Borel approximation of coanalytic sets with Borel sections and the regularity properties for $\Sigma^1_2$ sets of reals

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In 2008, Fujita showed the following:

**Theorem 1 (Fujita [5]).** The following are equivalent:

1. If $A \subseteq \mathbb{R} \times \mathbb{R}$ is $\Pi^1_1$ and for any real $x$, $A_x$ is Borel where $A_x = \{y \in \mathbb{R} | (x, y) \in A\}$, then there is a comeager Borel set $D \subseteq \mathbb{R}$ such that $A \cap (D \times \mathbb{R})$ is Borel, and
2. every $\Sigma^1_2$ set of reals has the Baire property.

We show that one can generalize the above theorem to a wide class of tree-type ccc forcings. More precisely:

**Theorem 2.** Let $\mathbb{P}$ be a strongly arboreal, $\Sigma^1_1$, provably ccc forcing. Then the following are equivalent:

1. If $A \subseteq \mathbb{R} \times \mathbb{R}$ is $\Pi^1_1$ and for any real $x$, $A_x$ is Borel, then there is a Borel set $D \subseteq \mathbb{R}$ such that $D$ is of $\mathbb{P}$-measure one and $A \cap (D \times \mathbb{R})$ is Borel, and
2. every $\Sigma^1_2$ set of reals is $\mathbb{P}$-measurable.

We also show that this equivalence fails for non-ccc forcings. In fact, for Sacks forcing, the corresponding statement to the first fails in ZFC while the one for the second is consistent with ZFC.

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Throughout this paper, we assume the basic knowledge of descriptive set theory and forcing which can be obtained from e.g., [9], [8], and [2]. By \textit{reals}, we mean elements of the Cantor space or those of the Baire space and we use $\mathbb{R}$ to denote the set of reals.

To prove Theorem 2, let us start with basic definitions. Let $n$ be a natural number with $n \geq 1$. A partial order $\mathbb{P}$ is $\Sigma^1_n$ if the sets $P$, $\leq P$, and $\perp P$ are $\Sigma^1_n$, where $\mathbb{P} = (P, \leq P)$ and $\perp P$ is the incompatibility relation in $\mathbb{P}$. A partial order $\mathbb{P}$ is \textit{provably ccc} if there is a formula $\phi$ defining $\mathbb{P}$ and the statement "$\phi$ defines a ccc partial order" is provable in ZFC. A partial order $\mathbb{P}$ is \textit{arboreal} if its conditions are perfect trees on $\omega$ (or on 2) ordered by inclusion. But this class of forcings contains some trivial forcings such as $\mathbb{P} = \langle \omega \rangle$. We need the following stronger notion:

\textbf{Definition 3.} A partial order $\mathbb{P}$ is \textit{strongly arboreal} if it is arboreal and the following holds:

$$(\forall T \in \mathbb{P}) (\forall t \in T) \ T_t \in \mathbb{P},$$

where $T_t = \{s \in T \mid \text{either } s \subseteq t \text{ or } s \supseteq t\}$.

With strongly arboreal forcings, one can code generic objects by reals in the standard way: Let $\mathbb{P}$ be strongly arboreal and $G$ be $\mathbb{P}$-generic over $V$. Let $x_G = \bigcup\{\text{stem}(T) \mid T \in G\}$, where stem$(T)$ is the longest $t \in T$ such that $T_t = T$. Then $x_G$ is a real and $G = \{T \in \mathbb{P} \mid x_G \in [T]\}$, where $[T]$ is the set of all infinite paths through $T$. Hence $V[x_G] = V[G]$. We call such real $x_G$ a $\mathbb{P}$-\textit{generic real over $V$}.

Almost all typical forcings related to regularity properties are strongly arboreal:

\textbf{Example 4.}

1. Cohen forcing $\mathbb{C}$: Let $T_0$ be $\omega \omega$. Consider the partial order $\{(T_0)_s \mid s \in \omega \omega, \subseteq\}$. Then this is strongly arboreal and equivalent to Cohen forcing.

