Remarks on the coloring number of graphs †

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Abstract

We give two characterizations of graphs with coloring number $\leq \kappa$ in terms of elementary submodels; one under ZFC and another under SSH and the version of very weak square principle of [8].

These characterizations suggest that the graphs with coloring number $\leq \kappa$ behave very much like the Boolean algebras with $\kappa$-Freese-Nation property (see [5], [8]).

1 Introduction

A graph $G = \langle G, K \rangle$ ($K \subseteq [G]^2$) has coloring number $\leq \kappa$ (notation: $\text{col}(G) \leq \kappa$) if there is a well ordering $\Subset$ on $G$ such that $K_\Subset^a = \{ b \in G : b \Subset a \}$ and $\{a, b\} \in $
$K$ has cardinality $< \kappa$ for all $a \in G$ ([3]). The coloring number $\text{col}(G)$ of $G$ is then defined as the minimum of such $\kappa$'s. It is easy to see that the chromatic number $\chi(G)$ of $G$ is less or equal to $\text{col}(G)$.

The purpose of this note is to show that the graphs with coloring number $\leq \kappa$ behave quite similarly to the Boolean algebras with $\kappa$-Freese-Nation property (see e.g. [5], [8]).

In Section 2 we give a characterization of graphs with coloring number $\leq \kappa$ in terms of elementary submodels (Theorem 2.4). As an application of the characterization, we present in Section 3 a short proof of the countability of the coloring number of the plane.

In Section 4, we show that the characterization of Section 2 can be yet sharpened under SSH and the version of the very weak square principle introduced in [8] (Theorem 4.2).

Both Theorems 2.4 and 4.2 find their parallels in the theory of Boolean algebras with $\kappa$-Freese-Nation property (see Proposition 3 and Theorem 10 in [8]).

The following theorem also underlines the analogy between the Boolean algebras with the $\kappa$-Freese-Nation property and the graphs with coloring number $\leq \kappa$ in the case of $\kappa = \aleph_0$. Note that Boolean algebras with $\aleph_0$-Freese Nation property are also called *openly generated*.

If $G = \langle G, K \rangle$ is graph then we identify any subset $H$ of $G$ with the graph $G \upharpoonright H = \langle H, K \cap [H]^2 \rangle$.

**Theorem 1.1** ([6] and [7]). The following assertions are equivalent over ZFC:

1. $(\alpha)$ For any Boolean algebra $B$ if there are club many subalgebras of $B$ of cardinality $\aleph_1$ which are openly generated then $B$ is openly generated.
2. $(\beta)$ For any graph $G$ if $\text{col}(H) \leq \aleph_0$ for every $H \in [G]^{\aleph_1}$ then $\text{col}(G) \leq \aleph_0$.

Theorem 1.1 in the formulation as above is a sort of bluff since we actually proved that each of the assertions $(\alpha)$ and $(\beta)$ is equivalent to the set-theoretic principle FRP introduced in [4].

**2 A characterization of graphs with coloring number $\leq \kappa$**

We use here the following notations. The first one was already used in the introduction:

For a linear ordering $\Subset$ on a graph $G = \langle G, K \rangle$ we denote
\[(2.1) \quad K^a_{\Subset} = \{b \in G : b \Subset a \text{ and } \{a, b\} \in K\}.\]

If \( H \subseteq G \) then we write
\[(2.2) \quad K^{H,a}_{\Subset} = \{b \in H : b \Subset a \text{ and } \{a, b\} \in K\}.\]

For a graph \( G = \langle G, K \rangle \), \( H \subseteq G \) and \( a \in G \), let
\[(2.3) \quad K^a_H = \{b \in H : \langle a, b \rangle \in K\}.\]

We write \( H \subseteq_{\kappa} G \) if \( |K^a_H| < \kappa \) for all \( a \in G \setminus H \).

A mapping \( f : G \rightarrow [G]^{<\kappa} \) is a \( \kappa \)-coloring mapping on \( G \) if for any \( a, b \in G \) with \( \{a, b\} \in K \), at least one of \( a \in f(b) \) or \( b \in f(a) \) holds.

