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On the nonuniqueness of positive solutions of boundary value problems for superlinear Emden-Fowler equations

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We consider the two-point boundary value problem

$$(1)\qquad \begin{cases} u'' + h(x)u^p = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases}$$

where $p > 1$, $h \in C^1([-1,0) \cup (0,1]) \cup C[-1,1]$ and $h(x) > 0$ for $x \in [-1,0) \cup (0,1]$.

A large number of studies have been made on the Emden-Fowler differential equation

$$(2)\qquad u'' + h(x)u^p = 0.$$

See, for example, Naito [8] and Wong [17]. Equation (2) is one of the origins of various equations such as the equation with one-dimensional p-Laplacian

$$(|u'|^{p-2}u')' + h(x)f(u) = 0$$

and elliptic partial differential equations of the form

$$\Delta u + K(|x|)u^p = 0.$$

It is well-known that if $p > 0$ and $p \neq 1$, then problem (1) has at least one positive solution. See, for example, [6], [9] and [16]. It is also well-known that if $0 < p < 1$, then the positive solution is unique. See, for example, [10]. In the case $p > 1$, sufficient conditions for the uniqueness of positive solutions were obtained in [2], [3], [4], [5], [7], [11], [13], [14] and [18]. However, there are still unknown cases whether the positive solution is unique or not. Therefore, in this paper, we concentrate on the case where $h(x)$ is an even function, that is,

$$h(-x) = h(x), \quad -1 \leq x \leq 1. \tag{3}$$

Then we can see that problem (1) has at lease one even positive solution. In the case (3), if one of the following conditions (4) or (5) is satisfied, then the positive solution of problem (1) is unique:

$$h'(x) \leq 0, \quad 0 \leq x \leq 1; \tag{4}$$

$$-\frac{2}{x+1} \leq \frac{h'(x)}{h(x)} \leq \frac{2}{1-x}, \quad 0 \leq x < 1; \tag{5}$$

By the result of Moroney [7], we can obtain condition (4). Kwong [4] established condition (5). It should be noted that (4) and (5) are the conditions for more general equations such as $u'' + h(x)f(u) = 0$ or $u'' + f(x,u) = 0$. In [15], by studying only the special problem (1), the following sufficient condition for the uniqueness of positive solutions is obtained:

$$p \max_{-1 \leq x \leq 1} \min \left\{ \frac{(x+1)^p}{\int_{-1}^{x} (s+1)^{p+1} h(s) ds}, \frac{(1-x)^p}{\int_{x}^{1} (1-s)^{p+1} h(s) ds} \right\} \leq \lambda_2, \tag{6}$$

where λ_2 is the second eigenvalue of

$$\begin{cases} \varphi'' + \lambda h(x) \varphi = 0, & -1 < x < 1, \\ \varphi(-1) = \varphi(1) = 0, \end{cases} \tag{7}$$

It is well-known that problem (7) has infinitely many eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \lambda_{k+1} < \cdots, \quad \lambda_k \to \infty \quad \text{as } k \to \infty$$

and no other eigenvalues.

There are only a few nonuniqueness results. Moore and Nehari [6] showed that if $h(x) = 1$ on $[-1, -c] \cup [c, 1]$ and $h(x) = 0$ on $(-c, c)$, then (1) has at least three positive solutions, where $c \in (-1, 1)$ is some number. See also [14]. It is shown by Smets, Willem and Su [12] that, for each $p \in (1, p^*)$, there exists $l^* > 0$ such that if $l \geq l^*$, then

$$\text{(8)} \qquad \begin{cases} \Delta u + |x|^l u^p = 0 & \text{in } B, \\ u = 0 & \text{on } \partial B \end{cases}$$

has a radial positive solution and a nonradial positive solution, where B denotes the unit ball in $\mathbf{R}^N$, $N \geq 2$,

$$p^* = \begin{cases} \infty, & N = 2, \\ (N+2)/(N-2), & N \geq 3. \end{cases}$$

Later, for the case $N \geq 1$, the existence of nonradial positive solutions of (8) has been established by Byeon and Wang [1].

The main result of this paper is as follows:

Theorem 1. *Suppose that* (3) *and the following conditions* (9) *and* (10) *hold:*

$$\text{(9)} \qquad \frac{xh'(x)}{h(x)} \text{ is nonincreasing on } [0,1],$$

$$\text{(10)} \qquad \frac{h'(x)}{h(x)} \geq \frac{4-(p+3)x}{(1-x)^2}, \quad x \in [0,1).$$

Then (1) *has an even positive solution and two non-even positive solutions, and the even positive solution is unique.*

We note here that the uniqueness of even positive solutions has proved by Yanagida [18] when (3) and (9). Roughly speaking, from (4), (5) and Theorem 1, we find that if $h'(x)/h(x)$ is negative or small, then the positive solution of (1) is unique, and if $h'(x)/h(x)$ is large, then (1) has three positive solutions.

By applying a general theory on the continuous dependence of solutions on initial conditions and parameters, we can obtain the following result.