2. Random forcing $\mathbb{B}$: Consider the set of all perfect trees $T$ on 2 such that for any $t \in T$, $[T_t]$ has a positive Lebesgue measure, ordered by inclusion. Then this forcing is strongly arboreal and equivalent to random forcing.

3. Hechler forcing $\mathbb{D}$: For $(n, f) \in \mathbb{D}$, let

$$T_{(n,f)} = \{t \in \omega \mid \text{either } t \subseteq f \upharpoonright n \text{ or} \ t \supseteq f \upharpoonright n \text{ and } (\forall m \in \text{dom}(t)) \ t(m) \geq f(m)\}.$$
Then the partial order \( \{ T_{(n,f)} \mid (n,f) \in D \}, \subseteq \) is strongly arboreal and equivalent to Hechler forcing.

4. Mathias forcing \( R_M \): For a condition \((s, A)\) in \( R_M \), let

\[
T_{(s,A)} = \{ t \in \langle \omega, \omega \rangle \mid t \text{ is strictly increasing and } s \subseteq \text{ran}(t) \subseteq s \cup A \}.
\]

Then \( \{ T_{(s,A)} \mid (s, A) \in R_M \} \) is a strongly arboreal forcing equivalent to Mathias forcing.

5. Sacks forcing \( S \), Silver forcing \( V \), Miller forcing \( M \), Laver forcing \( L \): These forcings can be naturally seen as strongly arboreal forcings.

The following is as expected:

**Lemma 5.** Let \( P \) be a strongly arboreal, \( \Sigma^1_1 \), provably ccc forcing and \( M \) be an inner model of ZFC containing parameter defining \( P \) with a \( \Sigma^1_1 \)-formula. Then if \( x \) is \( P \)-generic over \( V \), then \( x \) is \( P^M \)-generic over \( M \).

**Proof.** Since \( P \) is \( \Sigma^1_1 \), \( P^M = P \cap M \). So it suffices to show that if \( A \subseteq P^M \) is a maximal antichain in \( M \), so is in \( V \). Let \( A \subseteq P^M \) be a maximal antichain in \( M \). Since \( P \) is provably ccc, \( M \) thinks \( P^M \) is ccc. So \( A \) is countable in \( M \) and there is a real \( r \) coding \( A \) in \( M \). Since \( P \) is \( \Sigma^1_1 \), the statement "a real \( r \) codes a maximal antichain in \( P \)" is \( \Pi^1_2 \). So the real \( r \) also codes the maximal antichain \( A \) in \( V \), as desired. \( \blacksquare \)

We now introduce a \( \sigma \)-ideal \( I_P \) on the reals expressing "smallness" for each strongly arboreal forcing \( P \).

**Definition 6.** Let \( P \) be a strongly arboreal forcing. A set of reals \( A \) is \( P \)-null if for any \( T \) in \( P \) there is a \( T' \leq T \) such that \([T'] \cap A = \emptyset\). Let \( N_P \) denote the set of all \( P \)-null sets and \( I_P \) denote the \( \sigma \)-ideal generated by \( P \)-null sets, i.e., the set of all countable unions of \( P \)-null sets. A set of reals \( A \) is of \( P \)-measure one if \( R \setminus A \) is in \( I_P \).

**Example 7.**

1. Cohen forcing \( \mathbb{C} \): \( \mathbb{C} \)-null sets are the same as nowhere dense sets of reals and \( I_\mathbb{C} \) is the ideal of meager sets of reals.

2. Random forcing \( \mathbb{B} \): \( \mathbb{B} \)-null sets are the same as Lebesgue null sets in the Baire space and \( I_\mathbb{B} \) is the Lebesgue null ideal.
3. Hechler forcing $D$: $D$-null sets are the same as nowhere dense sets in the dominating topology, i.e., the topology generated by $\{[s, f] \mid (s, f) \in D\}$ where 

$$[s, f] = \{x \in {}^{\omega}\omega \mid s \subseteq x \text{ and } (\forall n \geq \text{dom}(s)) x(n) \geq f(n)\}.$$ 

Hence $I_D$ is the meager ideal in the dominating topology.

