数理解析研究所講究録
第 1750 巻 2011 年 83-90

Exact Multiplicity of Rapidly Decaying Solutions for a Semilinear Elliptic Equation with a Critical Exponent

東京工業大学・大学院理工学研究科　柳田　英二 (Eiji Yanagida)
Department of Mathematics
Tokyo Institute of Technology

1 Introduction

This article is based a joint work with Noriyuki Murai (Tohoku University). We consider radially symmetric solutions of the equation

$$\Delta u+K(|x|)u^{p}=0 \quad \text{in } \mathbb{R}^{n}, \tag{1.1}$$

where $n>2$, $p>1$. $K\geq 0$ and $K\in C^{1}([0,\infty))$, Any radially symmetric solution $u=u(r)$, $r=|x|$, of (1.1) satisfies

$$\begin{cases} u_{rr}+\dfrac{n-1}{r}u_{r}+K(r)u^{p}=0, & r>0,\\ u(0)=\alpha>0, \quad u_{r}(0)=0. \end{cases} \tag{1.2}$$

We denote by $u(r;\alpha)$ the unique solution of this initial value problem. According to [7], we can classify the solutions of (1.2) as follows:

- Rapidly decaying solution: $u(r)>0$ for all $r>0$ and $r^{n-2}u\rightarrow\beta\in(0,\infty)$ as $r\rightarrow\infty$.
- Slowly decaying solution: $u(r)>0$ for all $r>0$ and $r^{n-2}u\rightarrow\infty$ as $r\rightarrow\infty$.
- Crossing solution: $u(z)=0$ at some $z\in(0,\infty)$.

In what follows, we consider the critical case in the Sobolev sense:

$$p = \frac{n+2}{n-2}.$$

We note that the critical case is related to the Yamabe problem in differential geometry [9]. In the context of the Yamabe problem, any rapidly decaying solution corresponds to the complete metirc, while the slowly decaying solution corresponds to the incomplete metric.

Here we collect known facts about the existence of rapidly decaying solutions of (1.2) in the case of $p = \dfrac{n+2}{n-2}$. First, it is easy to show that if $K \equiv 1$, then

$$u = \varphi(r;\alpha) := \alpha\left\{1 + \frac{\alpha^{4/(n-2)}}{n(n-2)}r^2\right\}^{-(n-2)/2}$$

satisfies (1.2). We note that the solution satisfies

$$r^{n-2}\varphi(r;\alpha) \to \{n(n-2)\}^{(n-2)/n}\alpha^{-1} \quad (r \to \infty),$$

so that $u = \varphi(r;\alpha)$ is a rapidly decaying solution for any $\alpha \in (0,\infty)$. Ding-Ni [3] proved that if $K(r)$ is monotone and nonconstant, then there is no rapidly decaying solution. More precisely, they proved the following.

- If $K(r)$ is non-constant and non-increasing in r, then $u(r;\alpha)$ is a slowly decaying solution for all $\alpha > 0$.
- If $K(r)$ is non-constant and non-decreasing in r, then $u(r;\alpha)$ is a crossing solution for all $\alpha > 0$.

When $K(r)$ is not monotone, Bianchi-Egnell [1] showed the existence of a rapidly decaying solution by assuming that K satisfies $K(0) \leq K(\infty)$ and some other asymptotic conditions at $r = 0$ and $r = \infty$. Also, Sasahara-Tanaka [8] studied the case where $K(0) = K(\infty)$ and K has a minimum, and proved that there exists at least one rapidly decaying solution. See also Yanagida-Yotsutani [11] for a sufficient condition on the existence of a rapidly decaying solution.

Concerning the uniqueness, Yanagida-Yotsutani [10] proved that if $K(r)$ is nonconstant, non-decreasing in $(0,a)$, non-increasing in $r \in (a,\infty)$, and $K(0) = K(\infty)$, then there exists a unique rapidly decaying solution. In fact, there exists a unique $\alpha^* \in (0,\infty)$ such that

- $u(r;\alpha)$ is a slowly decaying solution for every $\alpha \in (0,\alpha^*)$.
- $u(r;\alpha^*)$ is a rapidly decaying solution.
- $u(r;\alpha)$ is a slowly decaying solution for every $\alpha \in (\alpha^*,\infty)$.

On the other hand, concerning the multiple existence of rapidly decaying solutions, it was shown numerically by Morishita-Yanagida-Yotsutani [6] that for some K, there may exist multiple rapidly decaying solutions. Kabeya [4] obtained a condition on K such that there exist at least two rapidly decaying solutions. Finally, Chen-Lin [2] considered the case $n \geq 7$ and found that for some K, there exists infinitely many rapidly decaying solutions.

