrt{J(s)} + \sqrt{B_\alpha(s)})} ds + \int_0^\epsilon \frac{(1 - s^q)K_\alpha(s)}{\sqrt{J(s)}\sqrt{B_\alpha(s)}(\sqrt{J(s)} + \sqrt{B_\alpha(s)})} ds
\]

\[
= M_{1,\alpha} + M_{2,\alpha} + M_{3,\alpha}.
\]

Lemma 3.1. For \( \alpha \gg 1 \)

\[
|M_{1,\alpha}| = O(g(\|u_\alpha\|_\infty)^{-1/2}).
\]

Proof. By (3.1),

\[
\frac{K_\alpha'(s)}{2} = -\frac{f(\|u_\alpha\|_\infty s)}{\lambda\|u_\alpha\|_\infty} + s^p.
\]

This implies that

\[
\frac{K_\alpha'(1)}{2} = \frac{\xi_\alpha}{\lambda}.
\]
Since $f(u) = g(u)u$, for $1 - \epsilon \leq s \leq 1$, by Taylor expansion, we obtain

$$\frac{K''(s)}{2} = \frac{f'\left(\|u_\alpha\|_\infty s\right)}{\lambda} + ps^{p-1}$$

(3.6)$$= \frac{g'\left(\|u_\alpha\|_\infty s\right)\|u_\alpha\|_\infty s + g\left(\|u_\alpha\|_\infty s\right)}{g\left(\|u_\alpha\|_\infty\right)} + \xi_\alpha + ps^{p-1}$$

$$= \frac{g'\left(\|u_\alpha\|_\infty s\right)\|u_\alpha\|_\infty s + g\left(\|u_\alpha\|_\infty s\right)}{g\left(\|u_\alpha\|_\infty\right)} \left(1 - \frac{\xi_\alpha}{g\left(\|u_\alpha\|_\infty\right)} (1 + o(1))\right) + ps^{p-1}.$$  

We put

(3.7) $$H(s, u) = ps^{p-1} \frac{h(us)}{h(u)} + us^{p} \frac{h'(us)}{h(u)}.$$  

For $u \gg 1$,

(3.8) $$g'(u) = (p-1)u^{p-2}h(u) + u^{p-1}h'(u).$$  

By this and (3.6), we obtain

(3.9) $$\frac{K''(s_{1})}{2} = -H(s_{1}, \|u_\alpha\|_\infty) \left(1 - \frac{\xi_\alpha}{g\left(\|u_\alpha\|_\infty\right)} (1 + o(1))\right) + ps^{p-1}.$$  

$$= ps^{p-1} \left(1 - \frac{h\left(\|u_\alpha\|_\infty s_1\right)}{h\left(\|u_\alpha\|_\infty\right)}\right) - \|u_\alpha\|_\infty s_1 \frac{h'\left(\|u_\alpha\|_\infty s_1\right)}{h\left(\|u_\alpha\|_\infty\right)}$$

$$+ \frac{\xi_\alpha}{g\left(\|u_\alpha\|_\infty\right)} H(s_{1}, \|u_\alpha\|_\infty)(1 + o(1)).$$

By this and mean value theorem, for $1 - \epsilon < s < s_1 < s_2 < 1$, we obtain

(3.10) $$\frac{K''(s_1)}{2} = ps^{p-1} \left(1 - \frac{h\left(\|u_\alpha\|_\infty s_1\right)}{h\left(\|u_\alpha\|_\infty\right)}\right) - \|u_\alpha\|_\infty s_1 \frac{h'\left(\|u_\alpha\|_\infty s_1\right)}{h\left(\|u_\alpha\|_\infty\right)}$$

$$+ \frac{\xi_\alpha}{g\left(\|u_\alpha\|_\infty\right)} H(s_{1}, \|u_\alpha\|_\infty)(1 + o(1))$$

$$= ps^{p-1} \left(\frac{h'\left(\|u_\alpha\|_\infty s_2\right)}{h\left(\|u_\alpha\|_\infty\right)}\right) \|u_\alpha\|_\infty (1 - s_1) - \|u_\alpha\|_\infty s_1 \frac{h'\left(\|u_\alpha\|_\infty s_1\right)}{h\left(\|u_\alpha\|_\infty\right)}$$

$$+ \frac{\xi_\alpha}{g\left(\|u_\alpha\|_\infty\right)} H(s_{1}, \|u_\alpha\|_\infty)(1 + o(1))$$

$$= O(g(\|u_\alpha\|_\infty)^{-1/2})) + O\left(\frac{\xi_\alpha}{g(\|u_\alpha\|_\infty)}\right)$$

$$= O(g(\|u_\alpha\|_\infty)^{-1/2})).$$

Since $K_\alpha(1) = 0$, by (3.5), (3.10) and Taylor expansion, for $1 - \epsilon \leq s \leq 1$,

(3.11) $$\frac{K_\alpha(s)}{2} = \frac{1}{2} \left(K_\alpha(1) + K'_\alpha(1)(s - 1) + \frac{1}{2} \frac{K''(s_1)}{2} (s - 1)^2\right)$$

$$= \frac{\xi_\alpha}{2\lambda} (s - 1) + O(g(\|u_\alpha\|_\infty)^{-1/2}))(s - 1)^2.$$
By this, (3.1) and Taylor expansion, for $1 - \epsilon \leq s \leq 1$,

\begin{align}
J(s) & \geq (p - 1 - \delta_1)(1-s)^2, \\
B_\alpha(s) & = J(s) - K_\alpha(s) \geq \frac{\xi_\alpha}{\lambda} (1-s) + \frac{\delta_1}{2} (1-s)^2.
\end{align}

