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Singular integral operators on $B^{p,\lambda}$ with Morrey-Campanato norms

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This is an announcement of our recent work.

1 Definitions

For $r > 0$, let $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$ and $B_r = B(0,r)$, and for $B \subset \mathbb{R}^n$, let

$$f_B = \fint_B f(y)\,dy = \frac{1}{|B|}\int_B f(y)\,dy,$$

where $|B|$ is the Lebesgue measure of B, and let

$$m(B,f,t) = |\{x \in B : |f(x)| > t\}|$$

and

$$m_B(f,t) = \frac{m(B,f,t)}{|B|},$$

where $0 \le t < \infty$.

First, we define the Morrey-Campanato norms on balls.

Definition 1. *For $1 \le p < \infty$, $\lambda \in \mathbb{R}^n$, $0 < \alpha \le 1$ and the ball B_r, let*

$$\|f\|_{L_{p,\lambda}(B_r)} = \sup_{B(x,s)\subset B_r} \frac{1}{s^\lambda}\left(\fint_{B(x,s)} |f(y)|^p\,dy\right)^{1/p},$$

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$$\|f\|_{WL_{p,\lambda}(B_r)} = \sup_{B(x,s)\subset B_r} \frac{1}{s^\lambda} \sup_{t>0} t\, m_{B(x,s)}(f,t)^{1/p},$$

$$\|f\|_{\mathcal{L}_{p,\lambda}(B_r)} = \sup_{B(x,s)\subset B_r} \frac{1}{s^\lambda} \left(\fint_{B(x,s)} |f(y) - f_{B(x,s)}|^p \, dy \right)^{1/p}$$

and

$$\|f\|_{\mathrm{Lip}_\alpha(B_r)} = \sup_{x,y\in B_r, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

Then, the following relation between the Campanato spaces and the Lipschitz spaces is shown.

Theorem 1 (Meyers [M], Spanne [S]). *If $1 \le p < \infty$, $0 < \lambda = \alpha \le 1$ and $r > 0$, then $\mathcal{L}_{p,\lambda}(B_r) = \mathrm{Lip}_\alpha(B_r)$ modulo null-functions and there exists a constant $C > 0$, dependent only on n and λ, such that*

$$C^{-1}\|f\|_{\mathcal{L}_{p,\lambda}(B_r)} \le \|f\|_{\mathrm{Lip}_\alpha(B_r)} \le C\|f\|_{\mathcal{L}_{p,\lambda}(B_r)}.$$

The same conclusion holds on $\mathbb{R}^n$.

Next, we introduce "new" function spaces B^σ spaces, i.e. $B^{p,\lambda}$ with Morrey-Campanato norms (see [MN] for details, and cf. [KM_2]).

Definition 2. *For $0 \le \sigma < \infty$, $1 \le p < \infty$, $\lambda \in \mathbb{R}^n$ and $0 < \alpha \le 1$, let B^σ-$E_{\{\text{name}\}}$ spaces $B^\sigma(E)(\mathbb{R}^n)$ and $\dot{B}^\sigma$-$E_{\{\text{name}\}}$ spaces $\dot{B}^\sigma(E)(\mathbb{R}^n)$ be the sets of all functions f such that the following functionals are finite, respectively:*

$$\|f\|_{B^\sigma(E)} = \sup_{r\ge 1} \frac{1}{r^\sigma}\|f\|_{E(B_r)} \quad \text{and} \quad \|f\|_{\dot{B}^\sigma(E)} = \sup_{r>0} \frac{1}{r^\sigma}\|f\|_{E(B_r)}$$

with

$$E = L^p,\ WL^p,\ L_{p,\lambda},\ WL_{p,\lambda},\ \mathcal{L}_{p,\lambda} \text{ and } \mathrm{Lip}_\alpha.$$

