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Asymptotic behavior of solutions for p-Laplace parabolic equations

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This lecture is based on the joint work with Professor Goro Akagi. We study the asymptotic behavior of solutions for the one-dimensional p-Laplace parabolic equation

$$
\begin{aligned}
&u_t = \Delta_p u := (|u_x|^{p-2}u_x)_x \quad \text{in } (0,1)\times(0,\infty),\\
&u(0,t) = u(1,t) = 0 \quad \text{in } (0,\infty),\\
&u(x,0) = u_0(x) \quad \text{in } (0,1),
\end{aligned}
\tag{1}
$$

where $p>2$ and $u_0 \in W_0^{1,p}(0,1)\setminus\{0\}$.

Definition 1. We call $u(x,t)$ a solution of (1) if $u\in C([0,\infty),W_0^{1,p}(0,1))\cap W_{loc}^{1,2}(0,\infty;L^2(0,1))$, $\Delta_p u \in L_{loc}^2(0,\infty;L^2(0,1))$, $u(x,0)=u_0(x)$ and $u(x,t)$ satisfies the first equation of (1) a.e. $t\in(0,\infty)$.

We denote the $L^q(0,1)$ and $W_0^{1,q}(0,1)$ norms by

$$\|u\|_q := \left(\int_0^1 |u(x)|^q dx\right)^{1/q} \quad \text{for } u\in L^q(0,1),$$

$$\|u\|_{1,q} := \left(\int_0^1 |u'(x)|^q dx\right)^{1/q} \quad \text{for } u\in W_0^{1,q}(0,1).$$

The next theorem can be proved by using Theorem 3.6 of [1].

Theorem A. *Problem* (1) *has a unique solution.*

The next theorem is proved in [4, 5].

Theorem B. *Any nontrivial solution $u(x,t)$ of* (1) *decays as $t\to\infty$, more precisely, there exist constants $C_i>0$ such that*

$$C_1(t+1)^{-1/(p-2)} \le \|u(t)\|_2 \le C_2\|u(t)\|_{1,p} \le C_3(t+1)^{-1/(p-2)}$$

for $t \in [0, \infty)$.

We investigate the asymptotic behavior of solutions as $t \to \infty$. To this end, we use a change of variable

$$v(x,s) = (t+1)^{1/(p-2)} u(x,t), \quad s = \log(t+1).$$

Then (1) is reduced to

$$\begin{aligned} &v_s = \Delta_p v + \alpha v && \text{in } (0,1) \times (0,\infty), \\ &v(0,t) = v(1,t) = 0 && \text{in } (0,\infty), \\ &v(x,0) = u_0(x) && \text{in } (0,1), \end{aligned} \tag{2}$$

where $\alpha := 1/(p-2)$. The stationary problem for (2) is written in the following form:

$$\begin{aligned} &-(|\phi'(x)|^{p-2}\phi'(x))' = \alpha\phi(x), \qquad x \in (0,1), \\ &\phi(0) = \phi(1) = 0. \end{aligned} \tag{3}$$

The next theorem implies that each stationary solution is characterized by its nodal number.

Theorem C. *For each* $k \in \mathbb{N}$, *there exists a unique solution* ϕ_k *of* (3) *which has exactly* $k-1$ *zeros in* $(0,1)$ *and* $\phi_k'(0) > 0$. *Moreover, the set of all nontrivial solutions of* (3) *consists of* $\pm\phi_k$ *with* $k \in \mathbb{N}$.

