Asymptotic behavior of solutions for
$p$-Laplace parabolic equations

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This lecture is based on the joint work with Professor Goro Akagi. We study the asymptotic behavior of solutions for the one-dimensional $p$-Laplace parabolic equation

$$
\begin{align*}
    u_t &= \Delta_p u := (|u_x|^{p-2}u_x)_x \quad \text{in} \ (0, 1) \times (0, \infty), \\
    u(0, t) &= u(1, t) = 0 \quad \text{in} \ (0, \infty), \\
    u(x, 0) &= u_0(x) \quad \text{in} \ (0, 1),
\end{align*}
$$

where $p > 2$ and $u_0 \in W_0^{1,p}(0, 1) \setminus \{0\}$.

**Definition 1.** We call $u(x, t)$ a solution of (1) if $u \in C([0, \infty), W_0^{1,p}(0, 1)) \cap W^{1,2}_{loc}(0, \infty; L^2(0, 1))$, $\Delta_p u \in L^2_{loc}(0, \infty; L^2(0, 1))$, $u(x, 0) = u_0(x)$ and $u(x, t)$ satisfies the first equation of (1) a.e. $t \in (0, \infty)$.

We denote the $L^q(0, 1)$ and $W_0^{1,q}(0, 1)$ norms by

$$
\|u\|_q := \left( \int_0^1 |u(x)|^q dx \right)^{1/q} \quad \text{for} \ u \in L^q(0, 1),
$$

$$
\|u\|_{1,q} := \left( \int_0^1 |u'(x)|^q dx \right)^{1/q} \quad \text{for} \ u \in W_0^{1,q}(0, 1).
$$

The next theorem can be proved by using Theorem 3.6 of [1].

**Theorem A.** Problem (1) has a unique solution.

The next theorem is proved in [4, 5].

**Theorem B.** Any nontrivial solution $u(x, t)$ of (1) decays as $t \to \infty$, more precisely, there exist constants $C_i > 0$ such that

$$
C_1(t + 1)^{-1/(p-2)} \leq \|u(t)\|_2 \leq C_2\|u(t)\|_{1,p} \leq C_3(t + 1)^{-1/(p-2)}
$$
for \( t \in [0, \infty) \).

We investigate the asymptotic behavior of solutions as \( t \to \infty \). To this end, we use a change of variable

\[
v(x, s) = (t + 1)^{1/(p-2)}u(x, t), \quad s = \log(t + 1).
\]

Then (1) is reduced to

\[
\begin{align*}
  v_s &= \Delta_p v + \alpha v \quad \text{in } (0, 1) \times (0, \infty), \\
  v(0, t) &= v(1, t) = 0 \quad \text{in } (0, \infty), \\
  v(x, 0) &= u_0(x) \quad \text{in } (0, 1),
\end{align*}
\]

where \( \alpha := 1/(p - 2) \). The stationary problem for (2) is written in the following form:

\[
\begin{align*}
  -(|\phi'(x)|^{p-2}\phi'(x))' &= \alpha \phi(x), \quad x \in (0, 1), \\
  \phi(0) &= \phi(1) = 0.
\end{align*}
\]

The next theorem implies that each stationary solution is characterized by its nodal number.

**Theorem C.** For each \( k \in \mathbb{N} \), there exists a unique solution \( \phi_k \) of (3) which has exactly \( k - 1 \) zeros in \( (0, 1) \) and \( \phi_k'(0) > 0 \). Moreover, the set of all nontrivial solutions of (3) consists of \( \pm \phi_k \) with \( k \in \mathbb{N} \).