We note that $B^\sigma(L_{p,\lambda})(\mathbb{R}^n)$ unifies $L_{p,\lambda}(\mathbb{R}^n)$ and $B^{p,\lambda}(\mathbb{R}^n)$ and that $B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$ unifies $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ and $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)$. Actually, we have the following relations:

$$B^0(L_{p,\lambda})(\mathbb{R}^n) = L_{p,\lambda}(\mathbb{R}^n), \quad B^0(\mathcal{L}_{p,\lambda})(\mathbb{R}^n) = \mathcal{L}_{p,\lambda}(\mathbb{R}^n) \tag{1}$$

and

$$B^{\lambda+n/p}(L_{p,-n/p})(\mathbb{R}^n) = B^{p,\lambda}(\mathbb{R}^n), \quad B^{\lambda+n/p}(\mathcal{L}_{p,-n/p})(\mathbb{R}^n) = \mathrm{CMO}^{p,\lambda}(\mathbb{R}^n). \tag{2}$$

We also have the same properties for $\dot{B}^\sigma(L_{p,\lambda})(\mathbb{R}^n)$ and $\dot{B}^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$.

Remark . We recall the definitions of several function spaces on $\mathbb{R}^n$ (see [AGL], [FLL], [LY$_1$], [LY$_2$] and [MN]): For $1 \le p < \infty$, $\lambda \in \mathbb{R}^n$ and $0 < \alpha \le 1$,

$$B^{p,\lambda}(\mathbb{R}^n) = \left\{ f : \|f\|_{B^{p,\lambda}} = \sup_{r \ge 1} \frac{1}{r^\lambda} \left(\fint_{B_r} |f(y)|^p \, dy \right)^{1/p} < \infty \right\},$$

$$\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n) = \left\{ f : \|f\|_{\mathrm{CMO}^{p,\lambda}} = \sup_{r \ge 1} \frac{1}{r^\lambda} \left(\fint_{B_r} |f(y) - f_{B_r}|^p \, dy \right)^{1/p} < \infty \right\},$$

$$\dot{B}^{p,\lambda}(\mathbb{R}^n) = \left\{ f : \|f\|_{\dot{B}^{p,\lambda}} = \sup_{r > 0} \frac{1}{r^\lambda} \left(\fint_{B_r} |f(y)|^p \, dy \right)^{1/p} < \infty \right\},$$

$$\mathrm{CBMO}^{p,\lambda}(\mathbb{R}^n) = \left\{ f : \|f\|_{\mathrm{CBMO}^{p,\lambda}} = \sup_{r > 0} \frac{1}{r^\lambda} \left(\fint_{B_r} |f(y) - f_{B_r}|^p \, dy \right)^{1/p} < \infty \right\},$$

$$L_{p,\lambda}(\mathbb{R}^n) = \left\{ f : \|f\|_{L_{p,\lambda}} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{r^\lambda} \left(\fint_{B(x,r)} |f(y)|^p \, dy \right)^{1/p} < \infty \right\},$$

$$WL_{p,\lambda}(\mathbb{R}^n) = \left\{ f ; \|f\|_{WL_{p,\lambda}} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{r^\lambda} \sup_{t>0} t \, m_{B(x,r)}(f,t)^{1/p} < \infty \right\},$$

$$\mathcal{L}_{p,\lambda}(\mathbb{R}^n) = \left\{ f : \|f\|_{\mathcal{L}_{p,\lambda}} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{r^\lambda} \left(\fint_{B(x,r)} |f(y) - f_{B(x,r)}|^p \, dy \right)^{1/p} < \infty \right\}$$

and

$$\mathrm{Lip}_\alpha(\mathbb{R}^n) = \left\{ f : \|f\|_{\mathrm{Lip}_\alpha} = \sup_{x, y \in \mathbb{R}^n, x \ne y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty \right\}.$$

2 Results

We consider a standard singular integral operator T and its modified version $\widetilde{T}$ defined by the following:

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y) f(y) dy,$$

where

$$|K(x)| \le \frac{C_K}{|x|^n} \quad \text{and} \quad |\nabla K(x)| \le \frac{C_K}{|x|^{n+1}}, \quad x \ne 0,$$

$$\int_{\epsilon < |x| < N} K(x) dx = 0 \ \text{ for all } \ 0 < \epsilon < N;$$

$$\tilde{T}f(x) = \text{p.v.} \int_{\mathbb{R}^n} \{K(x-y) - K(-y)(1 - \chi_{B_1}(y))\} f(y) dy,$$

where χ_E is the characteristic function of a set $E \subset \mathbb{R}^n$.