Proof. This theorem is a known result, but for the reader's convenience we give a sketch of proof. Observe that if ϕ satisfies the first equation of (3), so is $\lambda^{-p/(p-2)}\phi(\lambda x)$ for any $\lambda > 0$. We consider the first equation of (3) with the initial condition,

$$\phi(0) = 0, \quad \phi'(0) = 1.$$

This problem has a unique solution, which is denoted by $\phi_0(x)$. Moreover, $\phi_0(x)$ is a periodic solution and it has the first zero $T > 0$. Thus kT with $k \in \mathbb{Z}$ are all the zeros of $\phi_0(x)$. Then we put

$$\phi_k(x) := (kT)^{-p/(p-2)}\phi_0(kTx) \quad \text{with } k \in \mathbb{N}, \tag{4}$$

which is the desired solution. Furthermore, it is easy to check that the set of all nontrivial solutions of (3) consists of $\pm\phi_k$ with $k \in \mathbb{N}$. $\square$

By using Theorem C with the same way as in Berryman and Holland [2], we can prove the next theorem.

Theorem D. *For any nontrivial solution $v(s)$ of* (2)*, there exists a unique nontrivial stationary solution ϕ (i.e., $\phi = \phi_k$ or $-\phi_k$ with a certain $k \in \mathbb{N}$) such that*

$$\lim_{s\to\infty} \|v(s) - \phi\|_{1,p} = 0.$$

We give a definition of the stability of stationary solutions.

Definition 2. Let ϕ be a nontrivial solution of (3).

(i) ϕ is called stable if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\sup_{0\le s<\infty} \|v(s) - \phi\|_{1,p} < \varepsilon \quad \text{when } \|v(0) - \phi\|_{1,p} < \delta.$$

(ii) ϕ is called asymptotically stable if it is stable and moreover there exists a $\delta_0 > 0$ such that

$$\lim_{s\to\infty} \|v(s) - \phi\|_{1,p} = 0 \quad \text{when } \|v(0) - \phi\|_{1,p} < \delta_0.$$

We state our main result.

Theorem 1. *The positive solution ϕ_1 and the negative solution $-\phi_1$ of* (3) *are asymptotically stable and $\pm\phi_k$ with $k \ge 2$ are unstable.*

To prove Theorem 1, we define the energy

$$J(v) := \int_0^1 \left(\frac{1}{p}|v'(x)|^p - \frac{\alpha}{2}v(x)^2\right) dx \quad \text{for } v \in W_0^{1,p}(0,1).$$

Then J becomes a Lyapunov functional for (2). Indeed, multiplying (2) by v_s and integrating it over $(0,1)$, we have

$$-\|v_s\|_2^2 = (|v_x|^{p-2}v_x, (v_x)_s) - \alpha(v, v_s).$$

Here (u,v) denotes the duality pairing of u and v. The above expression is rewritten as

$$\frac{d}{ds}J(v(s)) = -\|v_s\|_2^2 \le 0.$$

Thus, if $v(s)$ is a solution of (2), then $J(v(s))$ is decreasing. Consequently J becomes a Lyapunov functional.

Lemma 1. *Each stationary solution is isolated from each other. Moreover, we have*

$$J(\pm\phi_1) < J(\pm\phi_2) < J(\pm\phi_3) < \cdots \nearrow 0. \tag{5}$$

Proof. Multiplying the first equation of (3) by $\phi(x)$ and integrating it over $(0,1)$, we have

$$\int_0^1 |\phi'|^p dx = \alpha \int_0^1 \phi^2 dx.$$

Using this relation with $\alpha = 1/(p-2)$, we get

$$J(\phi) = -\frac{1}{2p}\int_0^1 \phi^2 dx,$$

provided that ϕ is a solution of (3). Substituting (4) into the relation above, we obtain

$$\begin{aligned} J(\phi_k) &= -\frac{1}{2p}(kT)^{-2p/(p-2)} \int_0^1 \phi_0(kTx)^2 dx \\ &= -\frac{1}{2p}(kT)^{-2p/(p-2)} T^{-1} \int_0^T \phi_0(x)^2 dx. \end{aligned}$$

This expression assures (5), which implies that each stationary solution is isolated from each other. □