**Lemma 2.1** ([7]). For any graph \( G \) and any infinite cardinal \( \kappa \), the following are equivalent:

(a) \( \text{col}(G) \leq \kappa \).

(b) There is a \( \kappa \)-coloring mapping on \( G \).

**Proof.** (a) \( \Rightarrow \) (b): Suppose that \( \text{col}(G) \leq \kappa \) and let \( \Subset \) be a well-ordering on \( G \) such that \( |K^a_{\Subset}| < \kappa \) for all \( a \in G \). Then \( f : G \rightarrow [G]^{<\kappa} \) defined by \( f(a) = K^a_{\Subset} \) for \( a \in G \) is a \( \kappa \)-coloring mapping.

(b) \( \Rightarrow \) (a): Suppose that \( f : G \rightarrow [G]^{<\kappa} \) is a \( \kappa \)-coloring mapping on \( G \). Let \( \Subset \) be a well-ordering on \( G \) such that all initial segments of \( G \) of order-type of the form \( \kappa \cdot \alpha \) with respect to \( \Subset \) are closed with respect to \( f \). Then \( \Subset \) is as desired:

**Claim 2.1.1.** \( |K^a_{\Subset}| < \kappa \) for all \( a \in G \).

\( \vdash \) Suppose that \( a \in G \) is the \( \kappa \cdot \alpha + \beta \)'th element with respect to \( \Subset \) where \( \beta < \kappa \). Then the first \( \kappa \cdot \alpha \) elements of \( G \) are closed with respect to \( f \) and hence if \( b \) is among them and \( \{a, b\} \in K \) then we have \( b \in f(a) \). Thus

\[
K^a_{\Subset} \subseteq \{b \in G : b \text{ is the } \gamma \text{'th element for some } \kappa \cdot \alpha \leq \gamma < \kappa \cdot \alpha + \beta \}
\cup f(a).
\]

The right side of the inclusion has size \( < \kappa \) (note that we need here the infinity of \( \kappa \)). Hence \( |K^a_{\Subset}| < \kappa \).

\( \dashv \) (Claim 2.1.1)

\( \square \) (Lemma 2.1)

**Lemma 2.2.** Suppose that \( \langle G_\alpha : \alpha < \delta \rangle \) is a filtration of a graph \( G = \langle G, K \rangle \) and \( \kappa \) is an infinite cardinal. If \( G_\alpha \subseteq_{\kappa} G \) and \( \text{col}(G_{\alpha+1}) \leq \kappa \) for all \( \alpha < \delta \), then we have \( \text{col}(G) \leq \kappa \).
Proof. For $a \in G$ let $o(a) = \min\{\alpha < \delta : a \in G_{\alpha+1}\}$. For $\alpha < \delta$, let $\preceq_{\alpha+1}$ be a well-ordering of $G_{\alpha+1}$ witnessing $col(G_{\alpha+1}) \leq \kappa$. Let $\preceq$ be the ordering on $G$ defined by:

\[(2.4) \quad a \preceq b \iff o(a) < o(b) \text{ or } (o(a) = o(b) \text{ and } a \preceq_{o(a)+1} b)\]

Then $\preceq$ is a well ordering on $G$. The following claim shows that $\preceq$ witnesses that $G$ has coloring number $< \kappa$.

Claim 2.2.1. $|K^a_{\preceq}| < \kappa$ for all $a \in G$.

\[
\vdash \text{For } a \in G, \text{ we have } K^a_{\preceq} \subseteq K^a_{G_{o(a)}} \cup K^G_{o(a)+1,a}. \text{ Since the right side of the inclusion is of cardinality } < \kappa, \text{ it follows that } |K^a_{\preceq}| < \kappa. \]

\[\square\text{ (Lemma 2.2)}\]

Lemma 2.3. Suppose that $H_0$ and $H_1$ are subsets of $G$ with $H_0 \subseteq \kappa G$ and $H_1 \subseteq \kappa G$. Then we have $H_0 \cap H_1 \subseteq \kappa G$.