Here, it is known that

$$T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), \quad 1 < p < \infty,$$

$$T : L^1(\mathbb{R}^n) \to WL^1(\mathbb{R}^n),$$
$$\tilde{T} : \mathrm{BMO}(\mathbb{R}^n) \to \mathrm{BMO}(\mathbb{R}^n)$$

and

$$\tilde{T} : \mathrm{Lip}_\alpha(\mathbb{R}^n) \to \mathrm{Lip}_\alpha(\mathbb{R}^n), \quad 0 < \alpha < 1.$$

Also, the following two theorems, which show the extension of boundedness properties of T and $\tilde{T}$ to the Morrey spaces and the Campanato spaces, respectively, are well-known.

Theorem 2 (Peetre [P], Chiarenza and Frasca [CF], Nakai [N_1]). *Let $1 < p < \infty$, $-n/p \le \lambda < 0$ and T be a standard singular integral operator. Then T is bounded on $L_{p,\lambda}(\mathbb{R}^n)$, i.e. there exists a constant $C > 0$ such that*

$$\|Tf\|_{L_{p,\lambda}} \le C\|f\|_{L_{p,\lambda}}, \quad f \in L_{p,\lambda}(\mathbb{R}^n).$$

And also T is bounded from $L_{1,\lambda}(\mathbb{R}^n)$ to $WL_{1,\lambda}(\mathbb{R}^n)$, i.e. there exists a constant $C > 0$ such that

$$\|Tf\|_{WL_{1,\lambda}} \le C\|f\|_{L_{1,\lambda}}, \quad f \in L_{1,\lambda}(\mathbb{R}^n).$$

Theorem 3 (Peetre [P], Nakai [N_2]). *Let $1 < p < \infty$, $-n/p \le \lambda < 1$ and T be a standard singular integral operator. Then $\tilde{T}$ is bounded on $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)/\mathcal{C}$ and $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$, i.e. there exist constants $C_1 > 0$ and $C_2 > 0$ such that*

$$\|\tilde{T}f\|_{\mathcal{L}_{p,\lambda}} \le C_1\|f\|_{\mathcal{L}_{p,\lambda}}, \quad f \in \mathcal{L}_{p,\lambda}(\mathbb{R}^n)/\mathcal{C},$$

and

$$\|\tilde{T}f\|_{\mathcal{L}_{p,\lambda}} + |(\tilde{T}f)_{B_1}| \le C_2\left(\|f\|_{\mathcal{L}_{p,\lambda}} + |f_{B_1}|\right) \quad f \in \mathcal{L}_{p,\lambda}(\mathbb{R}^n),$$

respectively, where $\mathcal{C}$ is the space of all constant functions.

Furthermore, we can extened the boundedness properties of T and $\tilde{T}$ to B^σ-Morrey spaces and B^σ-Campanato spaces, respectively.

Theorem 4. *Let $1 < p < \infty$, $-n/p \le \lambda < 0$, $0 \le \sigma < -\lambda$ and T be a standard singular integral operator. Then T is bounded on $B^\sigma(L_{p,\lambda})(\mathbb{R}^n)$, i.e. there exists a constant $C > 0$ such that*

$$\|Tf\|_{B^\sigma(L_{p,\lambda})} \le C\|f\|_{B^\sigma(L_{p,\lambda})}, \quad f \in B^\sigma(L_{p,\lambda})(\mathbb{R}^n).$$

The same conclusion holds for the boundedness on $\dot{B}^\sigma(L_{p,\lambda})(\mathbb{R}^n)$.

In the above theorem, if $\lambda = -n/p$ and $\sigma = \lambda + n/p$, then by the relation (2), we have the result in [FLL].

Corollary 5 (Fu, Lin and Lu [FLL]). *Let $1 < p < \infty$, $-n/p \leq \lambda < 0$ and T be a standard singular integral operator. Then T is bounded on $B^{p,\lambda}(\mathbb{R}^n)$, i.e. there exists a constant $C > 0$ such that*

$$\|Tf\|_{B^{p,\lambda}} \leq C\|f\|_{B^{p,\lambda}}, \quad f \in B^{p,\lambda}(\mathbb{R}^n).$$

The same conclusion holds for the boundedness on $\dot{B}^{p,\lambda}(\mathbb{R}^n)$.

