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A topology of vector lattices

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Abstract

In [3] we define and study the Denjoy and Henstock-Kurzweil integrals in a vector lattice. Moreover in [5, 6, 7] we show some fixed point theorems in a vector lattice. In those theories we need a topology in a vector lattice and some assumptions. The purpose of this paper is to show criteria for the assumptions.

1 Introduction

There are many fixed point theorems in a topological vector space, for instance, Kirk's fixed point theorem in a Banach space, and so on; see for example [11].

We consider a derivative and fixed point theorems in a vector lattice. As known well every topological vector space has a linear topology. On the other hand, although every vector lattice does not have a topology, it has two lattice operators, which are the supremum $\vee$ and the infimum $\wedge$, and also an order is introduced from these operators; see also [9, 12] about vector lattices. There are some methods how to introduce a topology to a vector lattice. One method is to assume that the vector lattice has a linear topology [1]. On the other hand, there is another method to make up a topology in a vector lattice, for instance, in [3] one method is introduced in case of the vector lattice with unit.

In [3] we define and study the Denjoy and Henstock-Kurzweil integrals in a vector lattice. Moreover in [5, 6, 7] we show some fixed point theorems in a vector lattice. In those theories we need a topology in a vector lattice and some assumptions. The purpose of this paper is to show criteria for the assumptions.

2 Topology in a vector lattice

First we introduce a topology in a vector lattice introduced by [3]; see also [6, 7].

Let X be a vector lattice. $e \in X$ is said to be an unit if $e \wedge x > 0$ for any $x \in X$ with $x > 0$. Let $\mathcal{K}_X$ be the class of units of X. In the case where X is the set of real numbers $\mathbf{R}$, $\mathcal{K}_{\mathbf{R}}$ is the set of positive real numbers. Let X be a vector lattice with unit and let Y be a subset of X. Y is said to be open if for any $x \in Y$ and for any $e \in \mathcal{K}_X$ there exists

$\varepsilon \in \mathcal{K}_{\mathbf{R}}$ such that $[x - \varepsilon e, x + \varepsilon e] \subset Y$. Let $\mathcal{O}_X$ be the class of open subsets of X. For $e \in \mathcal{K}_X$ and for an interval $[a, b]$ we consider the following subset

$$[a, b]^e = \{x \mid \text{ there exists some } \varepsilon \in \mathcal{K}_{\mathbf{R}} \text{ such that } x - a \geq \varepsilon e \text{ and } b - x \geq \varepsilon e\}.$$

By the definition of $[a, b]^e$ it is easy to see that $[a, b]^e \subset [a, b]$. Every mapping from $X \times \mathcal{K}_X$ into $(0, \infty)$ is said to be a gauge. Let Δ_X be the class of gauges in X. For $x \in X$ and $\delta \in \Delta_X$, $O(x, \delta)$ is defined by

$$O(x, \delta) = \bigcup_{e \in \mathcal{K}_X} [x - \delta(x, e)e, x + \delta(x, e)e]^e.$$

$O(x, \delta)$ is said to be a δ-neighborhood of x. Suppose that for any $x \in X$ and for any $\delta \in \Delta_X$ there exists $U \in \mathcal{O}_X$ such that $x \in U \subset O(x, \delta)$.

Let X be a vector lattice with unit and Y a vector lattice. Let $\mathcal{U}_Y^s(\mathcal{K}_X, \geq)$ be the class of $\{v_e \mid e \in \mathcal{K}_X\}$ satisfying the following conditions:

(U1) $v_e \in Y$ with $v_e > 0$;

$(\mathrm{U2})^d$ $v_{e_1} \geq v_{e_2}$ if $e_1 \geq e_2$;

$(\mathrm{U3})^s$ For any $e \in \mathcal{K}_X$ there exists $\theta(e) \in \mathcal{K}_{\mathbf{R}}$ such that $v_{\theta(e)e} \leq \frac{1}{2} v_e$.