**Proof.** This theorem is a known result, but for the reader's convenience we give a sketch of proof. Observe that if \( \phi \) satisfies the first equation of (3), so is \( \lambda^{-p/(p-2)}\phi(\lambda x) \) for any \( \lambda > 0 \). We consider the first equation of (3) with the initial condition,

\[
\phi(0) = 0, \quad \phi'(0) = 1.
\]

This problem has a unique solution, which is denoted by \( \phi_0(x) \). Moreover, \( \phi_0(x) \) is a periodic solution and it has the first zero \( T > 0 \). Thus \( kT \) with \( k \in \mathbb{Z} \) are all the zeros of \( \phi_0(x) \). Then we put

\[
\phi_k(x) := (kT)^{-p/(p-2)}\phi_0(kTx) \quad \text{with } k \in \mathbb{N},
\]

which is the desired solution. Furthermore, it is easy to check that the set of all nontrivial solutions of (3) consists of \( \pm \phi_k \) with \( k \in \mathbb{N} \). \( \square \)
By using Theorem C with the same way as in Berryman and Holland [2], we can prove the next theorem.

**Theorem D.** For any nontrivial solution $v(s)$ of (2), there exists a unique nontrivial stationary solution $\phi$ (i.e., $\phi = \phi_k$ or $-\phi_k$ with a certain $k \in \mathbb{N}$ ) such that

$$\lim_{s \to \infty} \|v(s) - \phi\|_{1,p} = 0.$$ 

We give a definition of the stability of stationary solutions.

**Definition 2.** Let $\phi$ be a nontrivial solution of (3).

(i) $\phi$ is called stable if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\sup_{0 \leq s < \infty} \|v(s) - \phi\|_{1,p} < \varepsilon \quad \text{when} \quad \|v(0) - \phi\|_{1,p} < \delta.$$ 

(ii) $\phi$ is called asymptotically stable if it is stable and moreover there exists a $\delta_0 > 0$ such that

$$\lim_{s \to \infty} \|v(s) - \phi\|_{1,p} = 0 \quad \text{when} \quad \|v(0) - \phi\|_{1,p} < \delta_0.$$ 

We state our main result.

**Theorem 1.** The positive solution $\phi_1$ and the negative solution $-\phi_1$ of (3) are asymptotically stable and $\pm \phi_k$ with $k \geq 2$ are unstable.

To prove Theorem 1, we define the energy

$$J(v) := \int_0^1 \left( \frac{1}{p} |v'(x)|^p - \frac{\alpha}{2} v(x)^2 \right) dx \quad \text{for} \quad v \in W_{0}^{1,p}(0,1).$$ 

Then $J$ becomes a Lyapunov functional for (2). Indeed, multiplying (2) by $v_s$ and integrating it over $(0,1)$, we have

$$-\|v_s\|_2^2 = (|v_x|^{p-2}v_x, (v_x)_s) - \alpha(v, v_s).$$

Here $(u,v)$ denotes the duality pairing of $u$ and $v$. The above expression is rewritten as

$$\frac{d}{ds} J(v(s)) = -\|v_s\|_2^2 \leq 0.$$ 

Thus, if $v(s)$ is a solution of (2), then $J(v(s))$ is decreasing. Consequently $J$ becomes a Lyapunov functional.
Lemma 1. Each stationary solution is isolated from each other. Moreover, we have
\[ J(\pm \phi_1) < J(\pm \phi_2) < J(\pm \phi_3) < \cdots \searrow 0. \] (5)

Proof. Multiplying the first equation of (3) by \( \phi(x) \) and integrating it over \((0,1)\), we have
\[ \int_0^1 |\phi'|^p \, dx = \alpha \int_0^1 \phi^2 \, dx. \]
Using this relation with \( \alpha = 1/(p-2) \), we get
\[ J(\phi) = -\frac{1}{2p} \int_0^1 \phi^2 \, dx, \]
provided that \( \phi \) is a solution of (3). Substituting (4) into the relation above, we obtain
\[ J(\phi_k) = -\frac{1}{2p} (kT)^{-2p/(p-2)} \int_0^1 \phi_0(kTx)^2 \, dx \]
\[ = -\frac{1}{2p} (kT)^{-2p/(p-2)} T^{-1} \int_0^T \phi_0(x)^2 \, dx. \]
This expression assures (5), which implies that each stationary solution is isolated from each other.
\[ \square \]

Lemma 2. \( J \) has a global minimizer and it is equal to either \( \phi_1 \) or \(-\phi_1\).

