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# Existence and non-existence of the nonlinear Schrödinger equations for one and high dimensional case

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## 0. Introduction

In this report, we will introduce the results of [**S**] and related results. We consider the following nonlinear Schrödinger equations:

$$
\begin{aligned}
&-\Delta u+(1+b(x))u=f(u) \quad \text{in } \mathbf{R}^N, \\
&u\in H^1(\mathbf{R}^N).
\end{aligned}
\tag{$*$}
$$

We mainly considered the one-dimensional case in [**S**] but, in this report, we consider not only one-dimensional case but also the high-dimensional case. Here, we assume that the potential $b(x)\in C(\mathbf{R},\mathbf{R})$ satisfies the following assumptions:

(b.1) $1+b(x)\geq 0$ for all $x\in \mathbf{R}^N$.

(b.2) $\lim_{|x|\to\infty} b(x)=0$.

(b.3) There exist $\beta_0>2$ and $C_0>0$ such that $b(x)\leq C_0e^{-\beta_0|x|}$ for all $x\in \mathbf{R}^N$.

We also assume that the nonlinearity $f(u)\in C(\mathbf{R},\mathbf{R})$ satisfies the following

(f.0) $f(u)=|u|^{p-1}u$ for $p\in(1,\frac{N+2}{N-2})$ when $N\geq 3$ and $p\in(1,\infty)$ when $N=2$.

(f.1) There exists $\eta_0>0$ such that $\lim_{|u|\to 0}\frac{f(u)}{|u|^{1+\eta_0}}=0$.

(f.2) There exists $u_0>0$ such that

$$
\begin{aligned}
&F(u)<\frac{1}{2}u^2 \quad \text{for all } u\in(0,u_0), \\
&F(u_0)=\frac{1}{2}u_0^2, \quad f(u_0)>u_0.
\end{aligned}
$$

(f.3) There exists $\mu_0>2$ such that $0<\mu_0F(u)\leq uf(u)$ for all $u\neq 0$.

To consider the $(*)$, the following equation plays an important roles:

$$-\Delta u + u = f(u) \quad \text{in } \mathbf{R}^N, \quad u \in H^1(\mathbf{R}^N). \tag{0.1}$$

From (b.2), the equation $-\Delta u + u = f(u)$ appears as a limit when $|x|$ goes to $\infty$ in $(*)$. To show the existence of positive solution of $(*)$ in our arguments, the uniqueness (up to translation) of positive solutions of (0.1) is also important. Under the condition (f.0), it is well-known that the uniqueness (up to translation) of the positive solutions of (0.1). When $N = 1$, it is known that the conditions (f.1) and (f.2) are sufficient conditions for (0.1) to have an unique (up to translation) positive solution:

**Remark 0.1.** In Section 5 of [**BeL1**], Berestycki-Lions showed that if $f(u)$ is of locally Lipschitz continuous and $f(u) = 0$, then (f.2) is a necessary and sufficient condition for the existence of a non-trivial solution of (1.0). Moreover, it also was shown that the uniqueness (up to translation) of positive solutions under the (f.2). In Section 2 of [**JT1**], Jeanjean-Tanaka showed that when $f(u)$ is of continuous, (f.1) and (f.2) are sufficient conditions for (0.1) to have an unique positive solution.

The condition (f.3) is so called Ambrosetti-Rabinowitz condition, which guarantees the boundedness of (PS)-sequences for the functional corresponding to the equation $(*)$ and (0.1). To state an our result for one-dimensional case, we also need the following assumption for $b(x)$.

(b.4) When $N = 1$, there exists $x_0 \in \mathbf{R}$ such that

$$\int_{-\infty}^{\infty} b(x - x_0)e^{2|x|}\, dx \in [-\infty, 2).$$

Our first theorem is the following.

**Theorem 0.2.** *When $N \geq 2$, we assume that (b.1)–(b.3) and (f.0) hold. Then $(*)$ has at least a positive solution. When $N = 1$, we assume that (b.1)–(b.4) and (f.1)–(f.3) hold. Then $(*)$ has at least a positive solution.*

In [**S**], we give a proof of Theorem 0.2 for the one-dimensional case. To prove the Theorem 0.2, we developed the arguments of [**BaL**] and [**Sp**]. We remark that, for high-dimensional case, the proof of Theorem 0.2 almost are parallel to the proof of [**BaL**]. However, for the proof of the one-dimensional case, we essentially developed the arguments of [**BaL**] and [**Sp**]. Bahri-Li [**BaL**] showed that there exists a positive solution of

$$-\Delta u + u = (1 - b(x))|u|^{p-1}u \quad \text{in } \mathbf{R}^N, \qquad u \in H^1(\mathbf{R}^N), \tag{0.2}$$

where $N \geq 3$, $1 < p < \frac{N+2}{N-2}$ and $b(x) \in C(\mathbf{R}, \mathbf{R})$ satisfies (b.2)-(b.3) and

(b.1)' $1 - b(x) \geq 0$ for all $x \in \mathbf{R}^N$.

For one dimensional case, Spradlin [**Sp**] proved that there exists a positive solution of the equation

$$-u'' + u = (1 - b(x))f(u) \quad \text{in } \mathbf{R}, \qquad u \in H^1(\mathbf{R}). \tag{0.3}$$

They also assumed that $b(x) \in C(\mathbf{R}, \mathbf{R})$ satisfies (b.1)' and (b.2)–(b.3) and $f(u)$ satisfies (f.1)–(f.3) and

(f.4) $\frac{f(u)}{u}$ is an increasing function for all $u > 0$.

When (f.0) or (f.4) holds, we can consider the Nehari manifold and they argued on Nehari manifold in [**BaL**] and [**Sp**]. In our situation, when $N = 1$, we can not argue on Nehari manifold. This was one of the difficulties which had to overcome in [**S**].

From the above results and Theorem 0.2, it seems that, when $N = 1$, Theorem 0.2 holds without condition (b.4). However (b.4) is an essential assumption for $(*)$ to have non-trivial solutions. In what follows, we will show a result about the non-existence of nontrivial solutions for $(*)$.

In next our result, we will assume that $N = 1$ and $b(x)$ satisfies the following condition:

(b.5) There exist $\mu > 0$ and $m_2 \geq m_1 > 0$ such that

$$m_1 \mu e^{-\mu|x|} \leq b(x) \leq m_2 \mu e^{-\mu|x|} \quad \text{for all } x \in \mathbf{R}.$$

Here, we remark that, if (b.5) holds for $\mu > 2$, then $b(x)$ satisfies (b.1)–(b.3) and

$$\frac{2\mu}{\mu - 2} m_1 \leq \int_{-\infty}^{\infty} b(x) e^{2|x|}\, dx \leq \frac{2\mu}{\mu - 2} m_2.$$

Thus, when $m_2 < 1$ and $\mu$ is very large, the condition (b.4) also holds.