Then we obtain

\begin{align}
|M_{1,\alpha}| & \leq \int_{1-\epsilon}^{1} \frac{(1-s^\alpha)|K_\alpha(s)|}{J(s)\sqrt{B_\alpha(s)}} ds \\
& \leq C \int_{1-\epsilon}^{1} \left( \frac{\xi_\alpha}{\lambda} + O(g(||u_\alpha||_\infty^{-1/2})) \right) (1-s) + \frac{\delta_1}{2}(1-s)^2 \\
& = C \int_{1-\epsilon}^{1} \frac{1}{\lambda \sqrt{1-s}} ds + O(g(||u_\alpha||_\infty^{-1/2}) \int_{1-\epsilon}^{1} \frac{1-s}{\sqrt{(\delta_1/2)(1-s)^2}} ds \\
& \leq C \left( \frac{\xi_\alpha}{\lambda} + O(g(||u_\alpha||_\infty^{-1/2})) \right) = O(g(||u_\alpha||_\infty^{-1/2}).
\end{align}

Thus the proof is complete.

**Lemma 3.2.** $M_{2,\alpha} = O(g(||u_\alpha||_\infty^{-1/2})$ as $\alpha \to \infty$.

**Proof.** Since $f(u) = u^p h(u)$, for $0 \leq s \leq 1 - \epsilon$,

\begin{align}
K_\alpha(s) & = \frac{1}{\lambda ||u_\alpha||_\infty^2} \int_{||u_\alpha||_\infty s}^{||u_\alpha||_\infty} t^p h(t) dt - \frac{1}{p+1} (1-s^{p+1}) \\
& = \frac{1}{(p+1)\lambda ||u_\alpha||_\infty^2} \left\{ \left[ t^{p+1}h(t) \right]_{||u_\alpha||_\infty s}^{||u_\alpha||_\infty} - \int_{||u_\alpha||_\infty s}^{||u_\alpha||_\infty} t^{p+1}h'(t) dt \right\} \\
& - \frac{1}{p+1} (1-s^{p+1}).
\end{align}

Since $\xi_\alpha > 0$, for $\epsilon \leq s \leq 1 - \epsilon$,

\begin{align}
\frac{1}{\lambda ||u_\alpha||_\infty^2} \int_{||u_\alpha||_\infty s}^{||u_\alpha||_\infty} t^{p+1}h'(t) dt & \leq \frac{1}{h(||u_\alpha||_\infty)||u_\alpha||_{p+1}^\infty} \int_{||u_\alpha||_\infty}^{||u_\alpha||_\infty} t^{p+1}h'(t) dt \\
& \leq \max_{\epsilon \leq s \leq 1} \frac{||u_\alpha||_\infty h'(||u_\alpha||_\infty s)}{h(||u_\alpha||_\infty)} (1-s) \\
& = O(g(||u_\alpha||_\infty^{-1/2}).
\end{align}

By this and mean value theorem, for $\epsilon \leq s < s_1 < 1 - \epsilon$,

\begin{align}
\left| \frac{K_\alpha(s)}{2} \right| & \leq \frac{1}{(p+1)\lambda ||u_\alpha||_\infty^2} \left\{ ||u_\alpha||_{p+1}^\infty h(||u_\alpha||_\infty) - ||u_\alpha||_{p+1}^\infty s^{p+1} h(||u_\alpha||_\infty s) \right\}
\end{align}
$+O(g(||u_{\alpha}\|_{\infty})^{-1/2}) - \frac{1}{p+1}(1-s^{p+1})$

$\leq \frac{1}{p+1}(1-s^{p+1})\left(\frac{||u_{\alpha}||_{\infty}^{-p-1}h(||u_{\alpha}||_{\infty})}{\lambda} - 1\right) + \frac{||u_{\alpha}||_{\infty}^{-p-1}s^{p+1}}{\lambda(p+1)}(h(||u_{\alpha}||_{\infty}) - h(||u_{\alpha}||_{\infty}s) + O(g(||u_{\alpha}||_{\infty})^{-1/2})$

$\leq \frac{\xi_{\alpha}}{(p+1)\lambda}(1-s^{p+1}) + \frac{||u_{\alpha}||_{\infty}h'(||u_{\alpha}||_{\infty}s_{1})}{h(||u_{\alpha}||_{\infty})} + O(g(||u_{\alpha}||_{\infty})^{-1/2})$

$= O(g(||u_{\alpha}||_{\infty})^{-1/2}).$

Note that for $0 \leq s \leq 1 - \epsilon$,

(3.18) $J(s) \geq \delta_{2} > 0.$

By this and (3.14), for $\epsilon \leq s \leq 1 - \epsilon$ and $\alpha \gg 1$,

(3.19) $B_{\alpha}(s) \geq J(s) - K_{\alpha}(s) \geq \frac{\delta_{2}}{2} > 0.$

Then by this and direct calculation, we obtain

$$|M_{2,\alpha}| \leq C \int^{1-\epsilon}_{\epsilon} |K_{\alpha}(s)|(1-s^{q})ds = O(g(||u_{\alpha}||_{\infty})^{-1/2}).$$

Thus the proof is complete. 

Lemma 3.3. $M_{3,\alpha} = O(g(||u_{\alpha}||_{\infty})^{-1/2})$ as $\alpha \to \infty$.

The proof of Lemma 3.3 is similar to that of Lemma 3.2. So we omit the proof. Since $\alpha = ||u_{\alpha}||_{\infty}(1 + o(1))$, Lemma 2.2 follows from Lemmas 3.1–3.3. Thus the proof is complete.

References

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