Proof. Suppose that $a \in G \setminus (H_0 \cap H_1)$. Then we have $a \in G \setminus H_0$ or $a \in G \setminus H_1$.

If $a \in G \setminus H_0$, then $K^a_{H_0 \cap H_1} \subseteq K^a_{H_0}$. And hence $|K^a_{H_0 \cap H_1}| < \kappa$. If $a \in G \setminus H_1$, then $K^a_{H_0 \cap H_1} \subseteq K^a_{H_1}$. And hence again we have $|K^a_{H_0 \cap H_1}| < \kappa$.

This shows $H_0 \cap H_1 \subseteq \kappa G$. \[\square\text{ (Lemma 2.3)}\]

Theorem 2.4. For any graph $G = \langle G, K \rangle$ and an infinite cardinal $\kappa$, the following are equivalent:

(a) $col(G) \leq \kappa$.

(a') There is a well-ordering $\preceq$ of $G$ of order-type $|G|$ such that $|K^a_{\preceq}| < \kappa$ for all $a \in G$. 

(b) $G$ has a $\kappa$-coloring mapping.

(c) For $a$ all sufficiently large regular $\chi$ and for all $M < \mathcal{H}(\chi)$ such that $\langle G, K \rangle \in M$ and $\kappa + 1 \subseteq M$ we have $G \cap M \subseteq \kappa G$.

Proof. (a) $\Rightarrow$ (b) was already proved in Lemma 2.1. (a') $\Rightarrow$ (a) is trivial. The proof of (b) $\Rightarrow$ (a) in Lemma 2.1 actually proves (b) $\Rightarrow$ (a').

For (a) $\Rightarrow$ (c), suppose that $G = \langle G, K \rangle$ has coloring number $\leq \kappa$. Let $\chi$ be a sufficiently large regular cardinal and $M < \mathcal{H}(\chi)$ be such that $G \in M$ and $\kappa + 1 \subseteq M$. By elementarity and (a) $\Leftrightarrow$ (b), there is $f \in M$ such that $f$ is a $\kappa$-coloring mapping on $G$. Note that by $\kappa + 1 \subseteq M$ and by elementarity, $G \cap M$ is closed with respect to $f$. For $a \in G \setminus M$ and $b \in K^a_{G \cap M}$, since $a \notin f(b) \subseteq M$, we have $b \in f(a)$. Thus $K^a_{G \cap M} \subseteq f(a)$ and hence $|K^a_{G \cap M}| < \kappa$. This shows that $G \cap M \subseteq \kappa G$. 

\[\kappa\]
Now we prove (c) \(\Rightarrow\) (a) by induction on \(|G|\).

If \(|G| \leq \kappa\), then (c) \(\Rightarrow\) (a) holds since \(G\) then has coloring number \(\leq \kappa\) anyway — any well-ordering of \(G\) of order-type \(|G|\) will witness this.

Suppose that \(|G| > \kappa\) and we have shown the implication (c) \(\Rightarrow\) (a) for all graphs of cardinality \(< |G|\). Let \(\lambda = |G|, \lambda^* = \text{cf}(\lambda)\) and \(\langle M_\alpha : \alpha < \lambda^* \rangle\) an increasing chain of elementary submodels of \(\mathcal{H}(\chi)\) such that
\[
\begin{align*}
(2.5) & \quad G \in M_0; \ \kappa + 1 \subseteq M_0; \\
(2.6) & \quad |M_\alpha| < \lambda \text{ for all } \alpha < \lambda^*; \text{ and} \\
(2.7) & \quad G \subseteq \bigcup_{\alpha < \lambda^*} M_\alpha.
\end{align*}
\]

For \(\alpha < \lambda^*\), let \(G_\alpha = G \cap M_\alpha\). Then \(\langle G_\alpha : \alpha < \lambda^* \rangle\) is a filtration of \(G\) by (2.6) and (2.7). \(G_\alpha \subseteq G\) for all \(\alpha < \kappa\) by (2.5) and by the assumption of (c).

