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三角不等式の一考察

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1. Introduction and preliminaries.

The triangle inequality is one of the most fundamental inequalities in analysis and has been studied by several authors. In this note, we consider an another aspect of the classical triangle inequality of a normed linear space X, that is, for every $x, y \in X$,

$$\|x+y\| \leq \|x\| + \|y\|.$$

For an inner product space H we recall the parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad (x, y \in H).$$

This implies that the parallelogram inequality

$$\|x+y\|^2 \leq 2(\|x\|^2 + \|y\|^2) \quad (\forall x, y \in H) \tag{1}$$

holds. S. Saitoh noted the inequality (1) may be more suitable than the classical triangle inequality, and used the inequality (1) to the setting of a natural sum Hilbert space for two arbitrary Hilbert spaces.

In general, for any normed linear space X, we easily have

$$\|x+y\|^2 \leq 2(\|x\|^2 + \|y\|^2) \quad (\forall x, y \in X). \tag{2}$$

Recently, Belbachir, Mirzavaziri and Moslehian [1] introduced the notion of q-norm $(1 \leq q < \infty)$ in a vector space X over $\mathbb{K}(= \mathbb{R}$ or $\mathbb{C})$, where the definition of q-norm is a mapping $\|\cdot\|$ from X into $\mathbb{R}^+(=\{a \in \mathbb{R} : a \geq 0\})$ satisfying the following conditions:

$$\begin{aligned} &\text{(i)} \quad \|x\| = 0 \Leftrightarrow x = 0, \\ &\text{(ii)} \quad \|\alpha x\| = |\alpha|\|x\| \quad (x \in X, \alpha \in \mathbb{K}), \\ &\text{(iii)} \quad \|x+y\|^q \leq 2^{q-1}(\|x\|^q + \|y\|^q) \quad (x, y \in X). \end{aligned}$$

We easily show that every norm is a q-norm. Conversely, they proved that for all q with $1 \leq q < \infty$, every q-norm is a norm in the usual sense.

Let Ψ_2 of all continuous convex functions ψ on the unit interval $[0,1]$ satisfying $\psi(0) = \psi(1) = 1$ and $\max\{1-t, t\} \leq \psi(t) \leq 1$ for t with $0 \leq t \leq 1$.

In this note, we generalize the notion of q-norm, that is, we introduce the notion of ψ-norm by considering the fact that an absolute normalized norm on $\mathbb{R}^2$ corresponds to a continuous convex function ψ on the unit interval $[0,1]$ with some conditions. We show that a ψ-norm is a norm in the usual sense.

2. A ψ-norm is really a norm.

At first, we introduce the notion of ψ-norm on a vector space X.

Definition 1. Let X be a vector space and $\psi \in \Psi_2$. Then a mapping $\|\cdot\| : X \longrightarrow \mathbb{R}^+$ is called *ψ-norm on X* if it satisfies the following conditions:

$$\begin{aligned} &\text{(i)} \quad \|x\| = 0 \Leftrightarrow x = 0 \\ &\text{(ii)} \quad \|\alpha x\| = |\alpha|\|x\| \quad (x \in X, \alpha \in \mathbb{K}) \\ &\text{(iii)} \quad \|x+y\| \leq \frac{1}{\min\limits_{0\leq t\leq 1} \psi(t)} \|(\|x\|, \|y\|)\|_\psi \quad \text{for any } x, y \in X. \end{aligned}$$

Note that for all q with $1 \leq q < \infty$, any ψ_q-norm $\|\cdot\|$ is just a q-norm.

Indeed, since the function ψ_q takes the mininum at $t = 1/2$ and

$$\psi_q(1/2) = ((1/2)^q + (1/2)^q)^{1/q} = 2^{1/q-1},$$

the condition (iii) of Definition 1 implies

$$\|x+y\| \le \frac{1}{\psi_q(1/2)}\|(\|x\|, \|y\|)\|_{\psi_q} = 2^{1-1/q}(\|x\|^q + \|y\|^q)^{1/q}.$$

Thus we have $\|x+y\|^q \le 2^{q-1}(\|x\|^q + \|y\|^q)$ and so $\|\cdot\|$ becomes a q-norm.

If $\psi = \psi_1$, then the condition (iii) of Definition 1 is just a triangle inequality. Thus we suppose that $\psi \neq \psi_1$.

Proposition 2. Let X be a vector space and $\psi \in \Psi_2$ with $\psi \neq \psi_1$. Then every norm on X in the usual sense is a ψ-norm.

Conversely, we show that every ψ-norm is a norm in the usual sense. To do this, we need the following lemma given in [1].

Lemma 3. Let X be a vector space. Let $\|\cdot\| : X \to \mathbb{R}^+$ be a mapping satisfying the conditions (i) and (ii) in Definition 1. Then $\|\cdot\|$ is a norm if and only if the set $B_X = \{x \in X : \|x\| \le 1\}$ is convex.

Since every ψ_1-norm is just a usual norm, we suppose that $\psi \in \Psi_2$ with $\psi \neq \psi_1$. Put t_0 with $0 < t_0 < 1$ such that $\min_{0\le t\le 1}\psi(t) = \psi(t_0)$. Then we have the following lemma.

Lemma 4. Let $\|\cdot\|$ be a ψ-norm on X. Then, for every $x, y \in B_X$, $(1-t_0)x + t_0 y \in B_X$.

Here we define the set A_n for all $n = 1, 2, \cdots$, by

$$A_0 = \{0, 1\}, \quad A_n = \{(1-t_0)a + t_0 b : a, b \in A_{n-1}\} \quad (n = 1, 2, \cdots).$$

Put $A = \cup_{n=0}^{\infty} A_n$. It is clear that $\overline{A} = [0, 1]$. We also define a function f by $f(x, y, t) = (1-t)x + ty$ for all $x, y \in B_X$ and all $t \in [0, 1]$.

Lemma 5. For every $x, y \in B_X$ we have $f(x, y, t) \in B_X$ for all $t \in A$.

By Lemma 5, we have the following theorem.

Theorem 6. Let X be a vector space and $\psi \in \Psi_2$ with $\psi \neq \psi_1$. Then every ψ-norm on X is a norm in the usual sense.

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