Our second result is the following:

**Theorem 0.3.** *Assume $N = 1$, (b.5) holds and $f(u) = |u|^{p-1}u$ $(p > 1)$.*

*(i) If $m_1 > 1$, there exists $\mu_1 > 0$ such that $(*)$ does not have non-trivial solution for all $\mu \geq \mu_1$.*

*(ii) If $m_2 < 1$, there exists $\mu_2 > 0$ such that $(*)$ has at least a non-trivial solution for all $\mu \geq \mu_2$.*

*(iii) There exists $\mu_3 > 0$ such that $(*)$ does not have sign-changing solutions for all $\mu \geq \mu_3$.*

From Theorem 0.3, we see that Theorem 0.2 does not hold except for condition (b.4). This is a drastically different situation from the high-dimensional cases. This is one of the interesting points in our results.

We remark that the condition (b.4) implies $\int_{-\infty}^{\infty} b(x)\,dx < 2$ and the assumption of (ii) of Theorem 0.3 also means $\int_{-\infty}^{\infty} b(x)\,dx < 2$. Thus we expect that the difference from existence and non-existence of non-trivial solutions of $(*)$ depends on the quantity of integrate of $b(x)$.

We can obtain this expectation from another viewpoint, which is a perturbation problem. Setting $b_\mu(x) = m\mu e^{-\mu|x|}$, $b_\mu(x)$ satisfies (b.5) and, when $\mu \to \infty$, $b_\mu(x)$ converges to the delta function $2m\delta_0$ in distribution sense. Thus $(*)$ approaches to the equation

$$-u'' + (1 + 2m\delta_0)u = |u|^{p-1}u \quad \text{in } \mathbf{R}, \quad u \in H^1(\mathbf{R}), \tag{0.4}$$

in distribution sense. Here, if $u$ is a solution of (0.4) in distribution sense, we can see that $u$ is of $C^2$-function in $\mathbf{R}\backslash\{0\}$ and continuous in $\mathbf{R}$ and $u$ satisfies

$$u'(+0) - u'(-0) = 2mu(0). \tag{0.5}$$

Moreover, since $u$ is a homoclinic orbit of $-u''+u = f(u)$ in $(-\infty, 0)$ or $(0, \infty)$, respectively, $u$ satisfies

$$-\frac{1}{2}u'(x)^2 + \frac{1}{2}u(x)^2 - \frac{1}{p+1}|u(x)|^{p+1} = 0 \quad \text{for} \quad x \neq 0. \tag{0.6}$$

When $x \to \pm 0$ in (0.6), from (f.1), we find

$$u'(-0) = -u'(+0), \quad |u'(\pm 0)| < |u(0)|. \tag{0.7}$$

Thus, from (0.5) and (0.7), it easily see that (0.4) has an unique positive solution when $|m| < 1$ and (0.4) has no non-trivial solutions when $|m| \geq 1$. Therefore we can regard Theorem 0.3 as results of a perturbation problem of (0.4).

To prove Theorem 0.3, we develop the shooting arguments which used in [**BE**]. Bianchi and Egnell [**BE**] argued about the existence and non-existence of radial solutions for

$$-\Delta u = K(|x|)|u|^{\frac{N+2}{N-2}}, \quad u > 0 \quad \text{in } \mathbf{R}^N, \quad u(x) = O(|x|^{2-N}) \quad \text{as } |x| \to \infty. \tag{0.8}$$

Here $N \geq 3$ and $K(|x|)$ is a radial continuous function. Roughly speaking their approach, by setting $u(r) = u(|x|)$, they reduce (0.8) to an ordinary differential equation and considered solutions of two initial value problems of that ordinary differential equation which have initial conditions $u(0) = \lambda$ and $\lim_{r\to\infty} r^{N-2}u(r) = \lambda$. And, examining whether those solutions have suitable matchings at $r = 1$, they argued about the existence and non-existence of radial solutions.

In [**S**], to prove Theorem 0.3, we also consider two initial value problems from $\pm\infty$, that is, for $\lambda_1, \lambda_2 > 0$, we consider the following two problems:

$$\begin{aligned} &-u'' + (1+b(x))u = f(u), \\ &\lim_{x\to-\infty} e^{-x}u(x) = \lim_{x\to-\infty} e^{-x}u'(x) = \lambda_1, \end{aligned} \tag{0.9}$$

and

$$\begin{aligned} &-u'' + (1+b(x))u = f(u), \\ &\lim_{x\to\infty} e^{x}u(x) = -\lim_{x\to\infty} e^{x}u(x) = \lambda_2. \end{aligned} \tag{0.10}$$

Then (0.9) and (0.10) have an unique solution respectively and write those solutions as $u_1(x;\lambda_1)$ and $u_2(x;\lambda_2)$ respectively. We set

$$\begin{aligned} \Gamma_1 &= \{(u_1(0;\lambda_1), u_1'(0;\lambda_1)) \in \mathbf{R}^2 \mid \lambda_1 > 0\}, \\ \Gamma_2 &= \{(u_2(0;\lambda_2), u_1'(0;\lambda_2)) \in \mathbf{R}^2 \mid \lambda_2 > 0\}. \end{aligned}$$

Then, $\Gamma_1 \cap \Gamma_2 = \emptyset$ is equivalent to the non-existence of solutions for $(*)$. Thus it is important to study shapes of $\Gamma_1$ and $\Gamma_2$. In respect to the details of proofs of Theorem 0.3, see [**S**].

In next sections, we state about the outline of the proof of Theorem 0.2. We will consider the one-dimensional case in Section 1 and treat the high-dimensional case in Section 2.

## 1. The outline of the proof of Theorem 0.2 for $N = 1$

In this section, we consider the case $N = 1$. We will developed a variational approach which was used in [**BaL**] and [**Sp**].

In what follows, since we seek positive solutions of $(*)$, without loss of generalities, we assume $f(u) = 0$ for $u < 0$. To prove Theorem 0.2, we seek non-trivial critical points of the functional

$$I(u) = \frac{1}{2}||u||^2_{H^1(\mathbf{R})} + \frac{1}{2}\int_{-\infty}^{\infty} b(x)u^2\,dx - \int_{-\infty}^{\infty} F(u)\,dx \in C^1(H^1(\mathbf{R}), \mathbf{R}),$$

whose critical points are positive solutions of $(*)$. Here we use the following notations:

$$\begin{aligned} ||u||^2_{H^1(\mathbf{R})} &= ||u'||^2_{L^2(\mathbf{R})} + ||u||^2_{L^2(\mathbf{R})}, \\ ||u||^p_{L^p(\mathbf{R})} &= \int_{\mathbf{R}} |u|^p\,dx \quad \text{for} \quad p > 1. \end{aligned}$$

From (f.1)–(f.2), we can see that $I(u)$ satisfies a mountain pass geometry (See Section 3 in [**JT2**].), that is, $I(u)$ satisfies

(i) $I(0)=0$.