By Lemma 2.3, \(G_\alpha\) also satisfies (c) for \(\alpha < \lambda^*\). Since \(|G_\alpha| < \lambda\), it follows that \(\text{col}(G_\alpha) \leq \kappa\) for all \(\alpha < \lambda^*\) by the induction hypothesis. Hence we have \(\text{col}(G) \leq \kappa\) by Lemma 2.2. \(\square\) (Theorem 2.4)

3 Coloring number of the plane

The plane, or the unit distance graph of the plane, is the graph \(G^1(\mathbb{R}^2)\) defined by \(G^1(\mathbb{R}^2) = \langle \mathbb{R}^2, K^1_{\mathbb{R}^2} \rangle\) where \(K^1 = \{\{x, y\} \in [\mathbb{R}^2]^2 : d(x, y) = 1\}\). Applying Theorem 2.4, we can show easily that the coloring number of the plane is equal to \(\aleph_0\).

**Theorem 3.1.** \(\text{col}(G^1(\mathbb{R}^2)) = \aleph_0\).

**Proof.** In [2] it is noted that the list-chromatic number \(\text{list}(G^1(\mathbb{R}^2))\) of \(G^1(\mathbb{R}^2)\) is infinite since finite regular graph of arbitrarily large degree \(d\) can be embedded in \(G^1(\mathbb{R}^2)\) (e.g., throwing down of \(n\)-dimensional cube onto the plane) and the list-chromatic number of such finite graph is \(d\) (see [1]). Thus we have \(\aleph_0 \leq \text{list}(G^1(\mathbb{R}^2)) \leq \text{col}(G^1(\mathbb{R}^2))\).

To prove the inequality \(\text{col}(G^1(\mathbb{R}^2)) \leq \aleph_0\), let \(\chi\) be sufficiently large and \(N < \mathcal{H}(\chi)\). Note that we have \(G^1(\mathbb{R}^2) \in N\) since the plane is definable. Suppose \(x \in \mathbb{R}^2 \setminus N\). Let us write simply \(K\) for \(K^1_{\mathbb{R}^2}\). By Theorem 2.4, it is enough to show that \(K^1_{\mathbb{R}^2 \cap N}\) is finite. Actually, we can show that \(|K^1_{\mathbb{R}^2 \cap N}| \leq 1\):

Toward a contradiction, suppose that \(|K^1_{\mathbb{R}^2 \cap N}| > 1\). Then there are two distinct \(y, z \in G \cap N\) such that \(d(x, y) = d(x, z) = 1\). But then \(X = \{u \in \mathbb{R}^2 : d(u, y) = d(u, z) = 1\}\) is a two element set definable with parameters from \(N\). It follows that \(x \in X \subseteq N\). This is a contradiction to the choice of \(x\). \(\square\) (Theorem 3.1)
With the same proof we can also show:
\[
\text{col}(G^\text{Odd}([R]^2)) = \text{col}(G^\text{N}([R]^2)) = \text{col}(G^\mathbb{Q}([R]^2)) = \text{col}(G^\text{algebraic}([R]^2)) = \cdots = \aleph_0.
\]

Theorem 3.1 may be already known. However I could not find any direct mention of the theorem in the literature. Also, in [2] the authors prove \(\text{list}(G^\text{Odd}([R]^2)) \leq \aleph_0\) directly and it seems that idea of the proof cannot be extended to a proof of \(\text{col}(G^\text{Odd}([R]^2)) \leq \aleph_0\).

I first learned a proof of \(\text{col}(G^1([R]^2)) \leq \aleph_0\) from Hiroshi Sakai in November 2009 who proved the inequality straightforwardly.

Theorem 2.4 is often quite useful to decide the coloring number of infinite graphs. For example, \(\text{col}(K(\kappa, \kappa)) = \kappa\) and \(\text{col}(K(\kappa, \lambda)) = \kappa^+\) for any \(\aleph_0 \leq \kappa < \lambda\); \(\text{col}(G^\text{Odd}([R]^3)) = \aleph_1\) etc. can be seen immediately by this theorem.