Proof. We use the Sobolev imbedding to get a constant \( C > 0 \) such that
\[ J(v) = \frac{1}{p} ||v'||_p^p - \frac{\alpha}{2} ||v||_2^2 \geq \frac{1}{p} ||v'||_p^p - C ||v'||_p^2, \] (6)
which shows the lower boundedness of \( J \) because \( p > 2 \). In the standard way, we can prove that \( J \) satisfies the Palais-Smale condition. Then \( J \) has a global minimizer (for the proof, refer to [3, Theorem 2.7]). If \( \phi \) is a global minimizer, so is \( |\phi| \), which becomes a critical point of \( J \). Hence \( |\phi| \) is a solution of (3). By the strong maximum principle, \( |\phi| > 0 \) in \((0,1)\). Thus \( \phi \) is a positive or negative solution. Since a positive solution is unique by Theorem C, \( \phi \) is equal to either \( \phi_1 \) or \(-\phi_1\). \[ \square \]
Lemma 3. For any $\varepsilon > 0$, there exists an $\alpha > J(\phi_1)$ such that

$$\{v \in W_0^{1,p}(0,1): J(v) \leq \alpha\} \subset B(\phi_1, \varepsilon) \cup B(-\phi_1, \varepsilon),$$

where

$$B(\phi_1, \varepsilon) := \{v \in W_0^{1,p}(0,1): \|v - \phi_1\|_{1,p} < \varepsilon\}.$$ (7)

**Proof.** Recall that $\pm\phi_1$ are minimizers of $J$ and the $W_0^{1,p}(0,1)$-norm is defined by $\|v\|_{1,p} = \|v'\|_p$. We use contradiction. Suppose that there exist $\varepsilon > 0$ and a sequence $v_n \in W_0^{1,p}(0,1)$ such that $J(v_n)$ converges to $J(\phi_1)$ but

$$\|v_n - \phi_1\|_{1,p} \geq \varepsilon, \quad \|v_n + \phi_1\|_{1,p} \geq \varepsilon.$$ (8)

By (6), $\|v_n'\|_p$ is bounded. Hence a subsequence (denoted by $v_n$ again) of $v_n$ converges to $v_\infty$ weakly in $W_0^{1,p}(0,1)$ and strongly in $L^2(0,1)$. Since $J(v_n)$ converges to $J(\phi_1)$, we have

$$\frac{1}{p}\|v_n'\|_p^p = J(v_n) + \frac{\alpha}{2}\|v_n\|_2^2 \to J(\phi_1) + \frac{\alpha}{2}\|v_\infty\|_2^2$$

Since $\phi_1$ is a global minimizer of $J$, we get

$$J(\phi_1) \leq J(v_\infty) = \frac{1}{p}\|v_\infty'\|_p^p - \frac{\alpha}{2}\|v_\infty\|_2^2.$$

From two inequalities above, it follows that $\limsup_{n \to \infty} \|v_n'\|_p \leq \|v_\infty'\|_p$. Moreover, $\liminf_{n \to \infty} \|v_n'\|_p \geq \|v_\infty'\|_p$ because $v_n$ weakly converges in $W_0^{1,p}(0,1)$. Since $W_0^{1,p}(0,1)$ is uniformly convex, $v_n$ converges strongly in $W_0^{1,p}(0,1)$. Letting $n \to \infty$ in (8), we have

$$\|v_\infty - \phi_1\|_{1,p} \geq \varepsilon, \quad \|v_\infty + \phi_1\|_{1,p} \geq \varepsilon.$$

On the other hand, since $J(v_\infty) = J(\phi_1)$, $v_\infty$ is equal to $\phi_1$ or $-\phi_1$ by Lemma 2. This is a contradiction. Thus the proof is complete. \(\square\)

To prove instability of $\phi_k$ with $k \geq 2$, we use

**Lemma 4.** Let $k \geq 2$. Then for any $\varepsilon > 0$ there exists a $v_0 \in W_0^{1,p}(0,1)$ such that

$$\|v_0 - \phi_k\|_{1,p} < \varepsilon \quad \text{and} \quad J(v_0) < J(\phi_k).$$

