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VISCOSITY APPROXIMATION METHODS FOR FAMILIES OF STRICTLY PSEUDOCONTRACTIVE MAPPINGS AND NONSELF NONEXPANSIVE MAPPINGS

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1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H. Then, a mapping $T: C \rightarrow C$ is called nonexpansive if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. Browder [3] introduced the following iterations and proved strong convergence theorem:

$$u_n = \alpha_n u + (1-\alpha_n) T u_n \quad \text{for every} \quad n = 1, 2, \ldots. \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ converging to 0, and $u \in C$. Reich [13] and Takahashi and Ueda [21] extended Browder's result to those of a Banach space. Wittmann [24] obtained a strong convergence theorem in Hilbert spaces by using the iteration procedure which was initially introduced by Halpern [6]:

$$\begin{aligned} &x_1 \in C \quad \text{and} \\ &x_{n+1} = \alpha_n x_1 + (1-\alpha_n) T x_n, \quad n = 1, 2, \ldots, \end{aligned} \tag{1.2}$$

where $\alpha_n \in [0,1]$ (see [24, 19] for the proof). Moudafi[8] generalize Browder's and Halpern's theorems [3, 6]. Moudafi's generalizations are called viscosity approximations. Xu extend Moudafi's theorems toe uniformly smooth Banach spaces (see also [20]). Petrusel and Yao [11] studied viscosity approximations with generalized contraction mappings and nonexpansive mappings, and they proved strong convergence theorems for the mappings. Wangkeeree [23] studied viscosity approximations with nonself nonexpansive mappings and proved strong convergence theorems for the mappings.

In this paper, we study implicit and explicit viscosity approximations with generalized contraction mappings and strictly pseudocontractive mapppings, and prove strong convergence theorems for the families of strictly pseudocontractive mappings Further, we study implicit and explicit viscosity approximations with generalized contraction mappings and nonself nonexpansive mappings. We prove strong convergence theorems for the nonself nonexpansive mappings.

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2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the set of all positive integers, the set of all real numbers, respectively. We also denote by $\mathbb{R}^+$ the set of all nonnegative real numbers. Let E be a real Banach space with norm $\|\cdot\|$. We denote by B_r the set $\{x \in E : \|x\| \leq r\}$. Let E^* be the dual space of a Banach space E. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$. Let E be a real Banach space and let C be a nonempty closed convex subset of E. We denote by I the identity operator on E. The multi-valued mapping J from E into E^* defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \quad \text{for every} \quad x \in E$$

is called the duality mapping of E. From the Hahn-Banach theorem, we see that $J(x) \neq \emptyset$ for all $x \in E$. For $q > 1$, we denote by J_q the generalized duality mapping,

$$J_q(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\} \quad \text{for every} \quad x \in E.$$

We recall that

$$J_q(x) = \|x\|^{q-2} J(x)$$

for $x \neq 0$. We recall that

$$\rho(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1, \|x\| < 1, \|y\| \leq t\right\}.$$

E is said to be uniformly smooth if $\lim_{t\to 0} \rho(t)/t = 0$. Let $q > 1$. E is said to be q-uniformly smooth if there is a constant $c > 0$ such that $\rho(t) < ct^q$ (see, for example, [10, 4]).

A Banach space E is said to be strictly convex if

$$\frac{\|x+y\|}{2} < 1$$

for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space, we have that if $\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\|$ for $x, y \in E$ and $\lambda \in (0,1)$, then $x = y$. For every ε with $0 \leq \varepsilon \leq 2$, we define the modulus $\delta(\varepsilon)$ of convexity of E by

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} \ : \ \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\right\}.$$

A Banach space E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If E is uniformly convex, then for r, ε with $r \geq \varepsilon > 0$, we have $\delta\left(\frac{\varepsilon}{r}\right) > 0$ and

$$\left\|\frac{x+y}{2}\right\| \leq r\left(1 - \delta\left(\frac{\varepsilon}{r}\right)\right)$$

for every $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x-y\| \geq \varepsilon$. It is well-known that a uniformly convex Banach space is reflexive and strictly convex. Banach space E is said to be smooth if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each x and y in S_1, where $S_1 = \{u \in E : \|u\| = 1\}$. The norm of E is said to be uniformly Gâteaux differentiable if for each y in S_1, the limit is attained uniformly for x in S_1. We know that if E is smooth, then the duality mapping is single-valued and norm to weak star continuous and that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping is single-valued and norm to weak star, uniformly continuous on each bounded subset of E.