We shall demonstrate the last equality. Recall \(G^\text{Odd}([R]^3) = ([R]^3, K^\text{Odd}_{[R]^3})\) where \(K^\text{Odd}_{[R]^3} = \{\langle \vec{x}, \vec{y} \rangle \in [R]^2 : d(\vec{x}, \vec{y}) \text{ is an odd (natural) number}\}.

**Theorem A.3.1.** \(\text{col}(G^\text{Odd}([R]^3)) = \aleph_1\).

**Proof.** For notational simplicity, let \(G = G^\text{Odd}([R]^3) = \langle G, K \rangle\) with \(G = [R]^3\) and \(K = K^\text{Odd}_{[R]^3}\). Suppose that \(\chi\) is sufficiently large. By Theorem 2.4, it is enough to show that \(G \cap M \subseteq \aleph_1\) for all \(M \prec \mathcal{H}(\chi)\) but \(G \cap M \not\subseteq \aleph_0\) for some \(M \prec \mathcal{H}(\chi)\).

Suppose that \(M \prec \mathcal{H}(\chi)\). If \([R] \subseteq M\) then \(G \subseteq M\) and we have \(G \cap M \subseteq \aleph_1\) \(G\) vacuously.

Otherwise, letting \(C = \{(x, y, 0) \in [R]^3 : d((x, y, 0), \vec{0}) = 1\}\), we have \(C \not\subseteq M\). Let \(\vec{x} \in C \setminus M\). Then, for any odd \(n \in \omega\), \(\sqrt{n^2 - 1} \in M\) and \(d(\vec{x}, (0, 0, \sqrt{n^2 - 1})) = n\). Thus \((0, 0, \sqrt{n^2 - 1}) \in K^\text{Odd}_{\mathcal{H}G\cap M}\). This shows that \(G \cap M \not\subseteq \aleph_0\) \(G\).

To show \(G \cap M \not\subseteq \aleph_1\) \(G\), assume for contradiction that there is \(\vec{x} \in G \setminus M\) such that \(K^\text{Odd}_{G \cap M}\) is uncountable. Then there is an odd \(n \in \omega\) such that \(X = \{\vec{y} \in G \cap M : d(\vec{x}, \vec{y}) = n\}\) is uncountable. Let \(y_0, y_1, y_3\) be three distinct elements of \(X\). \(Y = \{\vec{z} \in G : d(\vec{z}, y_0) = d(\vec{z}, y_1) = d(\vec{z}, y_2) = n\}\) is a two-elements set definable with parameters form \(M\). It follows that \(\vec{z} \in Y \subseteq M\). This is a contradiction to the choice of \(\vec{x}\). \(\square\) (Theorem A.3.1)

4 Coloring number under very weak square

The following version of the very weak square was introduced in [8].
For a regular cardinal $\kappa$ and $\mu > \kappa$, let $\Box_{\kappa, \mu}^{***}$ be the following assertion: there exists a sequence $(C_\alpha : \alpha < \mu^*)$ and a club set $D \subseteq \mu^+$ such that, for all $\alpha \in D$ with $\text{cf}(\alpha) \geq \kappa$, we have

(4.1)  $C_\alpha \subseteq \alpha$, $C_\alpha$ is unbounded in $\alpha$; and

(4.2)  $[\alpha]^{< \kappa} \cap \{C_{\alpha'} : \alpha' < \alpha\}$ dominates $[C_\alpha]^{< \kappa}$ (with respect to $\subseteq$).

For a (sufficiently large regular) cardinal $\chi$ and $M < \mathcal{H}(\chi)$, $M$ is $\kappa$-internally cofinal if $[M]^{< \kappa} \cap M$ is cofinal in $[M]^{< \kappa}$ with respect to $\subseteq$. For $\mathcal{D} \subseteq [\mathcal{H}(\chi)]^{< \kappa}$, $M$ is $\mathcal{D}$-internally cofinal if $\mathcal{D} \cap M$ is cofinal in $[M]^{< \kappa}$ with respect to $\subseteq$.