(ii) There exist $\delta>0$ and $\rho>0$ such that $I(u)\geq\delta$ for all $||u||_{H^1(\mathbf{R})}=\rho$.

(iii) There exists $u_0\in H^1(\mathbf{R})$ such that $I(u_0)<0$ and $||u_0||_{H^1(\mathbf{R})}>\rho$.

From the mountain pass geometry (i)–(iii), we can define a standard minimax value $c>0$ by

$$c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)), \tag{1.1}$$
$$\Gamma=\{\gamma(t)\in C([0,1],H^1(\mathbf{R}))\,|\,\gamma(0)=0,\ I(\gamma(1))<0\}.$$

And, by a standard way, we can construct $(PS)_c$-sequence $(u_n)_{n=1}^\infty$, that is, $(u_n)_{n=1}^\infty$ satisfies

$$I(u_n)\to c \qquad (n\to\infty),$$
$$I'(u_n)\to 0 \quad \text{in} \quad H^{-1}(\mathbf{R}) \qquad (n\to\infty).$$

Moreover, since $(u_n)_{n=1}^\infty$ is bounded in $H^1(\mathbf{R})$ from (f.3), $(u_n)_{n=1}^\infty$ has a subsequence $(u_{n_j})_{j=1}^\infty$ which weakly converges to some $u_0$ in $H^1(\mathbf{R})$. If $(u_{n_j})_{j=1}^\infty$ strongly converges to $u_0$ in $H^1(\mathbf{R})$, $c$ is a non-trivial critical value of $I(u)$ and our proof is completed. However, since the embedding $L^p(\mathbf{R})\subset H^1(\mathbf{R})$ $(p>1)$ is not compact, there may not exist a subsequence $(u_{n_j})_{j=1}^\infty$ which strongly converges in $H^1(\mathbf{R})$. Therefore, in our situation, we don't know $c$ is a critical value.

In our situation, from the lack of the compactness mentioned the above, we must use the concentration-compactness approach as [**BaL**] and [**Sp**]. In the concentration-compactness approach, we examine in detail what happens in bounded (PS)-sequence. When we state the concentration-compactness argument for the (PS)-sequences of $I(u)$, the limit problem (0.1) plays an important role. Setting

$$I_0(u)=\frac{1}{2}||u||^2_{H^1(\mathbf{R})}-\int_{-\infty}^{\infty}F(u)\,dx\in C^1(H^1(\mathbf{R}),\mathbf{R}),$$

the critical points of $I_0(u)$ correspond to the solutions of limit problem (0.1). The equation (0.1) has an unique positive solution, identifying ones which obtain by translations. Thus let $\omega(x)$ be an unique positive solution of (0.1) with $\max_{x\in\mathbf{R}}\omega(x)=\omega(0)$ and we set $c_0=I_0(\omega)$. Since $I_0$ also satisfies the mountain pass geometry (i)–(iii), we see $c_0>0$ and $c_0$ is an unique non-trivial critical value.

For the bounded (PS)-sequences of $I(u)$, we have the following:

**Proposition 1.1.** *Suppose (b.1)–(b.2) and (f.1)–(f.2) hold. If $(u_n)_{n=1}^\infty$ is a bounded (PS)-sequence of $I(u)$, then there exist a subsequence $n_j\to\infty$, $k\in\mathbf{N}\cup\{0\}$, $k$-sequences*

$(x_j^1)_{j=1}^\infty, \cdots, (x_j^k)_{j=1}^\infty \subset \mathbf{R}$, *and a critical point* $u_0$ *of* $I(u)$ *such that*

$$\begin{aligned}
&I(u_{n_j}) \to I(u_0) + kc_0 \quad (j \to \infty),\\
&\left\| u_{n_j}(x) - u_0(x) - \sum_{\ell=1}^k \omega(x - x_j^\ell) \right\|_{H^1(\mathbf{R})} \to 0 \qquad (j \to \infty),\\
&|x_j^\ell - x_j^{\ell'}| \to \infty \quad (j \to \infty) \quad (\ell \neq \ell'),\\
&|x_j^\ell| \to \infty \quad (j \to \infty) \quad (\ell = 1, 2, \cdots, k).
\end{aligned}$$

**Proof.** We can easily get Proposition 1.1 from Theorem 5.1 of [**JT1**]. Theorem 5.1 of [**JT1**] required the assumption $\lim_{u\to\infty} f(u)u^{-p} = 0$ $(p > 1)$. However we take off that assumption for one dimensional case by improving Step 2 of Theorem 5.1 of [**JT1**]. In fact we have only to change $\sup_{z\in\mathbf{R}^N} \int_{B_1(z)} |v_n^1|^2\,dx \to 0$ in Step2 to $||v_n^1||_{L^\infty(\mathbf{R})} \to 0$. ∎

If the minimax value $c$ satisfies $c \in (0, c_0)$, from Proposition 1.1, we see that $I(u)$ has at least a non-trivial critical point. In fact, let $(u_n)_{n=1}^\infty$ be a bounded $(\mathrm{PS})_c$-sequence of $I(u)$, from Proposition 1.1, there exists a subsequence $n_j \to \infty$, $k \in \mathbf{N} \cup \{0\}$ and a critical point $u_0$ of $I(u)$ such that

$$I(u_{n_j}) \to I(u_0) + kc_0 \quad (j \to \infty).$$

Here, if $u_0 = 0$, we get $I(u_{n_j}) \to kc_0$ as $j \to \infty$. However this contradicts to the fact that $I(u_n) \to c \in (0, c_0)$ as $n \to \infty$. Thus $u_0 \neq 0$ and $u_0$ is a non-trivial critical point of $I(u)$. From the above argument, we have the following corollary.

**Corollary 1.2.** *Suppose* $I(u)$ *has no non-trivial critical points and let* $(u_n)_{n=1}^\infty$ *be a (PS)-sequence of* $I(u)$. *Then, only* $kc_0$*'s* ($k \in \mathbf{N} \cup \{0\}$) *can be limit points of* $\{I(u_n) \,|\, n \in \mathbf{N}\}$.

