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On Positive Solutions of Nonlinear Elliptic Equations with Hardy Term

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Abstract

We consider the elliptic equation $\Delta u+\mu/|x|^{2}+|x|^{l}u^{p}=0$ in $\mathbf{R}^{n}\backslash\{0\}$. We explain the existence and the asymptotic behavior of regular and singular solutions with respect to μ and l.

Key Words: nonlinear elliptic equation; Hardy term; positive solution; singular solution; fast decay; slow decay; asymptotically self-similar solution; Delaunay-Fowler-type solution; separation.

1. Introduction

We study positive solutions of the elliptic equation with Hardy term

$$\Delta u+\frac{\mu}{|x|^{2}}u+|x|^{l}u^{p}=0 \quad \text{in} \quad \mathbb{R}^{n}\setminus\{0\}, \tag{1.1}$$

where $\mu<(\frac{n-2}{2})^{2}$ and $p>1$. Set

$$\nu=\nu_{\pm}:=\frac{n-2\pm\sqrt{(n-2)^{2}-4\mu}}{2},$$

the solutions of the quadratic equation, $\nu(n-2-\nu)=\mu$.

Assume that $l>\nu_{-}(p-1)-2$. If $|x|^{l}$ is replaced by $\tilde{K}(|x|)$, then the condition corresponds to the integrability

$$\int_{0}s^{1-(p-1)\nu_{-}}\tilde{K}(s)ds<\infty,$$

which is necessary to have local radial solutions satisfying $u(r)=O(r^{-\nu_{-}})$ at 0 where $r=|x|$. We call this type *regular* solution.

Let

$$L^{p-1}=m(n-2-m).$$

where $m=\frac{2+l}{p-1}$. When $L^{p-1}>0$, we set $L=[m(n-2-m)]^{\frac{1}{p-1}}$.

Note that (i) $m>\nu_{-}$ (see the condition of l); (ii) if $m\leq\frac{n-2}{2}$ (or $p\geq\frac{n+2+2l}{n-2}$), then $L^{p-1}>\mu$.

2. Nonexistence

We observe the nonexistence in terms of parameter μ.

Theorem 2.1. *If $\mu \geq L^{p-1}$, then (1.1) has no positive solution.*

In particular, $\mu \geq \left(\frac{n-2}{2}\right)^2 =: \bar{\mu}$, the Hardy constant. $L^{p-1} = \bar{\mu}$ when $p = \frac{n+2+2l}{n-2}$.
Hence, for any $p > 1$, the nonexistence holds if

- $n = 1$ and $\mu \geq \frac{1}{4}$;
- $n = 2$ and $\mu \geq 0$.

Now, we consider the nonexistence of regular radial solutions.

Theorem 2.2. *If $\mu < L^{p-1}$ and $p(n-2) < n + 2 + 2l$, then (1.1) has no regular radial solution.*

The second condition is restated for $n = 1, 2$ as follows:

- $n = 1$ and $p > -2l - 3$ (or $l > -\frac{1}{2}(p-1) - 2$);
- $n = 2$ and $l > -2$;
- $n \geq 3$ and $p < \frac{n+2+2l}{n-2}$.

By the radial symmetry of regular solutions, we conclude that

Theorem 2.3. *If $0 \leq \mu < L^{p-1}$ and $p(n-2) < n + 2 + 2l$ with $l \leq 0$, then (1.1) has no regular solution.*

It is natural question to ask whether (1.1) has nonradial solution for all $\mu < 0$. In [8], Jin, Li and Xu gave a partial answer: If $l = 0$, $\mu < -\frac{n-2}{4}$ and $p = \frac{n+2}{n-2}$, then (1.1) has nonradial solutions. However, it is still open for $-\frac{n-2}{4} \leq \mu < 0$.

3. Regular solution

Now, we consider the existence of solutions of the equation

$$u'' + \frac{n-1}{r}u' + \frac{\mu}{r^2}u + r^l u^p = 0, \quad \lim_{r \to 0} r^{\nu_-} u(r) = \alpha > 0. \tag{3.1}$$

When $l > \nu_-(p-1) - 2$, (3.1) has a unique local solution $u_\alpha \in \mathbf{C}^2(0, \delta)$ for $\delta > 0$ small.

Theorem 3.1. *Let $\mu < L^{p-1}$ and $p(n-2) \geq n + 2 + 2l$. Then, (3.1) has one-parameter family of regular solutions.*

In particular, we are interested in the critical case.