Let μ be a mean on positive integers $\mathbb{N}$, i.e., a continuous linear functional on l^∞ satisfying $\|\mu\| = 1 = \mu(1)$. We know that μ is a mean on $\mathbb{N}$ if and only if

$$\inf\{a_n : n \in \mathbb{N}\} \leq \mu(f) \leq \sup\{a_n : n \in \mathbb{N}\}$$

for each $f = (a_1, a_2, \ldots) \in l^\infty$. Occasionally, we use $\mu_n(a_n)$ instead of $\mu(f)$. So, a Banach limit μ is a mean on $\mathbb{N}$ satisfying $\mu_n(a_n) = \mu_n(a_{n+1})$. Let $f = (a_1, a_2, \ldots) \in l^\infty$ and let μ be a Banach limit on $\mathbb{N}$. Then,

$$\varliminf_{n\to\infty} a_n \leq \mu(f) = \mu_n(a_n) \leq \varlimsup_{n\to\infty} a_n.$$

Specially, if $a_n \to a$, then $\mu(f) = \mu_n(a_n) = a$ (see [17, 19]).

Let E be a real Banach space and let C be a nonempty closed convex subset of E. Then, a mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. A function ψ: $\mathbb{R}^+ \to \mathbb{R}^+$ is said to be L-function if $\psi(0) = 0$, $\psi(t) > 0$ for $t > 0$ and for any $s > 0$, there exists $u > s$ such that $\psi(t) \leq s$ for $t \in [s, u]$. A mapping f from E into E is said to be (ψ, L)-contraction if $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is L-function and $\|f(x) - f(x)\| < \psi(\|x - y\|)$ for all $x, y \in E$ with $x \neq y$. A mapping f: $C \to C$ is said to be Meir-Keeler type mapping if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for any $x, y \in E$ with $\|x - y\| < \varepsilon + \delta \quad \|f(x) - f(y)\| < \varepsilon$ (see [9]). If f is k-contractive, then f is a Meir-Keeler type mapping and (ϕ, L)-contraction. By a generalized contraction mapping we mean a Meir-Keeler type mapping or (ϕ, L)-contraction (see [2, 7, 9, 11, 12, 16]). Let $\mathcal{S} = \{T_i\}_{i=1}^r$ be a family of mappings from C into itself and let $F(\mathcal{S})$ be the set of common fixed points of $\{T_n\}$, i.e., $F = \cap_{n=1}^\infty F(T_n)$.

3. STRONG CONVERGENCE THEOREMS FOR FAMILIES OF STRICTLY PSEUDOCONTRACTIVE MAPPINGS

In this section, we study implicit and explicit viscosity approximations with families of strict pseudocontractive mappings (see also [4]).

A mapping $T : C \to C$ is called pseudocontractive if there exists some $j(x - y) \in J(x - y)$ such that $\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2$ for all $x, y \in C$. A mapping $T : C \to C$ is called strongly pseudocontractive if there exists a constant $\alpha \in (0, 1)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \alpha\|x - y\|^2 \quad (x, y \in C)$$

for some $j(x - y) \in J(x - y)$. A mapping $T : C \to C$ is called k-strictly pseudocontractive in the Browder-Petsyshin sense if $I - T$ is k-inversely strongly monotone, i.e., for all $x, y \in C$ and $j(x - y) \in J(x - y)$

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - k\|x - y - T(x - y)\|^2.$$

If E is a q-uniformly smooth Banach space with single-valued generalized duality mapping j_q, $T : C \to C$ is called (q)-k-strictly pseudocontractive if for all $x, y \in C$

$$\langle Tx - Ty, j_q(x-y)\rangle \le \|x-y\|^q - k\|x - y - T(x-y)\|^q.$$

We note that for $q = 2$, the class of (q)-k-strictly pseudocontractive mappings coincides with that of strictly pseudocontractive mappings (see also [10]).