Lemma 2. *J has a global minimizer and it is equal to either ϕ_1 or $-\phi_1$.*

Proof. We use the Sobolev imbedding to get a constant $C > 0$ such that

$$J(v) = \frac{1}{p}\|v'\|_p^p - \frac{\alpha}{2}\|v\|_2^2 \geq \frac{1}{p}\|v'\|_p^p - C\|v'\|_p^2, \tag{6}$$

which shows the lower boundedness of J because $p > 2$. In the standard way, we can prove that J satisfies the Palais-Smale condition. Then J has a global minimizer (for the proof, refer to [3, Theorem 2.7]). If ϕ is a global minimizer, so is $|\phi|$, which becomes a critical point of J. Hence $|\phi|$ is a solution of (3). By the strong maximum principle, $|\phi| > 0$ in $(0,1)$. Thus ϕ is a positive or negative solution. Since a positive solution is unique by Theorem C, ϕ is equal to either ϕ_1 or $-\phi_1$. □

Lemma 3. *For any $\varepsilon > 0$, there exists an $a > J(\phi_1)$ such that*

$$\{v \in W_0^{1,p}(0,1) : J(v) \leq a\} \subset B(\phi_1, \varepsilon) \cup B(-\phi_1, \varepsilon), \tag{7}$$

where

$$B(\phi_1, \varepsilon) := \{v \in W_0^{1,p}(0,1) : \|v - \phi_1\|_{1,p} < \varepsilon\}.$$

Proof. Recall that $\pm\phi_1$ are minimizers of J and the $W_0^{1,p}(0,1)$-norm is defined by $\|v\|_{1,p} = \|v'\|_p$. We use contradiction. Suppose that there exist $\varepsilon > 0$ and a sequence $v_n \in W_0^{1,p}(0,1)$ such that $J(v_n)$ converges to $J(\phi_1)$ but

$$\|v_n - \phi_1\|_{1,p} \geq \varepsilon, \qquad \|v_n + \phi_1\|_{1,p} \geq \varepsilon. \tag{8}$$

By (6), $\|v_n'\|_p$ is bounded. Hence a subsequence (denoted by v_n again) of v_n converges to v_∞ weakly in $W_0^{1,p}(0,1)$ and strongly in $L^2(0,1)$. Since $J(v_n)$ converges to $J(\phi_1)$, we have

$$\frac{1}{p}\|v_n'\|_p^p = J(v_n) + \frac{\alpha}{2}\|v_n\|_2^2 \to J(\phi_1) + \frac{\alpha}{2}\|v_\infty\|_2^2$$

Since ϕ_1 is a global minimizer of J, we get

$$J(\phi_1) \leq J(v_\infty) = \frac{1}{p}\|v_\infty'\|_p^p - \frac{\alpha}{2}\|v_\infty\|_2^2.$$

From two inequalities above, it follows that $\limsup_{n\to\infty} \|v_n'\|_p \leq \|v_\infty'\|_p$. Moreover, $\liminf_{n\to\infty} \|v_n'\|_p \geq \|v_\infty'\|_p$ because v_n weakly converges in $W_0^{1,p}(0,1)$. Since $W_0^{1,p}(0,1)$ is uniformly convex, v_n converges strongly in $W_0^{1,p}(0,1)$. Letting $n \to \infty$ in (8), we have

$$\|v_\infty - \phi_1\|_{1,p} \geq \varepsilon, \qquad \|v_\infty + \phi_1\|_{1,p} \geq \varepsilon.$$

On the other hand, since $J(v_\infty) = J(\phi_1)$, v_∞ is equal to ϕ_1 or $-\phi_1$ by Lemma 2. This is a contradiction. Thus the proof is complete. □

To prove instability of ϕ_k with $k \geq 2$, we use

Lemma 4. *Let $k \geq 2$. Then for any $\varepsilon > 0$ there exists a $v_0 \in W_0^{1,p}(0,1)$ such that*

$$\|v_0 - \phi_k\|_{1,p} < \varepsilon \quad \textit{and} \quad J(v_0) < J(\phi_k).$$

In other words, there is a point v_0 sufficiently close to ϕ_k whose energy is less than that of ϕ_k.