**Remark 1.3.** Corollary 1.2 essentially depends on the uniqueness of the positive solution of (0.1).

As mentioned the above, when $c \in (0, c_0)$, $I(u)$ has at least a non-trivial critical point. However, unfortunately, under the condition (b.1)–(b.4), it may be $c = c_0$. Thus we need consider another minimax value. To define another minimax value, we use a path $\gamma_0(t) \in C(\mathbf{R}, H^1(\mathbf{R}))$ which is defined as follows: for small $\epsilon_0 > 0$, we set

$$h(x) = \begin{cases} \omega(x) & x \in [0, \infty],\\ x^4 + u_0 & x \in [-\epsilon_0, 0),\\ \epsilon_0^4 + u_0 & x \in (-\infty, -\epsilon_0), \end{cases}$$

$$\gamma_0(t)(x) = \begin{cases} h(x - t) & x \geq 0,\\ h(-x - t) & x < 0. \end{cases}$$

Here, we remark that $u_0$ was given in (f.2). This path $\gamma_0(t)$ was introduced in [**JT2**]. Choosing a proper $\epsilon_0 > 0$ sufficiently small, $\gamma_0(t)$ achieves the mountain pass value of $I_0(u)$ and satisfies the followings:

**Lemma 1.4.** *Suppose (f.1)–(f.2) hold. Then $\gamma_0(t)$ satisfies*

*(i)* $\gamma_0(0)(x) = \omega(x)$.

*(ii)* $I_0(\gamma_0(t)) < I_0(\omega) = c_0$ *for all* $t \neq 0$.

*(iii)* $\lim_{t\to-\infty} ||\gamma_0(t)||_{H^1(\mathbf{R})} = 0,\ \lim_{t\to\infty} ||\gamma_0(t)||_{H^1(\mathbf{R})} = \infty$.

**Proof.** See Section 3 in [**JT2**].

**Remark 1.5.** When $f(u)/u$ is a increasing function, we can use a simpler path than $\gamma_0(t)$. In fact, setting $\widetilde{\gamma}_0(t) = t\omega : [0,\infty) \to H^1(\mathbf{R})$, we also have

(i) $\widetilde{\gamma}_0(1)(x) = \omega(x)$.

(ii) $I_0(\widetilde{\gamma}_0(t)) < I_0(\omega) = c_0$ for all $t \neq 1$.

(iii) $\widetilde{\gamma}_0(0) = 0,\ \lim_{t\to\infty} ||\widetilde{\gamma}_0(t)||_{H^1(\mathbf{R})} = \infty$.

Moreover, if $f(u)/u$ is a increasing function, in what follows, we can also construct a simpler proofs by aruging on Nehari manifold $N = \{u \in H^1(\mathbf{R}) \setminus \{0\} \mid I'(u)u = 0\}$. (See [**Sp**].)

Now, for $R > 0$, we consider a path $\gamma_R \in C(\mathbf{R}^2, H^1(\mathbf{R}))$ which is defined by

$$\gamma_R(s,t)(x) = \max\{\gamma_0(s)(x+R),\ \gamma_0(t)(x-R)\}.$$

In our proof of Theorem 0.2 in [**S**], the following proposition is a key proposition.

**Proposition 1.6.** *Suppose (b.1)–(b.3) and (f.1)–(f.2) hold. Then, for any $L > 0$, we have*

$$\lim_{R\to\infty} e^{2R}\left\{\max_{(s,t)\in[-L,L]^2} I(\gamma_R(s,t)) - 2c_0\right\} \leq \frac{\lambda_0^2}{2}\left(\int_{-\infty}^{\infty} b(x)e^{2|x|}\,dx - 2\right). \tag{1.2}$$

*Here* $\lambda_0 = \lim_{x\to\pm\infty} \omega(x)e^{|x|}$.

**Proof.** See [**S**].

By using a translation, without loss of generalities, we assume $x_0 = 0$ in (b.4). If (b.4) with $x_0 = 0$ holds, from Proposition 1.6, for any $L > 0$, there exists $R_0 > 0$ such that

$$\max_{(s,t)\in[-L,L]^2} I(\gamma_{R_0}(s,t)) < 2c_0.$$

To prove the Theorem 0.2, we also need a map $m : H^1(\mathbf{R}) \setminus \{0\} \to \mathbf{R}$ which is defined by the following: for any $u \in H^1(\mathbf{R}) \setminus \{0\}$, a function

$$T_u(s) = \int_{-\infty}^{\infty} \tan^{-1}(x-s)|u(x)|^2\,dx : \mathbf{R} \to \mathbf{R}$$

is strictly decreasing and $\lim_{s\to\infty} T_u(s) = -||u||^2_{L^2(\mathbf{R})} < 0$ and $\lim_{s\to-\infty} T_u(s) = ||u||^2_{L^2(\mathbf{R})} > 0$. Thus, from the theorem of the intermediate value, $T_u(s)$ has an unique $s = m(u)$ such that $T_u(m(u)) = 0$. We also find that $m(u)$ is of continuous by the implicit function theorem to $(u,s) \mapsto T_u(s)$. The map $m(u)$ was introduced in [**Sp**]. We remark that $m(u)$ is regarded as a kind of center of mass of $|u(x)|^2$ and we can check the followings.

**Lemma 1.7.** *We have*
*(i)* $m(\gamma_0(t)) = 0$ *for all* $t \in \mathbf{R}$.
*(ii)* $m(\gamma_R(s,t)) > 0$ *for all* $-R < s < t < R$.
*(iii)* $m(\gamma_R(s,t)) < 0$ *for all* $-R < t < s < R$.

**Proof.** Since $\gamma_0(t)(x)$ is a even function, we have (i). We Note that

$$\gamma_R(s,t)(x) = \begin{cases} \gamma_0(s)(x+R) & \text{for } x \in (-\infty, \frac{s-t}{2}], \\ \gamma_0(t)(x-R) & \text{for } x \in (\frac{s-t}{2}, \infty). \end{cases}$$

Since $\gamma_R(s,s)(x)$ is also a even function, we have

$$m(\gamma_R(s,s)) = 0 \quad \text{for all} \quad s \in \mathbf{R},$$

and we get (ii)–(iii). ∎

In what follows, we will complete the proof of Theorem 0.2 for $N = 1$.

**Proof of Theorem 0.2 for $N = 1$.** First of all, we defined a minimax value $c_1 > 0$ by

$$c_1 = \inf_{\gamma\in\Gamma_1} \max_{t\in[0,1]} I(\gamma(t)),$$
$$\Gamma_1 = \{\gamma(t) \in C([0,1], H^1(\mathbf{R})) \mid \gamma(0) = 0,\ I(\gamma(1)) < 0,\ |m(\gamma(t))| < 1\}.$$

Noting $\Gamma_1 \subset \Gamma$, we have

$$0 < c \le c_1.$$

Since $\Gamma_1$ is not invariant by standard deformation flows of $I(u)$, $c_1$ may not be a critical point of $I(u)$. We will use $c_1$ to divide the case. We divide the case into the following three cases:

(i) $c_1 < c_0$.