The aim of this article is to obtain a condition on K for the exact multiplicity of rapidly decaying solutions of (1.2). The following theorem is our main result.

Theorem 1. *Let $n > 2$ and $p = \dfrac{n+2}{n-2}$. Assume that*

(i) *$K(r) = 1 + \varepsilon k(r)$, where $k(r) \equiv 0$ for $r \in [0, a]$ and $k(r) \equiv Const.$ for $r \in [b, \infty]$ with some $0 < a < b < \infty$, and $\varepsilon > 0$ is a parameter, and*

(ii) *the function*

$$g(\alpha) := \int_0^\infty r^n \varphi(r; \alpha)^{\frac{2n}{n-2}} k_r(r) dr$$

has exactly m simple zeros in $(0, \infty)$.

If $\varepsilon > 0$ is sufficiently small, then the problem (1.2) has exactly m rapidly decaying solutions.

We also have the following result.

Theorem 2. *Let $n > 2$ and $p = \dfrac{n+2}{n-2}$. Then for any $m \in \mathbf{N}$, there exists $K = K_m(r)$ such that the problem (1.2) has exactly m rapidly decaying solutions.*

2 Preliminaries

In this section we describe some preliminary results about the problem (1.2). Hereafter we always assume that $n > 2$ and $p = \frac{n+2}{n-2}$.

First we introduce the Pohozaev identity, which is obtained by direct computations and (1.2).

Lemma 1. *Define*

$$P[r; u] := \frac{1}{2} r^{n-1} u_r (r u_r + (n-2) u) + \frac{n-2}{2n} r^n K(r) u^{p+1}.$$

Then

$$\frac{d}{dr} P[r; u] \equiv \frac{n-2}{2} r^n K_r(r) u^{p+1}.$$

In particular, this identity implies that if $K(r) \equiv Const.$ for $r \in [b, \infty)$, then $P[r; u]$ is also constant for $r \in [b, \infty)$.

The following characterization of solutions of (1.2) is proved by Kawano-Yanagida-Yotsutani [5].

Lemma 2. *Suppose that* $K(r) \equiv Const.$ *for* $r \in [b, \infty)$.

(i) *If* $P[b; u] > 0$, *then* $u(r; \alpha)$ *is a crossing solution.*

(ii) *If* $P[b; u] = 0$, *then* $u(r; \alpha)$ *is a rapidly decaying solution.*

(iii) *If* $P[b; u] < 0$, *then* $u(r; \alpha)$ *is a slowly decaying solution.*

Using this lemma, we will identify the type of $u(r; \alpha)$ for small $\alpha > 0$, large α and intermediate α as follows.

Lemma 3.

$$\frac{u(r; \alpha)}{\alpha} \to 1 \quad and \quad \frac{u_r(r; \alpha)}{\alpha} \to 0 \qquad (\alpha \to 0)$$

uniformly in $r \in [0, b]$.

Proof. Setting $v(r) = \alpha^{-1} u(r; \alpha)$, we have

$$\begin{cases} v_{rr} + \dfrac{n-1}{r} v_r + \alpha^{p-1} K(r) v^p = 0, & r > 0, \\ v(0) = 1 > 0, \quad v_r(0) = 0 \end{cases}$$

Hence $v = \alpha^{-1} u \to 1$ and $v_r = \alpha^{-1} u_r \to 0$ as $\alpha \to 0$ uniformly in $r \in [0, b]$. ∎

Lemma 4.

$$\frac{P[b; u]}{\alpha^{p+1}} \to \frac{n-2}{2} \int_0^b r^n K_r(r) dr \qquad (\alpha \to 0).$$

Proof. By the Pohozev identity and Lemma 3, we have

$$\begin{aligned} \frac{P[b; u]}{\alpha^{p+1}} &= \frac{n-2}{2} \int_0^b r^n K_r(r) \left\{ \frac{u(r; \alpha)}{\alpha} \right\}^{p+1} dr \\ &\to \frac{n-2}{2} \int_0^b r^n K_r(r) dr \qquad (\alpha \to 0) \end{aligned}$$

Thus for small α, $P[b; u]$ has the same sign as $\displaystyle\int_0^b r^n K_r(r) dr$. ∎

Lemma 5. *Suppose that* $K(r) \equiv 1$ *for* $r \in [0, a]$. *Then*

$$\alpha u(r;\alpha) \to C_0(n) r^{2-n} \quad \text{and} \quad \alpha u_r(r;\alpha) \to -(n-2) C_0(n) r^{2-n}$$

as $\alpha \to \infty$ *uniformly in* $[a, b]$, *where* $C_0(n) := \{n(n-2)\}^{(n-2)/2} > 0$.