Let C be a nonempty convex subset of a Banach space E. Let $T_1, T_2, \ldots, T_r$ be mappings of C into itself and let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be real numbers such that $0 \le \alpha_i \le 1$ for every $i = 1, 2, \ldots, r$. Then, we define a mapping W of C into itself as follows (see [18, 14]):

$$\begin{aligned} U_1 &= \alpha_1 T_1 + (1-\alpha_1)I, \\ U_2 &= \alpha_2 T_2 U_1 + (1-\alpha_2)I, \\ &\vdots \\ U_{r-1} &= \alpha_{r-1} T_{r-1} U_{r-2} + (1-\alpha_{r-1})I, \\ W = U_r &= \alpha_r T_r U_{r-1} + (1-\alpha_r)I. \end{aligned} \tag{3.1}$$

Such a mapping W is called the W-mapping generated by $T_1, T_2, \ldots, T_r$ and $\alpha_1, \alpha_2, \ldots, \alpha_r$. Let $\alpha_{n1}, \alpha_{n2}, \ldots, \alpha_{nr} (n = 1, 2, \ldots)$ be real numbers such that $0 \le \alpha_{ni} \le 1$ for every $i = 1, 2, \ldots, r$. Let $W_n (n = 1, 2, \ldots)$ be the W-mappings generated by $T_1, T_2, \ldots, T_r$ and $\alpha_{n1}, \alpha_{n2}, \ldots, \alpha_{nr}$.

Now consider the following implicit iteration scheme:

$$x_n = \beta_n f(x_n) + (1-\beta_n) W_n x_n \quad \text{for every} \quad n \in \mathbb{N},$$

where $\{\beta_n\}$ is a sequence of real numbers such that $0 < \beta_n < 1$ for every $n \in \mathbb{N}$. And we study the following explicit iteration scheme: $x_1 = x \in C$,

$$x_{n+1} = \beta_n f(x_n) + (1-\beta_n) W_n x_n \quad \text{for every} \quad n \in \mathbb{N},$$

where $\{\beta_n\}$ is a sequence of real numbers such that $0 < \beta_n < 1$ for every $n \in \mathbb{N}$. We can prove a strong convergence theorem by an implicit viscosity approximation method (see also [1, 4]).

Theorem 3.1. Let E be a q-uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let $\mathcal{S} = \{T_i\}_{i=1}^r$ be a family of (q)-k-strictly pseudocontractive mappings from C into itself such that $F(\mathcal{S}) = \bigcap_{i=1}^r F(T_i) \ne \emptyset$. Let f be a generalized contraction mapping. Let $\{\alpha_{ni}\}_{i=1}^r$ be a sequence of real numbers such that $\alpha_{ni} \in [a, b]$ for $0 < a < b < 1$ and let $\{\beta_n\}$ be a sequence of real numbers such that $0 < \beta_n < 1$ with $\lim_{n\to\infty} \beta_n = 0$. Let $W_n (n = 1, 2, \ldots)$ be the W-mappings of C into itself generated by $T_1, T_2, \ldots, T_r$ and $\alpha_{n1}, \alpha_{n2}, \ldots, \alpha_{nr}$. Let $\{x_n\}$ be a sequence defined by

$$x_n = \beta_n f(x_n) + (1-\beta_n) W_n x_n$$

for every $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $p \in F(\mathcal{S})$. Further, p is the unique solution of the variational inequality :

$$\langle (f-I)p, j(u-p)\rangle \le 0$$

for all $u \in F(\mathcal{S})$.

Now we can prove a strong convergence theorem by an explicit viscosity approximation method (see also [1, 4]).

Theorem 3.2. Let E be a q-uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let $\mathcal{S} = \{T_i\}_{i=1}^r$ be a family of (q)-k-strictly pseudocontractive mappings from C into itself such that $F(\mathcal{S}) = \bigcap_{i=1}^r F(T_i) \neq \emptyset$. Let f be a generalized contraction mapping. Let $\{\alpha_{ni}\}_{i=1}^r$ and $\{\beta_n\}$ be sequences of real numbers satisfying the following:

(i) $\alpha_{ni} \in [a, b]$ for $0 < a < b < 1$ and $\beta_n \in (0, 1)$;

(ii) $\lim_{n\to\infty} \beta_n = 0$;

(iii) $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iv) $\lim_{n\to\infty} \frac{\beta_n}{\beta_{n+1}} = 1$;

(v) $\lim_{n\to\infty} \frac{1}{\beta_n} \sum_{i=1}^{r} |\alpha_{n+1i} - \alpha_{ni}| = 0$.