(ii) $c_1 = c_0$.

(iii) $c_1 > c_0$.

**Proof of Theorem 0.2 for the case (i).** Since the inequality $c_1 < c_0$ implies $0 < c < c_0$, from Corollary 1.2, we can see $I(u)$ has at least a non-trivial critical point. ∎

**Proof of Theorem 0.2 for the case (ii).** In this case, if $c < c_1 = c_0$, then $I(u)$ has at least a non-trivial critical point from Corollary 1.2. Thus we may consider the case $c = c_1 = c_0$. In this case, for any $\epsilon > 0$, there exists $\gamma_\epsilon(t) \in \Gamma_1$ such that

$$c \leq \max_{t\in[0,1]} I(\gamma_\epsilon(t)) < c + \epsilon.$$

Since $\gamma_\epsilon \in \Gamma_1 \subset \Gamma$ and $\Gamma$ is an invariant set by standard deformation flows of $I(u)$, by a standard Ekland principle, there exists $u_\epsilon \in H^1(\mathbf{R})$ such that

$$\begin{aligned}
&c \leq I(u_\epsilon) \leq \max_{t\in[0,1]} I(\gamma_\epsilon(t)) < c + \epsilon,\\
&||I'(u_\epsilon)|| < 2\sqrt{\epsilon},\\
&\inf_{t\in[0,1]} ||u_\epsilon - \gamma_\epsilon(t)||_{H^1(\mathbf{R})} < \epsilon.
\end{aligned}\tag{1.3}$$

Then, from Proposition 1.1, there exist a subsequence $\epsilon_j \to 0$, $k \in \mathbf{N}\cup\{0\}$, $k$-sequences $(x_j^1)_{j=1}^\infty, \cdots, (x_j^k)_{j=1}^\infty \subset \mathbf{R}$, and a critical point $u_0$ of $I(u)$ such that

$$\begin{aligned}
&I(u_{\epsilon_j}) \to I(u_0) + kc_0 \quad (j \to \infty),\\
&\left|\left| u_{\epsilon_j}(x) - u_0(x) - \sum_{\ell=1}^{k} \omega(x - x_j^\ell) \right|\right|_{H^1(\mathbf{R})} \to 0 \qquad (j \to \infty),\\
&|x_j^\ell - x_j^{\ell'}| \to \infty \quad (j \to \infty) \quad (\ell \neq \ell'),\\
&|x_j^\ell| \to \infty \quad (j \to \infty) \quad (\ell = 1, 2, \cdots, k).
\end{aligned}\tag{1.4}$$

Now, if $u_0 \neq 0$, our proof is completed. So we suppose $u_0 = 0$. Then, from (1.4), it must be $k = 1$. Thus, we have

$$\begin{aligned}
&||u_{\epsilon_j}(x) - \omega(x - x_j^1)||_{H^1(\mathbf{R})} \to 0 \qquad (j \to \infty).\\
&|x_j^1| \to \infty \quad (j \to \infty).
\end{aligned}\tag{1.5}$$

On the other hand, we remark that, since $m(\omega) = 0$ and $m$ is of continuous, there exists $\delta > 0$ such that

$$|m(u)| < 1 \quad \text{for all} \quad u \in B_\delta(\omega) = \{v \in H^1(\mathbf{R}) \,|\, ||v - \omega||_{H^1(\mathbf{R})} < \delta\}.$$

Thus, from (1.3) and (1.5), for some $\epsilon_0 \in (0, \frac{\delta}{2})$ and $t_0 \in [0,1]$, we have

$$|m(\gamma_{\epsilon_0}(t_0)) - x_j^1| < 1.$$

This contradicts to $\gamma_{\epsilon_0} \in \Gamma_1$. Therefore $u_0 \neq 0$ and $I(u)$ has at least a non-trivial critical point. ∎

**Proof of the Theorem 0.2 for the case (iii).** First of all, we set $\delta = \frac{c_1 - c_0}{2} > 0$ and choose $L_0 > 0$ such that

$$\max_{(s,t) \in \overline{D_{2L_0} \setminus D_{L_0}}} I(\gamma_R(s,t)) < c_0 + \delta < c_1 \quad \text{for all} \quad R > 3L_0. \tag{1.6}$$

Here we set $D_L = [L, L] \times [L, L] \subset \mathbf{R}^2$. Next, from Proposition 1.6, we can choose $R_0 > 3L_0$ such that

$$\max_{(s,t) \in D_{L_0}} I(\gamma_{R_0}(s,t)) < 2c_0. \tag{1.7}$$

Here we fix $\gamma_{R_0}(s,t)$ and define the following minimax value:

$$c_2 = \inf_{\gamma \in \Gamma_2} \max_{(s,t) \in D_{2L_0}} I(\gamma(s,t)),$$

$$\Gamma_2 = \{\gamma(s,t) \in C(D_{2L_0}, H^1(\mathbf{R})) \,|\, \gamma(s,t) = \gamma_{R_0}(s,t) \text{ for all } (s,t) \in D_{2L_0} \setminus D_{L_0}\}.$$

Then we have the following lemma.

**Lemma 1.8.** *We have*

$$0 < c_0 < c_1 \leq c_2 < 2c_0.$$

We postpone the proof of Lemma 1.8 to end of this section. If Lemma 1.8 is true, then $\Gamma_2$ is an invariant set by the deformation flows of $I(u)$. Thus $I(u)$ has a (PS)-sequence $(u_n)_{n=1}^{\infty}$ such that

$$I(u_n) \to c_2 \in (c_0, 2c_0) \quad (n \to \infty).$$

From Corollary 1.2, we can see that $I(u)$ must have at least a non-trivial critical point. Combining the proofs of the cases (i)–(iii), we complete a proof of Theorem 0.2. ∎

Finally we show Lemma 1.8.