In other words, there is a point $v_0$ sufficiently close to $\phi_k$ whose energy is less than that of $\phi_k$. 
Proof. We denote by $\psi(x, (a, b))$ the unique positive solution of

$$- (|\psi'(x)|^{p-2}\psi'(x))' = \alpha \psi(x), \quad \psi(x) > 0, \quad x \in (a, b),$$
$$\psi(a) = \psi(b) = 0,$$

Recall that $\phi_1(x)$ is a positive solution of (3). Hence it holds that $\psi(x, (0, 1)) = \phi_1(x)$ and moreover we have the relation

$$\psi(x, (a, b)) = c^{-p/(p-2)} \phi_1(c(x-a)), \quad c := 1/(b-a). \quad (9)$$

For $\lambda \in (0, 2)$, we define

$$\Psi_\lambda(x) := \begin{cases} \psi(x, (0, \lambda/k)) & \text{if } x \in [0, \lambda/k], \\ -\psi(x, (\lambda/k, 2/k)) & \text{if } x \in [\lambda/k, 2/k], \\ \phi_k(x) & \text{if } x \in [2/k, 1]. \end{cases}$$

By using (9) with (4), we can prove that $\Psi_\lambda \to \phi_k$ as $\lambda \to 1$ and $J(\Psi_\lambda) < J(\phi_k)$ if $\lambda \neq 1$. When $\lambda \neq 1$ is sufficiently close to 1, $v_0 = \Psi_\lambda$ satisfies the assertion of Lemma 4.

Proof of Theorem 1. We prove that $\phi_1$ is asymptotically stable. Let $\varepsilon > 0$. We can assume that $\varepsilon$ satisfies

$$B(\phi_1, \varepsilon) \cap B(-\phi_1, \varepsilon) = \emptyset, \quad \pm \phi_k \notin \overline{B(\phi_1, \varepsilon)} \quad (k \geq 2).$$

Then by Lemma 3, we determine $a(> J(\phi_1))$ which satisfies (7). If $\|v(0) - \phi_1\|_{1,p}$ is small enough, then $J(v(0))$ is sufficiently close to $J(\phi_1)$ and hence $J(v(0)) < a$. Thus $J(v(s)) \leq J(v(0)) \leq a$. By Lemma 3, we get

$$v(s) \in B(\phi_1, \varepsilon) \cup B(-\phi_1, \varepsilon) \quad \text{for all } s \geq 0.$$ 

Since $B(\phi_1, \varepsilon) \cap B(-\phi_1, \varepsilon) = \emptyset$, $v(s)$ belongs to $B(\phi_1, \varepsilon)$ for $s \geq 0$. Therefore $\phi_1$ is stable. By Theorem D, $v(s)$ has a limit as $s \to \infty$. Since $\pm \phi_k \notin \overline{B(\phi_1, \varepsilon)}$ for $k \geq 2$, $v(s)$ must converge to $\phi_1$. Therefore $\phi_1$ is asymptotically stable.

In the same way as above, we can show the asymptotic stability of $-\phi_1$.

Let $k \geq 2$. We show the instability of $\phi_k$. Let $\varepsilon > 0$. Then we choose $v_0$ by Lemma 4. Let $v(s)$ be the solution starting from $v(0) = v_0$. Then $v(s)$ converges to a certain stationary point $v_\infty$. But $v_\infty \neq \phi_k$ because

$$J(v_\infty) = \lim_{s \to \infty} J(v(s)) \leq J(v_0) < J(\phi_k).$$
Since each stationary point is isolated from each other, we define
\[ d := \inf\{\|u - \phi_k\|_{1,p} : u \text{ is any stationary solution except for } \phi_k\}. \]

Then \( \|v_\infty - \phi_k\|_{1,p} \geq d > 0 \). The initial data \( v_0 \) is sufficiently close to \( \phi_k \) but the solution \( v(s) \) is away from \( \phi_k \) with at least distance \( d/2 \) for \( s \) large enough. Therefore \( \phi_k (k \geq 2) \) is unstable. \( \square \)

References