Proof. Setting $w(r) = \alpha u(r;\alpha)$, we have for $r \in [0, a]$

$$\begin{aligned} w(r) = \alpha\varphi(r;\alpha) &= \alpha^2 \Big\{ 1 + \frac{\alpha^{4/(n-2)}}{n(n-2)} r^2 \Big\}^{-(n-2)/2} \\ &= \Big\{ \alpha^{-4/(n-2)} + \frac{1}{n(n-2)} r^2 \Big\}^{-(n-2)/2} \\ &\to C_0(n) r^{2-n} \qquad (\alpha \to \infty) \end{aligned}$$

uniformly $r \in [\delta, a]$. Similarly

$$\begin{aligned} w_r(r) = \alpha\varphi_r(r;\alpha) &= -\frac{1}{n} \alpha^{2n/(n-2)} r \Big\{ 1 + \frac{\alpha^{4/(n-2)}}{n(n-2)} r^2 \Big\}^{-n/2} \\ &\to -(n-2) C_0(n) r^{1-n} \qquad (\alpha \to \infty) \end{aligned}$$

uniformly in $r \in [\delta, a]$. On the other hand, from (1.2) we see that$w$ satisfies

$$\begin{cases} w_{rr} + \dfrac{n-1}{r} w_r + \alpha^{-4/(n-2)} K(r) w^p = 0, & r > 0, \\ w(a) \to C_0(n), \quad w_r(a) \to C_0(n) a^{2-n} & (\alpha \to \infty). \end{cases}$$

This implies that $w \to C_0(n) r^{2-n}$ uniformly in $r \in [a, b]$ as $\alpha \to \infty$. ■

As a consequence of this lemma, we have

Lemma 6.

$$\alpha^{p+1} P[b; u] \to \frac{n-2}{2} C_0(n)^{p+1} \int_0^a r^{-n} K_r(r) dr \quad as \quad \alpha \to \infty.$$

Proof.

$$\begin{aligned} \alpha^{p+1} P[b; u] &= \frac{n-2}{2} \int_0^b r^n K_r(r) \{\alpha u(r;\alpha)\}^{p+1} dr \\ &\to \frac{n-2}{2} \int_a^b r^n K_r(r) \{C_0(n) r^{2-n}\}^{p+1} dr \qquad (\alpha \to \infty) \\ &= \frac{n-2}{2} C_0(n)^{p+1} \int_0^b r^{-n} K_r(r) dr. \end{aligned}$$

Thus for large α. $P[b; u]$ has the same sign as $\displaystyle\int_0^b r^{-n} K_r(r) dr$. ■

We write K as $K(r) = 1 + \varepsilon k(r)$.

Lemma 7. *Let $0 < \alpha_1 < \alpha_2 < \infty$ be fixed. Then*

$$u(r;\alpha) \to \varphi(r;\alpha) \quad \text{and} \quad u_r(r;\alpha) \to \varphi_r(r;\alpha)$$

as $\varepsilon \to 0$ uniformly in $(r,\alpha) \in [0,b] \times [\alpha_1, \alpha_2]$.

Proof. Since

$$\begin{cases} u_{rr} + \dfrac{n-1}{r} u_r + \{1 + \varepsilon k(r)\} u^p = 0, & r > 0, \\ u(0) = \alpha > 0, \quad u'(0) = 0, \end{cases}$$

the proof is clear. ∎

As a consequence of this lemma, we have

Lemma 8.

$$\frac{P[b:u]}{\varepsilon} \to \frac{n-2}{2} \int_0^b r^n k_r(r) \varphi(r;\alpha)^{p+1} dr.$$

as $\varepsilon \to 0$.