Let $W_n (n = 1, 2, \dots)$ be the W-mappings of C into itself generated by $T_1, T_2, \dots, T_r$ and $\alpha_{n1}, \alpha_{n2}, \dots, \alpha_{nr}$. Let $\{x_n\}$ be a sequence defined by $x_1 = x \in C$ and

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) W_n x_n$$

for every $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $p \in F(\mathcal{S})$. Further, p is the unique solution of the variational inequality :

$$\langle (f - I)p, j(u - p) \rangle \leq 0$$

for all $u \in F(\mathcal{S})$.

4. Strong convergence theorems for nonself mappings

In this section, we study implicit and explicit viscosity approximations with generalized contraction mappings and nonself nonexpansive mappings (see [1]). Now we can prove a strong convergence theorem by an implicit viscosity approximation method (see [1]).

Theorem 4.1. Let E be a uniformly convex Banach space which admits a weakly sequentially continuous duality mapping J from E to E^*. Let C be a nonempty closed convex subset of E. Suppose that C is a sunny nonexpansive retract of E. Let P be a sunny nonexpansive retraction of E onto C, let T be a nonself nonexpansive mapping of C into E such that $F(T) \neq \emptyset$ and let f be a generalized contraction mapping. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < \alpha_n < 1$ and $\lim_{n\to\infty} \alpha_n = 0$. If $\{x_n\}$ is given by

$$x_n = \frac{1}{n} \sum_{j=1}^{n} P(\alpha_n f(x_n) + (1 - \alpha_n)(TP)^j x_n)$$

for every $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to $p \in F(T)$. Further, p is the unique solution of the variational inequality :

$$\langle (f - I)p, j(u - p) \rangle \leq 0$$

for all $u \in F(T)$.

We can prove a strong convergence theorem by an explicit viscosity approximation method (see [1]).

Theorem 4.2. Let E be a uniformly convex Banach space which admits a weakly sequentially continuous duality mapping J from E to E^*. Let C be a nonempty closed convex subset of E. Suppose that C is a sunny nonexpansive retract of E. Let P be a sunny nonexpansive retraction of E onto C, let T be a nonself nonexpansive mapping of C into E such that $F(T) \neq \emptyset$ and let f be a generalized contraction mapping. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < \alpha_n < 1$, $\lim_{n \to \infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $\{x_n\}$ is given by $x_1 = x \in C$ and

$$x_{n+1} = \frac{1}{n} \sum_{j=1}^{n} P(\alpha_n f(x_n) + (1 - \alpha_n)(TP)^j x_n)$$

for every $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to $p \in F(T)$. Further, p is the unique solution of the variational inequality :

$$\langle (f - I)p, j(u - p) \rangle \leq 0$$

for all $u \in F(T)$.

We also have a strong convergence theorem by an explicit viscosity approximation method (see [1]).

Theorem 4.3. Let E be a uniformly convex Banach space which admits a weakly sequentially continuous duality mapping J from E to E^*. Let C be a nonempty closed convex subset of E. Suppose that C is a sunny nonexpansive retract of E. Let P be a sunny nonexpansive retraction of E onto C, let T be a nonself nonexpansive mapping of C into E such that $F(T) \neq \emptyset$ and let f be a generalized contraction mapping. Let $\{\alpha_n\}$ a sequence of real numbers such that $0 < \alpha_n < 1$, $\lim_{n \to \infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $\{x_n\}$ is given by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) \frac{1}{n} \sum_{j=1}^{n} (PT)^j x_n$$

for every $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to $p \in F(T)$. Further, p is the unique solution of the variational inequality :

$$\langle (f - I)p, j(u - p) \rangle \leq 0$$

for all $u \in F(T)$.

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