Theorem 6. *Let $-n \leq \lambda < 0$, $0 \leq \sigma < -\lambda$ and T be a standard singular integral operator. Then T is bounded from $B^\sigma(L_{1,\lambda})(\mathbb{R}^n)$ to $B^\sigma(WL_{1,\lambda})(\mathbb{R}^n)$, i.e. there exists a constant $C > 0$ such that*

$$\|Tf\|_{B^\sigma(WL_{1,\lambda})} \leq C\|f\|_{B^\sigma(L_{1,\lambda})}, \quad f \in B^\sigma(L_{1,\lambda})(\mathbb{R}^n).$$

The same conclusion holds for the boundedness from $\dot{B}^\sigma(L_{1,\lambda})(\mathbb{R}^n)$ to $\dot{B}^\sigma(WL_{1,\lambda})(\mathbb{R}^n)$.

In the above theorem, if $\lambda = -n$ and $\sigma = \lambda + n$, then we have the following.

Corollary 7. *Let $-n \leq \lambda < 0$ and T be a standard singular integral operator. Then T is bounded from $B^{1,\lambda}(\mathbb{R}^n)$ to $WB^{1,\lambda}(\mathbb{R}^n)$, i.e. there exists a constant $C > 0$ such that*

$$\|Tf\|_{WB^{1,\lambda}} \leq C\|f\|_{B^{1,\lambda}}, \quad f \in B^{1,\lambda}(\mathbb{R}^n).$$

The same conclusion holds for the boundedness from $\dot{B}^{1,\lambda}(\mathbb{R}^n)$ to $W\dot{B}^{1,\lambda}(\mathbb{R}^n)$. .

Theorem 8. *Let $1 < p < \infty$, $-n/p \leq \lambda < 1$, $0 \leq \sigma < -\lambda + 1$ and T be a standard singular integral operator. Then $\tilde{T}$ is bounded on $B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)/C$ and $B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$, i.e. there exist constants $C_1 > 0$ and $C_2 > 0$ such that*

$$\|\tilde{T}f\|_{B^\sigma(\mathcal{L}_{p,\lambda})} \leq C_1\|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})}, \quad f \in B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)/C,$$

and

$$\|\tilde{T}f\|_{B^\sigma(\mathcal{L}_{p,\lambda})} + |(\tilde{T}f)_{B_1}| \leq C_2\left(\|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})} + |f_{B_1}|\right) \quad f \in B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n),$$

respectively. The same conclusion holds for the boundedness on $\dot{B}^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)/C$ and $\dot{B}^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$.

In the above theorem, if $\lambda = -n/p$ and $\sigma = \lambda + n/p$, then as a corollary, we have the extension of result in [KM$_1$].

Corollary 9. *Let $1 < p < \infty$, $-n/p \le \lambda < 1$ and T be a standard singular integral operator. Then $\tilde{T}$ is bounded on $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)/C$ and $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)$, i.e. there exist constants $C_1 > 0$ and $C_2 > 0$ such that*

$$\|\tilde{T}f\|_{\mathrm{CMO}^{p,\lambda}} \le C_1 \|f\|_{\mathrm{CMO}^{p,\lambda}}, \quad f \in \mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)/C,$$

and

$$\|\tilde{T}f\|_{\mathrm{CMO}^{p,\lambda}} + |(\tilde{T}f)_{B_1}| \le C_2 \left(\|f\|_{\mathrm{CMO}^{p,\lambda}} + |f_{B_1}|\right), \quad f \in \mathrm{CMO}^{p,\lambda}(\mathbb{R}^n),$$

respectively. The same conclusion holds for the boundedness on $\mathrm{CBMO}^{p,\lambda}(\mathbb{R}^n)/C$ and $\mathrm{CBMO}^{p,\lambda}(\mathbb{R}^n)$.

And, if $\sigma = 0$ and $\lambda = 0$, then by the relation (1), $\tilde{T}$ is bounded on $\mathrm{BMO}(\mathbb{R}^n)$. Also, if $0 < \lambda < 1$, then by Theorem 1, the following corollary is obtained.