**Proof of Lemma 1.8.** The inequality $c_0 < c_1$ is an assumption of the case (iii). From $\gamma_{R_0} \in \Gamma_2$ and (1.6)–(1.7), $c_2 < 2c_0$ is obvious. Thus we show $c_1 \leq c_2$. For any $\gamma(s,t) \in \Gamma_2$, we have

$$m(\gamma(s,t)) > 0 \quad \text{for all} \quad (s,t) \in D_1, \tag{1.8}$$

$$m(\gamma(s,t)) < 0 \quad \text{for all} \quad (s,t) \in D_2. \tag{1.9}$$

Here we set $D_1 = \{(s,t) \in D_{2L_0} \setminus D_{L_0} \,|\, s < t\}$ and $D_2 = \{(s,t) \in D_{2L_0} \setminus D_{L_0} \,|\, s > t\}$. From (1.8)–(1.9), a set $\{(s,t) \in D_{2L_0} \,|\, |m(\gamma(s,t))| < 1\}$ have a connected component which contains a path joining two points $\gamma_{R_0}(-2L_0, -2L_0)$ and $\gamma_{R_0}(2L_0, 2L_0)$. Thus we construct a path $\gamma_1(t) \in \Gamma_1$ such that

$$\{\gamma_1(t) \,|\, t \in [1/3, 2/3]\} \subset \{\gamma(s,t) \,|\, (s,t) \in D_{2L_0}\},$$
$$\max_{t \in [0,1/3] \cup [2/3,1]} I(\gamma_1(t)) \leq c_0.$$

Thus we see

$$\begin{aligned} c_1 &\leq \max_{t \in [0,1]} I(\gamma_1(t)) \\ &\leq \max_{(s,t) \in D_{2L_0}} I(\gamma(s,t)). \end{aligned} \tag{1.10}$$

Since $\gamma(s,t) \in \Gamma_2$ is arbitrary, from (1.10), we have

$$c_1 \leq c_2.$$

Thus we get Lemma 1.8. ∎

**Remark 1.9.** In our proofs of Theorem 0.2, the path $\gamma_R(s,t)$ played an important role. In particular, the estimate (1.2) was an important. However, we don't know that $\gamma_R(s,t)$ is the best path to show the existence of positive solutions of $(*)$. Using other path, we might be able to get better estimate than (1.2). Instead of $\gamma_R(s,t)$, we can consider another path $\widetilde{\gamma}_R \in C(\mathbf{R}^2, H^1(\mathbf{R}))$ which is defined by

$$\widetilde{\gamma}_R(s,t)(x) = \gamma_0(s)(x+R) + \gamma_0(t)(x-R).$$

We remark that $\widetilde{\gamma}_R(s,t)$ is a natural path because we can regard $\widetilde{\gamma}_R(s,t)$ as one-dimensional version of the path which was used in the proof of the high-dimensional case. (See Proposition 2.2.) Estimating $\widetilde{\gamma}_R(s,t)$ by similar way to (1.2), for any $L > 0$, we have

$$\lim_{R \to \infty} e^{2R} \left\{ \max_{(s,t) \in [-L,L]^2} I(\widetilde{\gamma}_R(s,t)) - 2c_0 \right\} \leq \frac{\lambda_0^2}{2} \left( \int_{-\infty}^{\infty} b(x)(e^{2x} + e^{-2x} + 2)\, dx - 4 \right).$$

We see that, if $\int_{-\infty}^{\infty} b(x)(e^{2x} + e^{-2x} + 2)\, dx < 4$ holds, then $\int_{-\infty}^{\infty} b(x) e^{2|x|}\, dx < 2$ also holds. Thus $\gamma_R(s,t)$ provides a better estimate than $\widetilde{\gamma}_R(s,t)$.

## 2. The outline of the proof of Theorem 0.2 for $N \geq 2$

In this section, we consider the case $N \geq 2$. We remark that, when $N \geq 2$, our proofs almost are parallel to [**BaL**]. We assume $f(u) = u^p$ for $u \geq 0$ and $f(u) = 0$ for $u < 0$, where $p \in (1, \frac{N+2}{N-2})$ when $N \geq 3$, $p \in (1, \infty)$ when $N = 2$. We set

$$I(u) = \frac{1}{2}||u||^2_{H^1_b(\mathbf{R}^N)} - ||u_+||^{p+1}_{L^{p+1}(\mathbf{R}^N)} \in C^2(H^1(\mathbf{R}^N), \mathbf{R}),$$

where

$$||u||^2_{H^1_b(\mathbf{R}^N)} = ||u||^2_{H^1(\mathbf{R}^N)} + \int_{\mathbf{R}^N} b(x)u^2\, dx$$

By the standard ways, we reduce $I_b$ to a functional

$$J(v) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{||v||_{H^1_b(\mathbf{R}^N)}}{||v_+||_{L^{p+1}(\mathbf{R}^N)}}\right)^{\frac{2(p+1)}{p-1}}.$$

which is defined on

$$\Sigma = \{v \in H^1(\mathbf{R}^N) \,|\, ||v||_{H^1(\mathbf{R}^N)} = 1, v_+ \neq 0\}.$$

Then $J \in C^1(\Sigma, \mathbf{R})$ and, for any critical point $v \in \Sigma$ of $J(v)$, $t_v v$ is a non-trivial critical point of $I(u)$ where $t_v = ||v||^{\frac{2}{p-1}}_{H^1_b(\mathbf{R}^N)}||v_+||^{-\frac{p+1}{p-1}}_{L^{p+1}(\mathbf{R}^N)}$. Thus, in what follows, we seek non-trivial critical points of $J(v)$.

Let $\omega(x)$ be an unique radially symmetric positive solution of (0.1) for $f(u) = u^p$ and we set $c_0 = \frac{1}{2}||\omega||^2_{H^1(\mathbf{R}^N)} - \frac{1}{p+1}||\omega||_{H^1(\mathbf{R}^N)} > 0$. For the (PS)-sequences of $J(u)$, we have the following:

**Proposition 2.1.** *Suppose (b.1)–(b.2), (f.0) hold and let $(v_n)_{n=1}^\infty$ be a (PS)-sequence of $J(u)$. Then there exist a subsequence $n_j \to \infty$, $k \in \mathbf{N} \cup \{0\}$, $k$-sequences $(x_j^1)_{j=1}^\infty, \cdots, (x_j^k)_{j=1}^\infty \subset \mathbf{R}^N$, and a critical point $u_0$ of $I(u)$ such that*

$$J(v_{n_j}) \to I(u_0) + kc_0 \quad (j \to \infty),$$

$$v_{n_j}(x) - \frac{u_0(x) - \sum_{\ell=1}^k \omega(x - x_j^\ell)}{\left|\left|u_0(x) - \sum_{\ell=1}^k \omega(x - x_j^\ell)\right|\right|_{H^1(\mathbf{R}^N)}} \to 0 \quad in \quad H^1(\mathbf{R}^N) \qquad (j \to \infty),$$

$$|x_j^\ell - x_j^{\ell'}| \to \infty \quad (j \to \infty) \quad (\ell \neq \ell'),$$

$$|x_j^\ell| \to \infty \quad (j \to \infty) \quad (\ell = 1, 2, \cdots, k).$$

**Proof.** Let $(v_n)_{n=1}^\infty$ be a (PS)-sequence of $J(v)$. Then $(t_{v_n} v_n)_{n=1}^\infty$ is a (PS)-sequence of $I(u)$. Moreover we remark that the set of the critical points of the functional $\frac{1}{2}||u||^2_{H^1(\mathbf{R}^N)} - \frac{1}{p+1}||u_+||_{H^1(\mathbf{R}^N)} : H^1(\mathbf{R}^N) \to \mathbf{R}$ is written by $\{\omega(x+\xi) \,|\, \xi \in \mathbf{R}^N\} \cup \{0\}$ from the uniqueness of positive solutions of (1.0). Thus Proposition 2.1 easily follows applying Theorem 5.1 of [**JT1**] to $(t_{v_n} v_n)_{n=1}^\infty$. ∎

By the similar arguments of Section 1, we have the following corollary.