Suppose now that $\kappa$ is a regular cardinal and $\mu > \kappa$ is such that $\text{cf}(\mu) < \kappa$. Let $\mu^* = \text{cf}(\mu)$. For a sufficiently large $\chi$ and $x \in \mathcal{H}(\chi)$, let us call a sequence $\langle M_{\alpha, \beta} : \alpha < \mu^+, \beta < \mu^* \rangle$ a $(\kappa, \mu)$-dominating matrix (of elementary submodels of $\mathcal{H}(\chi)$) over $x$ if the following conditions (4.3) – (4.6) hold:

(4.3)  $M_{\alpha, \beta} < \mathcal{H}(\chi)$, $x \in M_{\alpha, \beta}$, $\kappa + 1 \subseteq M_{\alpha, \beta}$ and $|M_{\alpha, \beta}| < \mu$ for all $\alpha < \mu^+$ and $\beta < \mu^*$;

(4.4)  $\langle M_{\alpha, \beta} : \beta < \mu^* \rangle$ is an increasing sequence for each fixed $\alpha < \mu^+$;

(4.5)  if $\alpha < \mu^+$ is such that $\text{cf}(\alpha) \geq \kappa$, then there is $\beta^* < \mu^*$ such that, for every $\beta^* \leq \beta < \mu^*$, $M_{\alpha, \beta}$ is $\kappa$-internally cofinal.

For $\alpha < \mu^+$, let $M_\alpha = \bigcup_{\beta < \mu^*} M_{\alpha, \beta}$. By (4.3) and (4.4), we have $M_\alpha < \mathcal{H}(\chi)$.

(4.6)  $\langle M_\alpha : \alpha < \mu^+ \rangle$ is continuously increasing and $\mu^+ \subseteq \bigcup_{\alpha < \mu^+} M_\alpha$.

**Theorem 4.1** (Theorem 7 in [8]). Suppose that $\kappa$ is a regular cardinal and $\mu > \kappa$ is such that $\text{cf}(\mu) < \kappa$. If we have $\text{cf}(\lambda)^{< \kappa}, \subseteq = \lambda$ for cofinally many $\lambda < \mu$ and $\Box_{\kappa, \mu}^{***}$ holds, then, for any sufficiently large $\chi$ and $x \in \mathcal{H}(\chi)$, there is a $(\kappa, \mu)$-dominating matrix over $x$.

**Theorem 4.2.** Assume SSH and $\Box_{\kappa, \mu}^{***}$ for a regular uncountable $\kappa$ and all singular cardinal $\mu$ with $\text{cf}(\mu) < \kappa < \mu$.

Then, for any graph $G = (G, K)$ the following are equivalent:

(a)  $\text{col}(G) \leq \kappa$.

(d)  For a (all sufficiently large regular $\chi$ and $\kappa$-internally cofinal $M < \mathcal{H}(\chi)$ with $G \in M$ we have $G \cap M \subseteq \kappa G$.

(e)  For a (all sufficiently large regular $\chi$ there is $\mathcal{D} \subseteq [\mathcal{H}(\chi)]^{< \kappa}$ such that $\mathcal{D}$ is cofinal in $[\mathcal{H}(\chi)]^{< \kappa}$ and, for any $\mathcal{D}$-internally cofinal $M < \mathcal{H}(\chi)$, we have $G \cap M \subseteq \kappa G$.  


Proof. (a) $\Rightarrow$ (d) follows from Theorem 2.4, (d) $\Rightarrow$ (e) is trivial (just put $D = [\mathcal{H}(\chi)]^{<\kappa}$).

For (e) $\Rightarrow$ (a), we proceed with induction on $|G|$. If $|G| \leq \kappa$ then the implication (e) $\Rightarrow$ (a) is trivial since $\text{col}(G) \leq \kappa$ holds always for any graph of size $\leq \kappa$. Suppose now that $|G| > \kappa$ and we have shown the implication (e) $\Rightarrow$ (a) for all graphs of cardinality $< |G|$.