Let $x_0 \in Z \subset X$ and $f : Z \longrightarrow Y$. f is said to be continuous at x_0 if there exists $\{v_e\} \in \mathcal{U}_Y^s(\mathcal{K}_X, \geq)$ such that for any $e \in \mathcal{K}_X$ there exists $\delta \in \mathcal{K}_{\mathbf{R}}$ such that for any $x \in Z$ if $|x - x_0| \leq \delta e$, then $|f(x) - f(x_0)| \leq v_e$.

Let X and Y be vector lattices with unit, $Z \subset X$ and $f : Z \longrightarrow Y$. Suppose that there exists $P \subset Y$ satisfying the following conditions:

(P1) P is open and convex;

(P2) If $x \in P$ and $x \leq y$, then $y \in P$;

(P3) $0 \notin P$;

(P4) $\{x \mid x > 0\} \subset P$.

Let $\mathcal{P}_Y$ be the class of the above P's. f is said to be upper semi-continuous with respect to $P \in \mathcal{P}_Y$ if $\{x \mid y - f(x) \in P\} \in \mathcal{O}_X \cap Z$ for any $y \in Y$. f is said to be lower semi-continuous with respect to $P \in \mathcal{P}_Y$ if $\{x \mid f(x) - y \in P\} \in \mathcal{O}_X \cap Z$ for any $y \in Y$. f is said to be semi-continuous with respect to $P \in \mathcal{P}_Y$ if it is upper and lower semi-continuous with respect to $P \in \mathcal{P}_Y$.

A vector lattice is said to be Archimedean if it holds that $x = 0$ whenever there exists $y \in X$ with $y \geq 0$ such that $0 \leq rx \leq y$ for any $r \in \mathcal{K}_{\mathbf{R}}$.

Let X be an Archimedean vector lattice. Then there exists a positive homomorphism f from X into $\mathbf{R}$, that is, f satisfies the following conditions:

(H1) $f(\alpha x+\beta y)=\alpha f(x)+\beta f(y)$ for any $x,y\in X$ and for any $\alpha,\beta\in\mathbf{R}$;

(H2) $f(x)\geq 0$ for any $x\in X$ with $x\geq 0$;

see [7]*Example 3.1. Suppose that there exists a homomorphism f from X into $\mathbf{R}$ satisfying the following condition instead of (H2):

(H2)^s $f(x)>0$ for any $x\in X$ with $x>0$.

The following hold under the topology above; see [6, 7].

Lemma 2.1. *Let X be an Archimedean vector lattice with unit and $\{x_1,\dots,x_n\}$ a subset of X. Then $co\{x_1,\dots,x_n\}$ is homeomorphic to a compact and convex subset of $\mathbf{R}^n$.*

Lemma 2.2. *Let X be an Archimedean vector lattice with unit, Y a vector lattice with unit, $Z\subset X$ and f a mapping from Z into Y. Suppose that there exists a homomorphism from X into $\mathbf{R}$ satisfying condition* (H2)^s *and that $\mathcal{P}_Y\neq\emptyset$.*

Then f is semi-continuous with respect to any $P\in\mathcal{P}_Y$ if it is continuous at any $x\in Z$.

Lemma 2.3. *Let X be an Archimedean vector lattice with unit, Y a vector lattice with unit, $x_0\in Z\subset X$ and f a mapping from Z into Y. Suppose that there exists a homomorphism from X into $\mathbf{R}$ satisfying condition* (H2)^s.

Then f is continuous at x_0 in the sense of topology if it is continuous at x_0.