- For $n = 1$, assume $m^2 + m + \mu < 0$ and $1 < p \leq -2l - 3 < -2\nu_-(p-1) + 4$. The critical problem is

$$u'' + \frac{\mu}{r^2}u + \frac{1}{r^{\frac{p+3}{2}}}u^p = 0, \quad \lim_{r \to 0} r^{\nu_-} u(r) = \alpha > 0;$$

- For $n=2$, assume $m^2+\mu<0$ and $l\le -2$. The critical problem is

$$u''+\frac{1}{r}u'+\frac{\mu}{r^2}u+\frac{1}{r^2}u^p=0,\quad \lim_{r\to 0} r^{-\sqrt{-\mu}}u(r)=\alpha>0;$$

- For $n\ge 3$, assume $\mu<L^{p-1}$ and $p\ge\frac{n+2+2l}{n-2}$.

The asymptotic behavior of solutions has two types: the first is fast decay for the critical case; the second is slow decay for the supercritical case.

Theorem 3.2. *Let $\mu<L^{p-1}$ and $p(n-2)\ge n+2+2l$.*

If $p(n-2)=n+2+2l$, then $\lim_{r\to\infty} r^{\nu_+}u_\alpha=c>0$ and for $n\ge 3$ and for some $\epsilon>0$

$$\bar{u}_\epsilon(x)=\frac{\left[\frac{2(n+l)(\bar{\mu}-\mu)\epsilon}{\sqrt{\bar{\mu}}}\right]^{\sqrt{\bar{\mu}}/(2+l)}}{|x|^{\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}}(\epsilon+|x|^{\frac{(2+l)\sqrt{\bar{\mu}-\mu}}{\sqrt{\bar{\mu}}}})^{(n-2)/(2+l)}}$$

are the solutions. If $p(n-2)>n+2+2l$, then

$$\lim_{r\to\infty} r^m u_\alpha=\mathbf{L}:=\left(L^{p-1}-\mu\right)^{\frac{1}{p-1}}.$$

4. Singular solution

Now, we look for solutions which are not regular. We call this type singular solutions.

Theorem 4.1. *Let $\mu<L^{p-1}$ and $p(n-2)\ge n+2+2l$.*

If $p(n-2)=\frac{n+2+2l}{n-2}$, then there are two types: the first has the self-similar singularity, $r^{-m}\mathbf{L}$; the second is Delaunay-Fowler type:

$$0<d_1:=\min r^m u_s(r)<\mathbf{L}<d_2:=\max r^m u_s(r)$$

$$<D:=\left[\frac{(n+l)(n-2)}{4}-\frac{n+l}{n-2}\mu\right]^{\frac{n-2}{2(l+2)}},$$

and $r^m u_s(r)$ is periodic in $t=\log r$.

If $p(n-2)>n+2+2l$: $r^{-m}\mathbf{L}$ is the unique singular radial solution.

If $l\le -2$, then each problem for $p>1$ is a supercritical case. Hence, for $l=-2$, $\mathbf{L}=(-\mu)^{\frac{1}{p-1}}$ is the unique singular radial solution while for $\nu_-(p-1)-2<l<-2$, $r^{-m}\mathbf{L}$ is the unique singular radial solution.

5. Separation

We consider separation of solutions.

Theorem 5.1. *If $L^{p-1}>\mu\ge L^{p-1}-\frac{a^2}{4(p-1)}$ with $a=n-2-2m$, then any two solutions of (3.1) do not intersect.*

We analyze the assumption and explain the cases.

(i) $l > -2$ and $n \leq 10 + 4l$:

For given $p > \frac{n+2+2l}{n-2}$, there exist $\mu_-(n,p,l) < \mu_+(n,p,l) < \bar{\mu}$ such that separation happens for $0 < \mu_- \leq \mu < \mu_+$. Observe that

$$\lim_{p \to \frac{n+2+2l}{n-2}} \mu_\pm = \bar{\mu}, \qquad \lim_{p \to \infty} \mu_\pm = 0.$$

For given $0 < \mu < \bar{\mu}$, there exist $p_+ > p_- > \frac{n+2+2l}{n-2}$ such that separation happens for $p_- \leq p < p_+ = \frac{l+2}{\nu_-} + 1$. Moreover, $p_\pm$ is decreasing in $(0, \bar{\mu})$.

(ii) $l > -2$ and $n > 10 + 4l$:

$\mu_- \geq -\frac{(2n+l-2)(n-10-4l)^2}{108(l+2)} = \mu_*$ for $p \geq \frac{n+2+2l}{n-2}$. $\mu_- = \mu_*$ only when $p = \frac{n+2+2l}{n-10-4l} = p_*(m = \frac{n-10-4l}{6})$. Note that $\mu_* = -\frac{(n-1)(n-10)^2}{108}$ when $p = \frac{n+2}{n-10}$ and $l = 0$. In other words, for given $0 < \mu < \bar{\mu}$, there exist $p_+ > p_- > \frac{n+2+2l}{n-2}$ such that separation happens for $p_- \leq p < p_+ = \frac{l+2}{\nu_-} + 1$, while for $\mu = 0$, $p \geq p_c$ and for $\mu_* \leq \mu < 0$, $p_- \leq p \leq p_+$. p_- is decreasing in μ, and p_+ is decreasing only in $(0, \bar{\mu})$. $p_-(0) = p_c$ and p_+ is increasing in $[\mu_*, 0)$, $p_+(0) = \infty$.

$$\lim_{\mu \to \bar{\mu}} p_\pm = \frac{n+2+2l}{n-2}, \qquad \lim_{\mu \to \mu_*} p_\pm = p_*.$$

(iii) $l = -2$:

$0 > \mu \geq -\frac{\bar{\mu}}{p-1}$ and $1 < p \leq p_+ = -\frac{\bar{\mu}}{\mu} + 1$.