4. Mathias forcing $R_M$: A set of reals $A$ is $R_M$-null if and only if $\{\text{ran}(x) \mid x \in A \cap A_0\}$ is Ramsey null or meager in the Ellentuck topology, where $A_0$ is the set of strictly increasing infinite sequences of natural numbers. Hence $I_{R_M} = N_{R_M}$.

5. Sacks forcing $S$: In this case, $I_S = N_S$ by a standard fusion argument. The ideal $I_S$ is called the Marczewski ideal and often denoted by $s_0$.

As with Sacks forcing, all the typical non-ccc tree-type forcings admitting a fusion argument satisfy the equation $I_{\mathbb{P}} = N_{\mathbb{P}}$. In the case of ccc forcings, $I_{\mathbb{P}}$ is often different from $N_{\mathbb{P}}$ (e.g., Cohen forcing and Hechler forcing).

We now introduce $\mathbb{P}$-measurability:

**Definition 8.** Let $\mathbb{P}$ be strongly arboreal. A set of reals $A$ is $\mathbb{P}$-measurable if for any $T$ in $\mathbb{P}$ there is a $T' \leq T$ such that either $[T'] \cap A \in I_{\mathbb{P}}$ or $[T'] \setminus A \in I_{\mathbb{P}}$.

As is expected, $\mathbb{P}$-measurability coincides with a known regularity property for $\mathbb{P}$ when $\mathbb{P}$ is ccc:

**Proposition 9.** Let $\mathbb{P}$ be a strongly arboreal, ccc forcing and let $A$ be a set of reals. Then $A$ is $\mathbb{P}$-measurable if and only if there is a Borel set $B$ such that $A \triangle B \in I_{\mathbb{P}}$, where $A \triangle B$ is the symmetric difference between $A$ and $B$.

**Proof.** See Proposition 2.9 in [6].

Proposition 9 does not hold for non-ccc forcings such as Sacks forcing.\(^1\)

But $\mathbb{P}$-measurability is almost the same as the regularity properties for non-ccc forcings $\mathbb{P}$, e.g., for Mathias forcing, a set of reals $A$ is $R_M$-measurable if and only if $\{\text{ran}(x) \mid x \in A \cap A_0\}$ is completely Ramsey (or has the

\(^1\)For example, assuming every $\Pi^1_1$ set has the perfect set property (i.e., either the set is countable or contains a perfect subset), there is no $\Sigma^1_1$ Bernstein set (i.e., a set where neither it nor its complement contains a perfect subset) but for a $\Sigma^1_1$ set of reals $A$, $A$ is approximated by a Borel set modulo $I_S$ if and only if $A$ is Borel. This is because $I_S$ restricted to analytic sets (or co-analytic sets) is the set of all countable sets of reals by the assumption that every $\Pi^1_1$ set has the perfect set property.
Baire property in the Ellentuck topology), where $A_0$ is the set of all strictly increasing infinite sequences of natural numbers. Also, for Sacks forcing, the following holds:

**Proposition 10** (Brendle, Löwe). Let $\Gamma$ be a topologically reasonable point-class, i.e., it is a set of sets of reals closed under continuous preimages and any intersection between a set in $\Gamma$ and a closed set of reals. Then every set in $\Gamma$ is $S$-measurable if and only if there is no Bernstein set in $\Gamma$.$^2$

*Proof.* See [3, Lemma 2.1].

As expected, every $\Sigma^1_1$ set of reals is $P$-measurable:

**Theorem 11.** Let $P$ be a strongly arboreal, proper forcing. Then every $\Sigma^1_1$ set of reals is $P$-measurable.

*Proof.* It follows from the fact that every $\Sigma^1_1$ set of reals is universally Baire, that every universally Baire set of reals is $P$-Baire, and that every $P$-Baire set of reals is $P$-measurable. For the details, see [4] and Section 3 in [6].