Assume that $G$ satisfies (e) with $\chi$ and $\mathcal{D}$. Let $\chi^*$ be sufficiently large above $\chi$ such that we have in particular $\mathcal{H}(\chi) \in \mathcal{H}(\chi^*)$.

Claim 4.2.1. If $M$ is a $\kappa$-internal cofinal elementary submodel of $\mathcal{H}(\chi^*)$ such that

\[(4.7)\] $G, \chi, D \in M$ and $\kappa + 1 \subseteq M$,

then we have $G \cap M \subseteq_{\kappa} G$.

$\dashv$ Suppose not. Then there is $a \in G \setminus M$ such that $|K_{G \cap M}^a| \geq \kappa$. Let $N = \mathcal{H}(\chi) \cap M$. By elementarity we have $N \prec \mathcal{H}(\chi)$. Let $\langle N_{\alpha} : \alpha < \kappa \rangle$ be an increasing sequence such that, for all $\alpha < \kappa$, we have

\[(4.8)\] $N_{\alpha} \in D \cap M$;
\[(4.9)\] $N_{\alpha} \subseteq N_{\alpha+1};$
\[(4.10)\] there is $N_{\alpha}^* \in [N]^{<\kappa} \cap M$ such that $N_{\alpha}^* < N$ and $N_{\alpha} \subseteq N_{\alpha}^* \subseteq N_{\alpha+1};$ and
\[(4.11)\] $K_{G \cap N}^a \cap (N_{\alpha+1} \setminus N_{\alpha}) \neq \emptyset$.

The construction is possible by elementarity of $M$ and since $D$ is cofinal in $\mathcal{H}(\chi)$.

Let $N^* = \bigcup_{\alpha < \kappa} N_{\alpha}$. By (4.10) we have $N^* < N \prec \mathcal{H}(\chi)$. By (4.8) and (4.9) $N^*$ is $D$-internally cofinal. On the other hand, we have $|K_{G \cap N^*}^a| \geq \kappa$ by (4.11). This is a contradiction to the assumption of (e).

$\dashv$ (Claim 4.2.1)

Claim 4.2.2. If $H \subseteq_{\kappa} G$ then for every $D$-internally cofinal $M < \mathcal{H}(\chi)$ we have $H \cap M \subseteq_{\kappa} H$. In particular, $H$ also satisfies the condition (e).

Proof. Suppose that $M < \mathcal{H}(\chi)$ is $D$-internally approachable. For $a \in H \setminus (H \cap M)$, since $a \in G \setminus (G \cap M)$, we have $K_{H \cap M}^a \subseteq K_{G \cap M}^a$. The right side of the inclusion is of cardinality $< \kappa$ by the assumption of (e) on $G$. This shows that $H \cap M \subseteq_{\kappa} H$.

$\dashv$ (Claim 4.2.2)

Now we finish the induction step for the proof of (e) $\Rightarrow$ (a) in two cases. Let $\nu = |G|$.

Case I. $\nu$ is a limit cardinal or $\nu = \delta^+$ with $\text{cf}(\delta) \geq \kappa$.

Let $\nu^* = cf(\nu)$. Note that, in this case, we have that
the cardinals $\lambda < \nu$ such that $cf([\lambda]^{<\kappa}) = \lambda$ are cofinal among cardinals below $\nu$

by SSH.

Let $\langle M_\alpha : \alpha < \nu^* \rangle$ be an increasing sequence of elementary submodels of $\mathcal{H}(\chi^*)$ of cardinality $< \nu$ satisfying (4.7) and $G \subseteq \bigcup_{\alpha < \nu^*} M_\alpha$. We can find such a sequence by (4.12).

Let

$$G_\alpha = \begin{cases} G \cap M_\alpha & \text{if } \alpha = 0 \text{ or } \alpha \text{ is a successor ordinal;} \\ G \cap \left( \bigcup_{\beta < \alpha} M_\beta \right) & \text{otherwise} \end{cases}$$

for $\alpha < \nu^*$. Then $\langle G_\alpha : \alpha < \nu^* \rangle$ is a filtration of $G$.

Claim 4.2.3. $G_\alpha \subseteq \kappa G$ for all $\alpha < \nu^*$.