Proof. We denote by $\psi(x,(a,b))$ the unique positive solution of

$$
\begin{aligned}
&-(|\psi'(x)|^{p-2}\psi'(x))' = \alpha\psi(x), \quad \psi(x) > 0, \quad x \in (a,b),\\
&\psi(a) = \psi(b) = 0,
\end{aligned}
$$

Recall that $\phi_1(x)$ is a positive solution of (3). Hence it holds that $\psi(x,(0,1)) = \phi_1(x)$ and moreover we have the relation

$$
\psi(x,(a,b)) = c^{-p/(p-2)}\phi_1(c(x-a)), \quad c := 1/(b-a). \tag{9}
$$

For $\lambda \in (0,2)$, we define

$$
\Psi_\lambda(x) := \begin{cases} \psi(x,(0,\lambda/k)) & \text{if } x \in [0,\lambda/k],\\ -\psi(x,(\lambda/k,2/k)) & \text{if } x \in [\lambda/k,2/k],\\ \phi_k(x) & \text{if } x \in [2/k,1]. \end{cases}
$$

By using (9) with (4), we can prove that $\Psi_\lambda \to \phi_k$ as $\lambda \to 1$ and $J(\Psi_\lambda) < J(\phi_k)$ if $\lambda \neq 1$. When $\lambda \neq 1$ is sufficiently close to 1, $v_0 = \Psi_\lambda$ satisfies the assertion of Lemma 4. □

Proof of Theorem 1. We prove that ϕ_1 is asymptotically stable. Let $\varepsilon > 0$. We can assume that ε satisfies

$$
B(\phi_1,\varepsilon) \cap B(-\phi_1,\varepsilon) = \emptyset, \quad \pm\phi_k \notin \overline{B(\phi_1,\varepsilon)} \quad (k \geq 2).
$$

Then by Lemma 3, we determine $a(> J(\phi_1))$ which satisfies (7). If $\|v(0) - \phi_1\|_{1,p}$ is small enough, then $J(v(0))$ is sufficiently close to $J(\phi_1)$ and hence $J(v(0)) < a$. Thus $J(v(s)) \leq J(v(0)) \leq a$. By Lemma 3, we get

$$
v(s) \in B(\phi_1,\varepsilon) \cup B(-\phi_1,\varepsilon) \quad \text{for all } s \geq 0.
$$

Since $B(\phi_1,\varepsilon) \cap B(-\phi_1,\varepsilon) = \emptyset$, $v(s)$ belongs to $B(\phi_1,\varepsilon)$ for $s \geq 0$. Therefore ϕ_1 is stable. By Theorem D, $v(s)$ has a limit as $s \to \infty$. Since $\pm\phi_k \notin \overline{B(\phi_1,\varepsilon)}$ for $k \geq 2$, $v(s)$ must converge to ϕ_1. Therefore ϕ_1 is asymptotically stable. In the same way as above, we can show the asymptotic stability of $-\phi_1$.

Let $k \geq 2$. We show the instability of ϕ_k. Let $\varepsilon > 0$. Then we choose v_0 by Lemma 4. Let $v(s)$ be the solution starting from $v(0) = v_0$. Then $v(s)$ converges to a certain stationary point v_∞. But $v_\infty \neq \phi_k$ because

$$
J(v_\infty) = \lim_{s\to\infty} J(v(s)) \leq J(v_0) < J(\phi_k).
$$

Since each stationary point is isolated from each other, we define

$$d := \inf\{\|u - \phi_k\|_{1,p} : \ u \text{ is any stationary solution except for } \phi_k\}.$$

Then $\|v_\infty - \phi_k\|_{1,p} \geq d > 0$. The initial data v_0 is sufficiently close to ϕ_k but the solution $v(s)$ is away from ϕ_k with at least distance $d/2$ for s large enough. Therefore ϕ_k $(k \geq 2)$ is unstable. □

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