**Corollary 2.2.** *Suppose $I(u)$ has no non-trivial critical points and let $(v_n)_{n=1}^{\infty}$ be a (PS)-sequence of $J(v)$. Then, only $kc_0$'s ($k \in \mathbf{N}$) can be limit points of $\{J(v_n) \mid n \in \mathbf{N}\}$.*

We set

$$c = \inf_{v \in \Sigma} J(v).$$

Then we can easily see that $0 < c \leq c_0$. From the boundedness of $J(v)$ from below, we get also more strong corollary.

**Corollary 2.3.** *For any $b \in (-\infty, c_0) \cup (c_0, c_0 + c)$, $J(v)$ satisfies (PS)$_b$-condition.*

**Proof.** If (PS)$_b$-condition does not hold for $b \in \mathbf{R}$, then for some (PS)$_b$-sequence $(v_n)_{n=1}^{\infty}$, it must be $k \neq 0$ in Proposition 2.1. Thus we have

$$\lim_{n\to\infty} J(v_n) = b = kc_0 \quad \text{or} \quad \lim_{n\to\infty} J(v_n) = b \geq c + kc_0.$$

This implies Corollary 2.3. ∎

When $c < c_0$, from Corollary 2.3, $c$ is a critical value of $J(v)$. Thus this case is easy. Thus we consider the case $c = c_0$. When $c = c_0$, we must define another minimax value. To define another minimax value, the following proposition is important.

**Proposition 2.4.** *Suppose $N \geq 2$, (b.1)–(b.3) and (f.0) hold. Then, there exists $R_0 > 0$ such that for any $R \geq R_0$, we have*

$$\max_{(\zeta,\xi,t)\in \partial B_{\frac{1}{2}R}\times\partial B_R\times[0,1]} J\left(\frac{t\omega(x-\zeta)+(1-t)\omega(x-\xi)}{||t\omega(x-\zeta)+(1-t)\omega(x-\xi)||_{H^1(\mathbf{R}^N)}}\right) < 2c_0. \tag{2.1}$$

*Here $B_R = \{x \in \mathbf{R}^N \mid |x| \leq R\}$.*

**Proof.** To get (2.1), for large $R > 0$, it sufficient to show

$$\max_{(\zeta,\xi,s,t)\in \partial B_{\frac{1}{2}R}\times\partial B_R\times\mathbf{R}^2} I(s\omega(x-\zeta)+t\omega(x-\xi)) < 2c_0. \tag{2.2}$$

In many papers [**BaL**], [**A**], [**H1**], [**H2**], the estimates like (2.2) were obtained. In [**A**], [**H1**], [**H2**], they treated more general $f(u)$ including $u_+^p$. Since we can get (2.2) by similar ways to those calculations, we omit the proof of (2.2). ∎

**Remark 2.5.** When $N = 1$, the estimate (2.1) does not hold. (See Proposition 1.6 and [**S**].) We remark that, for some $C_0 > 0$, $\omega(x)$ satisfies

$$0 < \omega(x) \leq C_0|x|^{-\frac{N-1}{2}}e^{-|x|} \qquad \text{for all} \quad x \in \mathbf{R}^N. \tag{2.3}$$

Roughly explaining about the difference from $N = 1$ and $N \geq 2$, when $N \geq 2$, we can obtain (2.1) by the effect of $|x|^{-\frac{N-1}{2}}$ in (2.3). On the other hand, when $N = 1$, since the effect of $|x|^{-\frac{N-1}{2}}$ vanishes, (2.1) does not hold.

To prove the Theorem 0.2, we also define a map $m : H^1(\mathbf{R}^N) \setminus \{0\} \to \mathbf{R}^N$ which is an expansion of $m$ defined in Section 1. That is, for any $u \in H^1(\mathbf{R}^N) \setminus \{0\}$, we consider a map

$$\begin{aligned} T_u(\xi) = & \left( \int_{\mathbf{R}^N} \tan^{-1}(x_1 - \xi_1)|u(x)|^2\, dx, \cdots, \int_{\mathbf{R}^N} \tan^{-1}(x_N - \xi_N)|u(x)|^2\, dx \right) \\ & : \mathbf{R}^N \to \mathbf{R}^N . \end{aligned}$$

Then we can see that $T_u(\xi)$ has an unique $\xi_u \in \mathbf{R}^N$ such that $T_u(\xi_u) = 0$ because

$$DT_u = \begin{bmatrix} \int_{\mathbf{R}^N} \frac{1}{1+(x_1-\xi_1)^2}|u(x)|^2\, dx & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \int_{\mathbf{R}^N} \frac{1}{1+(x_N-\xi_N)^2}|u(x)|^2\, dx \end{bmatrix}.$$

Thus for any $u \in H^1(\mathbf{R}^N) \setminus \{0\}$, we define $m(u) = \xi_u$. We also find that $m(u)$ is of continuous by the implicit function theorem to $(u, \xi) \mapsto T_u(\xi)$. Since $\omega(x)$ is a radially symmetric function, from the definition of $m(u)$, we can easily see that

$$m(\omega(x - \xi)) = \xi \qquad \text{for all} \quad \xi \in \mathbf{R}^N . \tag{2.4}$$

In what follows, we will complete the proof of Theorem 0.2.

**Proof of Theorem 0.2 for $N \geq 2$.** We set

$$c = \inf_{v \in \Sigma} J(v).$$

When $c < c_0$, from Corollary 2.3, $c$ is a critical point of $J(v)$ and our proof is completed. Thus we must consider the case $c = c_0$. For $a \in \mathbf{R}^N$ we defined a minimax value $c_a > 0$ by

$$\begin{aligned} c_a &= \inf_{v \in \Sigma_a} J(v), \\ \Sigma_a &= \{v \in \Sigma \,|\, m(v) = a\}. \end{aligned}$$

Noting $\Sigma_a \subset \Sigma$ and $c = c_0$, we have

$$0 < c_0 \leq c_a.$$

We will show that $I(u)$ has at least a non-trivial critical point for the following both cases:

(i) For some $a \in \mathbf{R}^N$, $c_0 = c_a$.