Theorem 2. *Let* $\widetilde{h} \in C^1([-1,0)\cup(0,1])\cup C[-1,1]$ *satisfy*

$$\widetilde{h}(-x) = \widetilde{h}(x), \quad \widetilde{h}(x) > 0, \quad x \in (0,1], \tag{11}$$

$$\frac{x\widetilde{h}'(x)}{\widetilde{h}(x)} \text{ is nonincreasing on } [0,1], \tag{12}$$

$$\frac{\widetilde{h}'(x)}{\widetilde{h}(x)} \geq \frac{4-(p+3)x}{(1-x)^2}, \quad x \in [0,1). \tag{13}$$

Then there exists $\delta > 0$ *such that if*

$$|h(x) - \widetilde{h}(x)| \leq \delta, \quad x \in [-1,1],$$

then (1) *has at least three positive solutions.*

Applying Theorem 2, we can obtain the following corollary.

Corollary 1. *Let* $p > 1$, $l \geq 4/(p-1)$ *and* $\lambda \geq 0$. *Then there exists* $\lambda_* > 0$ *such that if* $0 \leq \lambda \leq \lambda_*$, *then the problem*

$$\begin{cases} u'' + (|x|^l + \lambda)u^p = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \tag{14}$$

has an even positive solution and two non-even positive solutions, and the even positive solution is unique.

Of course, Corollary 1 implies that (14) with $\lambda = 0$ has non-even positive solutions. As we mentioned above, it has already known that (14) with $\lambda = 0$ has non-even positive solutions if l is sufficiently large. However, it had been not known the specific number as $4/(p-1)$. On the other hand, by (6), we can see that if (p,l) is sufficiently close to $(1,0)$, then there is no non-even positive solution.

By Kwong's condition (5), we see that if

$$\lambda \geq \frac{1}{2}\left(\frac{l-1}{l+2}\right)^{l-1}, \quad l > 1,$$

then (14) has no non-even positive solutions. Hence it is natural to expect that, for each $l > 0$,u there exists $\overline{\lambda} > 0$ satisfying the following (i) and (ii):

(i) if $0 \leq \lambda < \overline{\lambda}$, then (14) has an even positive solution and two non-even positive solutions;

(ii) if $\lambda \geq \overline{\lambda}$, then the positive solution of (14) is unique.

We consider the linearized problem

$$(15) \qquad \begin{cases} w'' + ph(x)|u|^{p-1}w = 0, & -1 < x < 1, \\ w(-1) = 0, \quad w'(-1) = 1, \end{cases}$$

where u is a positive solution of (1).

The proof of Theorem 1 is based on the following proposition.

Proposition 1. *If the solution w of* (15) *satisfies $w(1) > 0$ for some positive solution u of* (1), *then* (1) *has at least three positive solutions.*

By using the Kolodner-Coffman method, we can obtain Proposition 1.

Hereafter, let u be an even positive solution of (1) and let w be the solution of (15).

Lemma 1. *Suppose that* (3) *and* (10) *hold. Then w has at least two zeros in* $(-1, 1)$.

Lemma 2. *Suppose that* (3), (9) *and* (10) *hold. Then w has at most two zeros in* $(-1, 1]$.

From Lemmas 1 and 2 it follows that w has exactly two zeros in $(-1, 1)$ and $w(1) \neq 0$. Since $w(-1) = 0$ and $w'(-1) = 1 > 0$, we conclude that $w(1) > 0$. Therefore Proposition 1 implies that problem (1) has at least three positive solutions. In the case (3) and (9), by Yanagida [18], we see that the even positive solution is unique, and hence there exist two non-even positive solutions of (1). Thus we have proved Theorem 1.

Now we give the proof of Lemma 1.

Put $y(x) = xu(x) - (x-1)^2 u'(x)$. Then y satisfies

$$y'' + ph(x)u^{p-1}y = \left(\frac{h'(x)}{h(x)} - \frac{4-(p+3)x}{(1-x)^2}\right)(1-x)^2 h(x)u^p \geq 0,$$

and $y(0) = y(1) = 0$ and $y(x) > 0$ on $(0,1)$. Hence we have

$$(y'w - yw')' = \left(\frac{h'(x)}{h(x)} - \frac{4-(p+3)x}{(1-x)^2}\right)(1-x)^2 h(x)u^p w. \tag{16}$$

Assume that w has no zero in $(0,1)$. Integrating (16) on $(0,1)$ and using (10), we find that

$$\int_0^1 (y'w - yw')'dx = \int_0^1 \left(\frac{h'(x)}{h(x)} - \frac{4-(p+3)x}{(1-x)^2}\right)(1-x)^2 h(x)u^p w dx > 0,$$

which implies

$$y'(1)w(1) - y'(0)w(0) > 0.$$

On the other hand, we see that $y'(1) < 0$, $y'(0) > 0$, $w(1) \geq 0$ and $w(0) \geq 0$. This is a contradiction. Hence w has a zero in $(0,1)$. By the similar way, we can show that w has a zero in $(-1,0)$. Then w has at least two zero in $(-1,1)$.

We see that $z(x) = cu(x) + xu'(x)$ satisfies

$$z'' + ph(x)u^{p-1}z = \left(-\frac{xh'(x)}{h(x)} + (p-1)c - 2\right)hu^p.$$

Using this identity for some $c > 0$, we can show Lemma 2.

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