(iv) $\sigma(p-1) - 2 < l < -2$:

$\mu_- \leq \mu < \mu_+$ and

$$\lim_{p \to 1} \mu_\pm = -\infty, \qquad \lim_{p \to \infty} \mu_\pm = \infty.$$

For given $-\infty < \mu < 0$, $\frac{l+2}{\nu_-} + 1 = p_- < p \leq p_+$.

(v) $n = 1$ and $l < -2$:

$\mu_- \leq \mu < \mu_+ \leq \bar{\mu} = \frac{1}{4}$ and

$$\lim_{p \to 1} \mu_\pm = -\infty, \qquad \lim_{p \to \infty} \mu_\pm = \infty.$$

For given $-\infty < \mu < \frac{1}{4}$, $\frac{l+2}{\nu_-} + 1 = p_- < p \leq p_+$.

$$\lim_{\mu \to \bar{\mu}} p_\pm = -2l - 3, \qquad \lim_{\mu \to -\infty} p_\pm = 1.$$

(vi) $n = 2$ and $l < -2$:

$\mu_- \leq \mu < \mu_+$ and

$$\lim_{p \to 1} \mu_\pm = -\infty, \qquad \lim_{p \to \infty} \mu_\pm = \infty.$$

For given $-\infty < \mu < 0$, $\frac{l+2}{\nu_-} + 1 = p_- < p \leq p_+$.

Theorem 5.2. *Let $p > \frac{n+2+2l}{n-2}$. Assume $\bar{\mu} \geq p\mathbf{L}^{p-1} + \mu (= pL^{p-1} - (p-1)\mu)$. Then, every radial regular steady state u satisfies*

$$|x|^2 u(x)^{p-1} \leq \mathbf{L}^{p-1} (= L^{p-1} - \mu)$$

and the operator $-\Delta - \frac{\mu}{|x|^2} - pu^{p-1}$ has no negative spectrum and u is linearly stable.

Proof. Suppose the inequality. Then,

$$-pu^{p-1} - \frac{\mu}{|x|^2} \geq (-pL^{p-1} + (p-1)\mu)\frac{1}{|x|^2}$$

and

$$\int |\nabla\phi|^2 - \frac{\mu}{|x|^2}\phi^2 - pu^{p-1}\phi^2 \geq 0$$

for $\phi \in H^1$.

Let $\mathbf{L}^{p-1} = Q(m) - \mu$ and $Q(m) = L^{p-1}$. We observe that

$$Q(m - \partial_t)V - \mu V = V^p,$$

$$V^p - \mathbf{L}^p \geq p\mathbf{L}^{p-1}(V - \mathbf{L}),$$

$$Q(m - \partial_t)V - \mu V - \mathbf{L}^p \geq p(Q(m) - \mu)(V - \mathbf{L}),$$

$$[pQ(m) - Q(m - \partial_t) - (p-1)\mu]W \leq 0,$$

where $W = V - \mathbf{L}$. The characteristic polynomial $\mathbf{P}(\lambda) = pQ(m) - Q(m-\lambda) - (p-1)\mu$ has two negative roots, λ_1, λ_2. Let $Y = W' - \lambda_1 W$. $Y' - \lambda_2 Y \leq 0$. Then, $e^{-\lambda_2 t}Y$ is decreasing and zero at $t = -\infty$. Hence, $W' - \lambda_1 W \leq 0$ and W is also zero at $t = -\infty$. Therefore, $W \leq 0$. The product of the two root is $\mathbf{P}(0) = (p-1)(Q(m) - \mu) = (p-1)\mathbf{L}^{p-1} > 0$.

$$\mathbf{P}(m - \frac{n-2}{2}) = pQ(m) - \bar{\mu} - (p-1)\mu \leq 0.$$

Hence, $\mathbf{P}(\lambda)$ has two negative roots. □

Theorem 5.3. *Let $p \geq \frac{n+2+2l}{n-2}$. Assume $\bar{\mu} < p\mathbf{L}^{p-1} + \mu$. Then, $-\Delta - \frac{\mu}{|x|^2} - pu^{p-1}$ has a negative eigenvalue.*

Note that if $\mu = L^{p-1} - \frac{a^2}{4(p-1)}$, then $\bar{\mu} = pL^{p-1} - (p-1)\mu$.

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