Proof. If $\alpha < \nu^*$ is 0 or a successor ordinal, this follows from Claim 4.2.1.

If $\alpha < \nu^*$ is a limit and $cf(\alpha) < \kappa$, then $G_\alpha$ is a union of less than $\kappa$ many $G_\beta$'s where $\beta < \alpha$ may be chosen to be a successor ordinal and hence $G_\beta \subseteq \kappa G$. It follows that we have $G_\alpha \subseteq \kappa G$ also in this case.

If $cf(\alpha) \geq \kappa$, then $\bigcup_{\beta < \alpha} M_\beta$ is $\kappa$-internally cofinal and hence we have $G_\beta \subseteq \kappa G$ again by Claim 4.2.1.

$\dashv$ (Claim 4.2.3)

Now by Claim 4.2.2 and by the induction hypothesis, all of $G_\alpha$, $\alpha < \nu^*$ are of coloring number $\leq \kappa$. By Lemma 2.2, it follows that $G$ also has coloring number $\leq \kappa$.

Case II. $\nu = \mu^+$ with $cf(\mu) < \kappa$. Let $\mu^* = cf(\mu)$.

By Theorem 4.1, there is a $(\kappa, \mu)$-dominating matrix $\langle M_{\alpha, \beta} : \alpha < \nu, \beta < \mu^* \rangle$ of submodels of $\mathcal{H}(\chi^*)$ over $x = \langle G, \mathcal{H}(\chi) \rangle$.

For $\alpha < \nu$ and $\beta < \mu^*$, let $G_{\alpha, \beta} = G \cap M_{\alpha, \beta}$ and $G_\alpha = \bigcup_{\beta < \mu^*} G_{\alpha, \beta} = G \cap \left( \bigcup_{\beta < \mu^*} M_{\alpha, \beta} \right)$. By (4.6), the sequence $\langle G_\alpha : \alpha < \nu \rangle$ is continuously increasing and $\bigcup_{\alpha < \nu} G_\alpha = G$. By (4.3), we have $|G_\alpha| \leq \mu < \nu$. Thus $\langle G_\alpha : \alpha < \nu \rangle$ is a filtration of $G$.

Let

$$C = \{ \alpha < \nu : cf(\alpha) \geq \kappa \text{ or } \alpha' < \alpha : cf(\alpha') \geq \kappa \} \text{ is cofinal in } \alpha \}.$$

$C$ is a club subset of $\nu$.

Claim 4.2.4. $G_\alpha \subseteq \kappa G$ for all $\alpha \in C$. 

(4.12) the cardinals $\lambda < \nu$ such that $cf([\lambda]^{<\kappa}) = \lambda$ are cofinal among cardinals below $\nu$.
Suppose $\alpha \in C$. If $\text{cf}(\alpha) \geq \kappa$, $M_{\alpha, \beta}$ is $\kappa$-internally cofinal for all sufficiently large $\beta < \mu^*$ by (4.5). Hence by Claim 4.2.1, we have $G_{\alpha, \beta} \subseteq_{\kappa} G$ for all such $\beta$. Since $\mu^* < \kappa$, it follows that $G_\alpha \subseteq_{\kappa} G$.

If $\text{cf}(\alpha) < \kappa$, then let $X \subseteq \alpha$ be a cofinal subset of $\alpha$ with $|X| < \kappa$ such that all $\alpha' \in X$ have cofinality $\geq \kappa$. Since $G_\alpha = \bigcup_{\alpha' \in X} G_{\alpha'}$ and $G_{\alpha'} \subseteq_{\kappa} G$ for all $\alpha' \in X$ by the first part of the proof, it follows that $G_\alpha \subseteq_{\kappa} G$. \hfill (Claim 4.2.4)

By Claim 4.2.2 and by the induction hypothesis, we have $\text{col}(G_\alpha) \leq \kappa$ for all $\alpha \in C$. Hence by Lemma 2.2 we can conclude that $\text{col}(G) \leq \kappa$.

\square (Theorem 4.2)

References