(ii) For some $a \in \mathbf{R}^N$, $c_0 < c_a$.

**Proof of Theorem 0.2 for the case (i).** For any $\epsilon > 0$, there exists $\tilde{v}_\epsilon \in \Sigma_a$ such that

$$c_0 \leq J(\tilde{v}_\epsilon) < c_0 + \epsilon.$$

Since $\tilde{v}_\epsilon \in \Sigma_a \subset \Sigma$ and $\Sigma$ is an invariant set by standard deformation flows of $J(v)$, by a standard Ekland principle, there exists $v_\epsilon \in \Sigma$ such that

$$\begin{aligned} &c_0 \leq J(v_\epsilon) \leq J(\tilde{v}_\epsilon) < c_0 + \epsilon, \\ &||J'(v_\epsilon)|| < 2\sqrt{\epsilon}, \\ &||v_\epsilon - \tilde{v}_\epsilon||_{H^1(\mathbf{R})} < \epsilon. \end{aligned} \tag{2.5}$$

Then, from Proposition 2.1, there exist a subsequence $\epsilon_j \to 0$, $k \in \mathbf{N} \cup \{0\}$, $k$-sequences $(x_j^1)_{j=1}^\infty, \cdots, (x_j^k)_{j=1}^\infty \subset \mathbf{R}^N$, and a critical point $u_0$ of $I(u)$ such that

$$J(v_{\epsilon_j}) \to I(u_0) + kc_0 \quad (j \to \infty), \tag{2.6}$$

$$v_{n_j}(x) - \frac{u_0(x) - \sum_{\ell=1}^k \omega(x - x_j^\ell)}{\left|\left|u_0(x) - \sum_{\ell=1}^k \omega(x - x_j^\ell)\right|\right|_{H^1(\mathbf{R}^N)}} \to 0 \quad \text{in} \quad H^1(\mathbf{R}^N) \qquad (j \to \infty),$$

$$|x_j^\ell - x_j^{\ell'}| \to \infty \quad (j \to \infty) \quad (\ell \neq \ell'),$$

$$|x_j^\ell| \to \infty \quad (j \to \infty) \quad (\ell = 1, 2, \cdots, k).$$

Now, if $u_0 \neq 0$, our proof is completed. So we suppose $u_0 = 0$. Then, from (2.6), it must be $k = 1$. Thus, we have

$$\begin{aligned} &\left|\left| v_{\epsilon_j}(x) - \frac{\omega(x - x_j^1)}{||\omega||_{H^1(\mathbf{R}^N)}} \right|\right|_{H^1(\mathbf{R}^N)} \to 0 \qquad (j \to \infty), \\ &|x_j^1| \to \infty \quad (j \to \infty). \end{aligned} \tag{2.7}$$

From (2.4), (2.5) and (2.7), we see that

$$|m(\tilde{v}_{\epsilon_j})| \to \infty \qquad \text{as} \quad j \to \infty.$$

This contradicts to $m(\tilde{v}_{\epsilon_j}) = a$. Therefore $u_0 \neq 0$ and $I(u)$ has at least a non-trivial critical point. ∎

**Proof of the Theorem 0.2 for the case (ii).** From Proposition 2.4, we set $\zeta_0 = (\frac{1}{2}R_0, 0, \cdots, 0)$ and $\delta = \frac{1}{2}(c_a - c_0) > 0$ and choose a large $R_0 > |a|$ such that

$$\max_{\xi \in \partial B_{R_0}} J(\omega(x-\xi)) < c_0 + \delta < c_a, \tag{2.8}$$

$$\max_{(\xi,t)\in\partial B_{R_0}\times[0,1]} J\left(\frac{t\omega(x-\zeta_0)+(1-t)\omega(x-\xi)}{||t\omega(x-\zeta_0)+(1-t)\omega(x-\xi)||_{H^1(\mathbf{R}^N)}}\right) < 2c_0. \tag{2.9}$$

Here we define the following minimax value:

$$c_2 = \inf_{\gamma\in\Gamma}\max_{\xi\in B_{R_0}} J(\gamma(\xi)),$$

$$\Gamma = \left\{\gamma(\xi)\in C(B_{R_0},\Sigma)\ \middle|\ \gamma(\xi)(x) = \frac{\omega(x+\xi)}{||\omega||_{H^1(\mathbf{R}^N)}} \text{ for all } \xi\in\partial B_{R_0}\right\}.$$

Then we have the following lemma.

**Lemma 2.6.** *We have*

$$0 < c_0 < c_a \le c_2 < 2c_0.$$

We postpone the proof of Lemma 2.6 to end of this section. If Lemma 2.6 is true, then $\Gamma$ is an invariant set by the deformation flows of $J(v)$. Thus $J(v)$ has a (PS)-sequence $(v_n)_{n=1}^{\infty}$ such that

$$J(v_n) \to c_2 \in (c_0, 2c_0) \quad (n\to\infty).$$

From Corollary 2.3, $J(u)$ satisfies $(\mathrm{PS})_{c_2}$-conditions. Thus $c_2$ is a critical value of $J(v)$. That is, $I(u)$ has at least a non-trivial critical point. Combining the proofs of the cases (i)–(ii), we complete a proof of Theorem 0.2. ∎

Finally we show Lemma 2.6.

**Proof of Lemma 2.6.** The inequality $c_0 < c_a$ is an assumption of the case (ii). From (2.9), $c_2 < 2c_0$ is obvious. Thus we show $c_a \le c_2$. For any $\gamma\in\Gamma$, from (2.10), we have

$$m(\gamma(\xi)) = \xi \quad \text{for all} \quad \xi\in\partial B_{R_0}.$$

Thus we can see

$$\deg(m\circ\gamma, B_{R_0}, a) = 1. \tag{2.10}$$

From (2.10), there exists $\xi_0\in B_{R_0}$ such that $m(\gamma(\xi_0)) = a$. Therefore, since $\gamma(\xi_0)\in\Sigma_a$, we find that

$$\begin{aligned} c_a &\le \inf_{v\in\Sigma_a} J(v) \\ &\le J(\gamma(\xi_0))) \\ &\le \max_{\xi\in B_{R_0}} I(\gamma(\xi)). \end{aligned} \tag{2.11}$$

Since $\gamma \in \Gamma$ is arbitrary, from (2.11), we have

$$c_a \leq c_2.$$

Thus we get Lemma 2.6. ∎

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