Corollary 10. *Let $0 < \alpha < 1$, $0 \le \sigma < -\alpha+1$ and T be a standard singular integral operator. Then $\tilde{T}$ is bounded on $B^\sigma(\mathrm{Lip}_\alpha)(\mathbb{R}^n)/C$ and $B^\sigma(\mathrm{Lip}_\alpha)(\mathbb{R}^n)$, i.e. there exist constants $C_1 > 0$ and $C_2 > 0$ such that*

$$\|\tilde{T}f\|_{B^\sigma(\mathrm{Lip}_\alpha)} \le C_1 \|f\|_{B^\sigma(\mathrm{Lip}_\alpha)}, \quad f \in B^\sigma(\mathrm{Lip}_\alpha)(\mathbb{R}^n)/C,$$

and

$$\|\tilde{T}f\|_{B^\sigma(\mathrm{Lip}_\alpha)} + |(\tilde{T}f)_{B_1}| \le C_2 \left(\|f\|_{B^\sigma(\mathrm{Lip}_\alpha)} + |f_{B_1}|\right), \quad f \in B^\sigma(\mathrm{Lip}_\alpha)(\mathbb{R}^n),$$

respectively. The same conclusion holds for the boundedness on $\dot{B}^\sigma(\mathrm{Lip}_\alpha)(\mathbb{R}^n)/C$ and $\dot{B}^\sigma(\mathrm{Lip}_\alpha)(\mathbb{R}^n)$.

In the above corollary, if $\sigma = 0$, then $\tilde{T}$ is bounded on $\mathrm{Lip}_\alpha(\mathbb{R}^n)$.

3 Proofs of theorems

In the following proofs of theorems, we use the symbol $A \lesssim B$ to denote that there exists a positive constant C such that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, we then write $A \sim B$.

Before proving Theorems 4, 6 and 8, we state the following lemma in [MN] (see also [N_2] for the first part of the lemma).

Lemma 11. *Let $1 \le p < \infty$, $r > 0$,*

$$h(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| \ge 2, \end{cases} \quad x \in \mathbb{R}^n, \quad \textit{such that} \quad \|h\|_{\mathrm{Lip}_1} \le 1, \tag{3}$$

and

$$h_r(\cdot) = h(\cdot/r).$$

(i) *If* $-n/p \le \lambda < 0$, *then for all* $f \in L^p_{loc}(\mathbb{R}^n)$ *with* $\|f\|_{L_{p,\lambda}(B_{3r})} < \infty$,

$$\|f\chi_r\|_{L_{p,\lambda}} \le \|f\|_{L_{p,\lambda}(B_{3r})}.$$

(ii) *If* $-n/p \le \lambda \le 1$, *then there exists a constant* $C > 0$, *dependent only on* n *and* λ, *such that for all* $f \in L^p_{loc}(\mathbb{R}^n)$ *with* $\|f\|_{\mathcal{L}_{p,\lambda}(B_{3r})} < \infty$,

$$\|(f - f_{B_{2r}})h_r\|_{\mathcal{L}_{p,\lambda}} \le C\|f\|_{\mathcal{L}_{p,\lambda}(B_{3r})}.$$

Now we prove the theorems. Here, we omit the proof of Theorem 4 due to the similarity with that of Theorem 6.

Proof of Theorem 6. Let $f \in B^\sigma(L_{1,\lambda})(\mathbb{R}^n)$ and $r \ge 1$. Then, we prove that for any ball B_r,

$$\|Tf\|_{WL_{1,\lambda}(B_r)} \lesssim r^\sigma \|f\|_{B^\sigma(L_{1,\lambda})}.$$

To prove this, let

$$Tf = T(f\chi_{B_{2r}}) + T(f(1-\chi_{B_{2r}})).$$

Now, for any ball $B(x,s) \subset B_r$, it follows that

$$\begin{aligned}
&\frac{1}{s^\lambda}\sup_{t>0} 2t\, m_{B(x,s)}(Tf, 2t) \\
&\le 2\left\{\frac{1}{s^\lambda}\sup_{t>0} t\, m_{B(x,s)}(T(f\chi_{B_{2r}}), t) + \frac{1}{s^\lambda}\sup_{t>0} t\, m_{B(x,s)}(T(f(1-\chi_{B_{2r}})), t)\right\} \\
&= 2\,(I_1 + I_2)\,, \quad \text{say.}
\end{aligned}$$