Proof. By the Pohozaev idenitity, we have

$$P[b:u] = \frac{n-2}{2} \int_0^b r^n \varepsilon k_r(r) u(r;\alpha)^{p+1} dr.$$

Since $u(r;\alpha) \to \varphi(r;\alpha)$ as $\varepsilon \to 0$, we obtain the conclusion. Thus for intermediate α and small $\varepsilon > 0$, $P[b;u]$ has the same sign as $\int_0^b r^n k_r(r) \varphi(r;\alpha)^{p+1} dr$. ∎

3 Outline of proofs

Proof of Theorem 1.
Step 1: For small $\alpha \in (0, \alpha_1)$, we can identify the type of $u(r;\alpha)$ by examining the sign of

$$\int_0^b r^n k_r(r) dr \qquad \text{(the same sign as } g(0).$$

Step 2: For large $\alpha \in (\alpha_2, \infty)$, we can identify the type of $u(r;\alpha)$ by examining the sign of

$$\int_0^a r^{-n} k_r(r) dr \qquad \text{(the same sign as } g(\infty)).$$

Step 3: Fix $0 < \alpha_1 < \alpha_2 < \infty$ as above. If we take $\varepsilon > 0$ sufficiently small, then we can identify the type of solutions for $\alpha \in (\alpha_1, \alpha_2)$ by examining the sign of

$$g(\alpha) := \int_0^\infty r^n \varphi(r;\alpha)^{p+1} k_r(r) dr$$

From these considerations and the simplicity of zeros of $g(\alpha)$, we can count the exact number of rapidly decaying solutions by counting the number of zeros of $g(\alpha)$.

In fact, if $\varepsilon > 0$ is sufficiently small, then the number of rapidly decaying solutions is the same as the number of zeros of $g(\alpha)$. ■

Proof of Theorem 2.

If we rewrite $g(\alpha)$ as

$$g(\alpha) := -\int_0^\infty \{r^n \varphi(r;\alpha)^{p+1}\}_r k(r) dr,$$

we can handle the case where $K(r) = 1 + \varepsilon k(r)$ is piece-wise constant (or $K_r = \varepsilon k_r(r)$ is a superposition of the delta functions). Then for every $m \in N$, we may control the locations of discontinuous points and gaps to find a piece-wise constant $K(r) = 1+\varepsilon k(r)$ such that $g(\alpha)$ has exactly m simple zeros.

In the last step, we approximate the piece-wise function by a smooth function. Then the number of zeros does not change because the zeros are simple. ■

References

[1] G. Bianchi and H. Egnell, Local existence and uniqueness of positive solutions of the equation $\Delta u + (1 + \varepsilon\varphi(r))u^{\frac{n+2}{n-2}} = 0$, in $\mathbf{R}^n$ and a related equation, in Nonlinear diffusion equations and their equilibrium states, **3** (N. G. Lloyd, W. M. Ni, L. A. Peletier, J. Serrin eds.), 111-128 , Birkhäuser, 1992.

[2] C.-C. Chen and C.-S. Lin, Blowing up with infinite energy of conformal metrics on S^n, Comm. Partial Differential Equations **24** (1999) 785–799.

[3] W.-Y. Ding and W.-M. Ni, On the elliptic equation $\Delta u + Ku^{\frac{n+2}{n-2}} = 0$ and related topics, Duke Math. J. 52 (1985), 485–506.

[4] Y. Kabeya, Perturbation of structures of radial solutions to elliptic equations, Asymptotic analysis and singularities—elliptic and paraboric PDEs and related problems, 589–599, Adv. Stud. Pure Math. 47-2, Math. Soc. Japan, Tokyo, 2007.

[5] N. Kawano, E. Yanagida, and S. Yotsutani, Structure theorems for positive radial solutions to $\Delta u + K(|x|)u^p = 0$ in $\mathbf{R}^n$, Funkcial. Ekvac. 30 (1993), 557–579.

[6] H. Morishita, E. Yanagida, and S. Yotsutani, Structural change of solutions for a scalar curvature equation, Differential Integral Equations 14 (2001), 273–288.

[7] W.-M. Ni and S. Yotsutani, Semilinear elliptic equations of Matukuma-type and related topics, Japan J. Appl. Math. 5 (1988), 1-32.

[8] Y.Sasahara and K.Tanaka, A remark on positive radial solutions of elliptic equation $\Delta u + K(|x|)u^{(n+2)/(n-2)} = 0$ in $\mathbf{R}^n$, Proc. Amer. Math. Soc. 123 (1995), 527–531.

[9] H. Yamabe, On a deformation of Riemannian structures on compact manifolds, Osaka Math. J. 12 (1960), 21–37.

[10] E. Yanagida and S. Yotsutani, Classification of the structure of positive radial solutions to $\Delta u + K(|x|)u^p = 0$ in $\mathbf{R}^n$, Arch. Rat. Mech. Anal. 124 (1993), 239–259.

[11] E. Yanagida and S. Yotsutani, Existence of positive radial solutions to $\Delta u + K(|x|)u^p = 0$ in $\mathbf{R}^n$, J. Differential Equations 115 (1995), 477–502.