3 Criteria for the condition (H2)^s

Theorem 3.1. *Let X be an Archimedean vector lattices with unit.*

Then the following are equivalent:

(1) *X satisfies the condition* (H2)^s;

(2) *$\mathcal{P}_X\neq\emptyset$;*

(3) *There exists $O\in\mathcal{O}_X$ such that $O\neq\emptyset$ and $\{x\mid x>0\}\subset co(O)\neq X$.*

Proof. (1) $\Rightarrow$ (2): Let $0<\beta<1$ and $\delta(x,e)=\frac{\beta f(x)}{f(e)}$ for any $x\in X$ with $x>0$ and for any $e\in\mathcal{K}_X$. Put $P=\bigcup_{x\in X \text{ with } x>0}\mathrm{int}(O(x,\delta))$. Then P is open and $\{x\mid x>0\}\subset P$. Note that by condition (H2)^s for any $x_1,x_2\in X$ with $x_1,x_2>0$ and $x_1\neq x_2$, $\frac{x_1}{f(x_1)}$ and $\frac{x_2}{f(x_2)}$ are incomparable mutually. Therefore $x-\delta(x,e)e\not\leq 0$ for any $x\in X$ with $x>0$ and for any $e\in\mathcal{K}_X$. Assume that $0\in P$. Then there exist $x\in X$ with $x>0$ and $e\in\mathcal{K}_X$ such that $0\in[x-\delta(x,e)e,x+\delta(x,e)e]^e$. It is a contradiction. Therefore $0\notin P$. Note that $x\in\mathrm{int}(A)$ if and only if there exists $\delta_x\in\Delta_X$ such that $O(x,\delta_x)\subset A$. Let $x\in P$ and $x\leq y$. Then there exist $z\in X$ with $z>0$ and $\delta_x\in\Delta_X$ such that $O(x,\delta_x)\subset O(z,\delta)$. Let $\delta_y(u,e)=\delta_x(u-y+x,e)$.

Since $\delta(x_2, e) \leq \delta(x_1+x_2, e)$ for any $x_1, x_2 \in X$ with $x_1, x_2 > 0$, it holds that $x_1+O(x_2, \delta) \subset O(x_1 + x_2, \delta)$. Therefore

$$\begin{aligned} O(y, \delta_y) = y - x + O(x, \delta_x) &\subset y - x + O(z, \delta) \\ &\subset O(z + y - x, \delta), \end{aligned}$$

that is, $y \in \mathrm{int}(O(z + y - x, \delta)) \subset P$. Let $x_0, x_1 \in P$ and $\alpha \in \mathbf{R}$ with $0 \leq \alpha \leq 1$. Then for $i = 0, 1$ there exist $y_i \in X$ with $y_i > 0$ and $\delta_i \in \Delta_X$ such that $O(x_i, \delta_i) \subset O(y_i, \delta)$. Let $\delta_\alpha(z, e) = (1 - \alpha)\delta_0(x_0, e) + \alpha\delta_1(x_1, e)$. Take $z \in O((1 - \alpha)x_0 + \alpha x_1, \delta_\alpha)$ arbitrary. Then there exists $e \in \mathcal{K}_X$ such that

$$\begin{aligned} z &\in [(1-\alpha)x_0 + \alpha x_1 - \delta_\alpha((1-\alpha)x_0 + \alpha x_1, e)e, \\ &\qquad (1-\alpha)x_0 + \alpha x_1 + \delta_\alpha((1-\alpha)x_0 + \alpha x_1, e)e]^e \\ &= (1-\alpha)[x_0 - \delta_0(x_0, e)e, x_0 + \delta_0(x_0, e)e]^e \\ &\qquad + \alpha[x_1 - \delta_1(x_1, e)e, x_1 + \delta_1(x_1, e)e]^e. \end{aligned}$$

Since $\delta(\alpha x, e) = \alpha\delta(x, e)$ for any $x \in X$ with $x > 0$ and for any $\alpha \in \mathcal{K}_{\mathbf{R}}$, it holds that $O(\alpha x, \delta) = \alpha O(x, \delta)$. Since

$$\begin{aligned} &\delta(z_0, e_0)e_0 + \delta(z_1, e_1)e_1 \\ &\quad = \delta\left(z_0 + z_1, \frac{f(z_0)}{f(e_0)}e_0 + \frac{f(z_1)}{f(e_1)}e_1\right)\left(\frac{f(z_0)}{f(e_0)}e_0 + \frac{f(z_1)}{f(e_1)}e_1\right) \end{aligned}$$