First, by applying the boundedness of T from $L_{1,\lambda}(\mathbb{R}^n)$ to $WL_{1,\lambda}(\mathbb{R}^n)$ (Theorem 2) and (i) of Lemma 11, we have

$$\begin{aligned}
I_1 &\le \|T(f\chi_{B_{2r}})\|_{WL_{1,\lambda}(B_r)} \le \|T(f\chi_{B_{2r}})\|_{WL_{1,\lambda}} \lesssim \|f\chi_{B_{2r}}\|_{L_{1,\lambda}} \\
&\le \|f\|_{L_{1,\lambda}(B_{6r})} \lesssim r^\sigma \|f\|_{B^\sigma(L_{1,\lambda})}.
\end{aligned}$$

Next, we estimate I_2. It follows that for $x \in B_r$,

$$|T(f(1-\chi_{B_{2r}}))(x)| \lesssim \int_{\mathbb{R}^n\setminus B_{2r}} \frac{|f(y)|}{|y|^n}\,dy \lesssim r^{\lambda+\sigma}\|f\|_{B^\sigma(L_{1,\lambda})}.$$

Therefore, since $\lambda < 0$, we obtain

$$I_2 \le \|T(f(1-\chi_{B_{2r}}))\|_{WL_{1,\lambda}(B_r)} \le r^{-\lambda}\|T(f(1-\chi_{B_{2r}}))\|_{L^\infty(B_r)} \lesssim r^\sigma\|f\|_{B^\sigma(L_{1,\lambda})}.$$

Thus, we have for any ball B_r,

$$\|Tf\|_{WL_{1,\lambda}(B_r)} \lesssim r^\sigma \|f\|_{B^\sigma(L_{1,\lambda})}.$$

This shows the conclusion.

The proof of the boundedness from $\dot{B}^\sigma(L_{1,\lambda})(\mathbb{R}^n)$ to $\dot{B}^\sigma(WL_{1,\lambda})(\mathbb{R}^n)$ is the same as above. □

Proof of Theorem 8. Let $f \in B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$ and $r \geq 1$. Then, we prove that that for any ball B_r,

$$\|\tilde{T}f\|_{\mathcal{L}_{p,\lambda}(B_r)} \lesssim r^\sigma \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})},$$

and then $|(\tilde{T}f)_{B_1}| \lesssim \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})} + |f_{B_1}|$.

Now, let $\tilde{f} = f - f_{B_{4r}}$ and let h be defined by (3). Then, for $x \in B_r$, it follows that

$$\begin{aligned}
\tilde{T}f(x) &= \tilde{T}\tilde{f}(x) + \tilde{T}(f_{B_{4r}})(x) \\
&= T(\tilde{f}h_{2r})(x) + \int_{\mathbb{R}^n} \tilde{f}(1-h_{2r})(y)\,(K(x-y) - K(-y))\,dy \\
&\quad + \int_{\mathbb{R}^n} \tilde{f}(\chi_{B_1} - h_{2r})(y)K(-y)\,dy + f_{B_{4r}}(\tilde{T}1)(x) \\
&= I_1(r)(x) + I_2(r)(x) + I_3(r) + I_4(r)(x), \quad \text{say.}
\end{aligned}$$

Here, note that $\tilde{T}1$ is a constant function and $I_3(r)$ is constant.

First, since $(\chi_{B_1} - h_{2r})/|\cdot|^n$ is in $L^{p'}(\mathbb{R}^n)$, it follows that

$$|I_3(r)| \leq \left\| \frac{\chi_{B_1} - h_{2r}}{|\cdot|^n} \right\|_{L^{p'}} \|\tilde{f}\|_{L^p(B_{4r})} \lesssim \|\tilde{f}\|_{L^p(B_{4r})} \lesssim \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})}. \tag{4}$$

To estimate $I_1(r)$, applying the boundedness of $\tilde{T}$ on $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)/C$ (Theorem 3) and (ii) of Lemma 11, we have

$$\|I_1(r)\|_{\mathcal{L}_{p,\lambda}(B_r)} \leq \|T(\tilde{f}h_{2r})\|_{\mathcal{L}_{p,\lambda}} \lesssim \|\tilde{f}h_{2r}\|_{\mathcal{L}_{p,\lambda}} \lesssim \|f\|_{\mathcal{L}_{p,\lambda}(B_{6r})} \lesssim r^\sigma \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})}.$$

Similarly, by the boundedness of $\tilde{T}$ on $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ (Theorem 3) and (ii) of Lemma 11, we obtain

$$\begin{aligned}
\|I_1(r)\|_{\mathcal{L}_{p,\lambda}(B_r)} + |(I_1(r))_{B_1}| &\leq \|T(\tilde{f}h_{2r})\|_{\mathcal{L}_{p,\lambda}} + |(T(\tilde{f}h_{2r}))_{B_1}| \\
&\lesssim \|\tilde{f}h_{2r}\|_{\mathcal{L}_{p,\lambda}} + |(\tilde{f}h_{2r})_{B_1}| \lesssim r^\sigma \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})} + |(\tilde{f}h_{2r})_{B_1}|.
\end{aligned} \tag{5}$$

Next, we get for $x \in B_r$,

$$|I_2(r)(x)| \leq r \int_{\mathbb{R}^n \setminus B_{2r}} \frac{|f(y) - f_{B_{4r}}|}{|y|^{n+1}}\,dy \lesssim r^{\lambda+\sigma} \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})}. \tag{6}$$

If $-n/p \leq \lambda \leq 0$, then we have

$$\|I_2(r)\|_{\mathcal{L}_{p,0}(B_r)} \lesssim \|I_2(r)\|_{L_{p,0}(B_r)} \leq \|I_2(r)\|_{L^\infty(B_r)} \lesssim r^\sigma \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})}.$$

If $0<\lambda<1$, then we have for any $x, z \in B_r$,

$$\begin{aligned}
|I_2(r)(x) - I_2(r)(z)| &\le \int_{\mathbb{R}^n\setminus B_{2r}} |\tilde{f}(y)||K(x-y) - K(z-y)|\,dy \\
&\lesssim |x-z| \int_{\mathbb{R}^n\setminus B_{2r}} \frac{|f(y) - f_{B_{4r}}|}{|y|^{n+1}}\,dy \\
&\lesssim |x-z|\, r^{-1+\lambda+\sigma} \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})},
\end{aligned}$$

and so

$$\frac{|I_2(r)(x) - I_2(r)(z)|}{|x-z|^\lambda} \lesssim \left(\frac{|x-z|}{r}\right)^{1-\lambda} r^\sigma \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})} \lesssim r^\sigma \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})}.$$

Therefore, by Theorem 1,

$$\|I_2(r)\|_{\mathcal{L}_{p,\lambda}(B_r)} \sim \|I_2(r)\|_{\mathrm{Lip}_\lambda(B_r)} \lesssim r^\sigma \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})}.$$

Thus, we have for any ball B_r,

$$\begin{aligned}
\|\tilde{T}f\|_{\mathcal{L}_{p,\lambda}(B_r)} &= \|I_1(r) + I_2(r) + I_3(r) + I_4(r)\|_{\mathcal{L}_{p,\lambda}(B_r)} \\
&\lesssim r^\sigma \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})},
\end{aligned}$$

which gives the conclusion

$$\|\tilde{T}f\|_{B^\sigma(\mathcal{L}_{p,\lambda})} \lesssim \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})}.$$

Finally, we estimate each term of right hand side in the inequality

$$|(\tilde{T}f)_{B_1}| \le |(I_1(1))_{B_1}| + |(I_2(1))_{B_1}| + |I_3(1)| + |I_4(1)|.$$

By taking $r=1$ in (4), (5) and (6), it follows that

$$|I_3(1)| \lesssim \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})},$$

$$|(I_1(1))_{B_1}| \lesssim \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})} + |(\tilde{f}h_2)_{B_1}| = \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})} + |f_{B_1} - f_{B_4}|$$

and

$$|(I_2(1))_{B_1}| \lesssim \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})},$$

respectively. Moreover,

$$|f_{B_1} - f_{B_4}| \lesssim \|f\|_{\mathcal{L}_{p,\lambda}(B_4)} \lesssim \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})}.$$

Therefore, we prove that

$$|(\tilde{T}f)_{B_1}| \lesssim \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})} + |f_{B_1}|.$$

Thus, we complete the proof of the desired coclusion.

The proof of the boundedness of $\tilde{T}$ on $\dot{B}^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)/C$ and on $\dot{B}^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$ is the same as above. □

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