for any $z_0, z_1 \in X$ with $z_0, z_1 > 0$, it holds that $O(z_0, \delta) + O(z_1, \delta) \subset O(z_0 + z_1, \delta)$. Then

$$\begin{aligned} z &\in (1-\alpha)O(x_0, \delta_0) + \alpha O(x_1, \delta_1) \\ &\subset (1-\alpha)O(y_0, \delta) + \alpha O(y_1, \delta) \\ &= O((1-\alpha)y_0, \delta) + O(\alpha y_1, \delta) \\ &\subset O((1-\alpha)y_0 + \alpha y_1, \delta). \end{aligned}$$

Therefore $O((1 - \alpha)x_0 + \alpha x_1, \delta_\alpha) \subset O((1 - \alpha)y_0 + \alpha y_1, \delta)$, that is, $(1 - \alpha)x_0 + \alpha x_1 \in \mathrm{int}(O((1 - \alpha)y_0 + \alpha y_1, \delta)) \subset P$.
(2) $\Rightarrow$ (3): $P \in \mathcal{P}_X$ satisfies $P \in \mathcal{O}_X$, $P \neq \emptyset$ and $\{x \mid x > 0\} \subset co(P) \neq X$.
(3) $\Rightarrow$ (1): Take $x_0 \in co(O)$. Let p be a mapping from X into $[0, \infty]$ defined by $p(x) = \inf\{r \mid r > 0, \frac{1}{r}x \in co(O) - x_0\}$. p satisfies the following:

(1) $p(x) < \infty$;

(2) $\forall \alpha \in \mathbf{R}$ with $\alpha > 0$, $p(\alpha x) = \alpha p(x)$;

(3) $p(x + y) \leq p(x) + p(y)$;

(4) $\{x \mid p(x) < 1\} = co(O) - x_0$.

Since $0 \notin co(O)$, $p(-x_0) \geq 1$. Let $Y = \{\lambda x_0 \mid \lambda \in \mathbf{R}\}$ and g a mapping from Y into $\mathbf{R}$ defined by $g(\lambda x_0) = -\lambda$. g is linear and $g(\lambda x_0) \leq p(\lambda x_0)$. By Hanh-Banach theorem there exists a mapping f from X into $\mathbf{R}$ satisfying $f \leq p$ and $f|_Y = g$. $-f$ is an answer of the proposition. $\square$

4 Criteria for the Hausdorffness

Let X be a vector lattice with unit. Let $|\mathcal{K}_X|$ be the class of x satisfying $|x| \in \mathcal{K}_X$. For any $x \in |\mathcal{K}_X|$ let $x_+^\perp = \{0 \vee x\}^\perp$, $x_-^\perp = \{0 \vee (-x)\}^\perp$,

$$Q(x) = \{x_1 \mid x_1 \in |\mathcal{K}_X|,\ (x_1)_+^\perp = x_+^\perp,\ (x_1)_-^\perp = x_-^\perp\}$$

and

$$\overline{Q}(x) = \left(\bigcup_{x_1, x_2 \in Q(x)} [0 \wedge x_1, 0 \vee x_2] \right) \setminus \{0\}.$$

Theorem 4.1. *Let X be a complete vector lattice with unit and satisfying $\mathcal{P}_X \neq \emptyset$. Then X is Hausdorff.*

Proof. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. It holds that for any $y \in X$ there exists $x \in |\mathcal{K}_X|$ such that $y \in \overline{Q}(x)$. Let $y = \frac{x_2 - x_1}{2}$. Let R_x be a mapping from X into X defined by $R_x(y_1 + y_2) = -y_1 + y_2$ for any $y_1 \in x_+^\perp$ and for any for any $y_2 \in x_-^\perp$. Let $O_1 = \left(\frac{x_1+x_2}{2} - R_x^{-1}(P)\right)$ and $O_2 = \left(\frac{x_1+x_2}{2} + R_x^{-1}(P)\right)$. Then O_1 and O_2 are answers of the proposition. $\square$

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