We are now ready to state the theorem characterizing the regularity properties for $\Sigma^2_1$ sets of reals in terms of the existence of many generic reals over $L[r]$ for a real $r$, which we will use for the proof of Theorem 2:

**Theorem 12.** Let $P$ be a strongly arboreal, $\Sigma^1_1$, provably ccc forcing. Then the following are equivalent:

1. Every $\Sigma^1_2$ set of reals is $P$-measurable, and

2. for any real $r$, the set of $P$-generic reals over $L[r]$ is of $P$-measure one.

*Proof.* See Definition 2.11, Lemma 2.13 (3), Definition 2.15, Proposition 2.17 (3), and Theorem 4.4 in [6].

We are now ready to prove Theorem 2:

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$^2$In general, the property not being a Bernstein set does not imply $S$-measurability while the converse is true. By using the axiom of choice, one can construct a set of reals which is not $S$-measurable but is not a Bernstein set.
Proof of Theorem 2. The argument is exactly the same as the one in Theorem 1 in [5]. For the sake of completeness, we will give the proof.

We first show the implication from 1. to 2. Let $P$ be a $\Sigma^1_2$ set of reals. We will show that $P$ is $\mathbb{P}$-measurable. Since $P$ is $\Sigma^1_2$, there is a $\Pi^1_1$ set $A \subseteq \mathbb{R} \times \mathbb{R}$ such that $P = \{x \in \mathbb{R} \mid (\exists y) (x, y) \in A\}$. By Kondô’s uniformization theorem, there is a $\Pi^1_1$ function $f: \mathbb{P} \to \mathbb{R}$ uniformizing $A$. Then for any real $x$, $f_x = \{y \mid (x, y) \in f\} = \{f(x)\}$ is Borel, so by applying the assumption for $f$, there is a Borel set $D$ of reals such that $D$ is of $\mathbb{P}$-measure one and $f \cap (D \times \mathbb{R})$ is Borel. Hence $P \cap D = \{x \mid (\exists y) (x, y) \in f \cap (D \times \mathbb{R})\}$ is $\Sigma^1_1$ and is $\mathbb{P}$-measurable by Theorem 11. So by Proposition 9, there is a Borel set $B$ such that $(P \cap D) \triangle B$ is in $I_{\mathbb{P}}$. Since $D$ is of $\mathbb{P}$-measure one, $P \triangle B$ is also in $I_{\mathbb{P}}$. Again by Proposition 9, $P$ is $\mathbb{P}'$-measurable, as desired.

We now show the implication from 2. to 1. Let $WO$ be the set of reals coding a well-order on $\omega$. It is well-known that $WO$ is a complete $\Pi^1_1$ set of reals. For an element $w$ of $WO$, $|w|$ denotes the countable ordinal that $w$ codes. We need the following notion and lemma for the proof:

**Definition 13.** Let $r$ be a real. A set $X \subseteq \mathbb{R} \times \omega_1$ is $\Pi^1_2(r)$ in the codes if the set

$$\{(x, w) \in \mathbb{R} \times \mathbb{R} \mid w \in WO \text{ and } (x, |w|) \in X\}$$

is $\Pi^1_2(r)$.

**Lemma 14.** Let $r$ be a real and $X \subseteq \mathbb{R} \times \omega_1$ be $\Pi^1_2(r)$ in the codes. Suppose that for any real $x$ there is a $\xi < \omega_1$ such that $(x, \xi) \in X$. Then there is a countable ordinal $\delta$ such that for any $\mathbb{P}$-generic real $x$ over $L[r]$, there is a $\xi < \delta$ such that $(x, \xi) \in X$.

**Proof of Lemma 14.** Since $X$ is $\Pi^1_2(r)$ in the codes, pick a $\Pi^1_2$-formula $\phi(x, w, v)$ such that

$$(\forall x, w) \ (\phi(x, w, r) \iff w \in WO \text{ and } (x, |w|) \in X).$$

Let $\tilde{\phi}(x, \xi, r)$ be the following:

$$\tilde{\phi}(x, \xi) \iff (\forall w \in WO) \ |w| = \xi \rightarrow \phi(x, w, r).$$

Then $\tilde{\phi}$ is absolute among all the transitive proper class models of ZFC in which $\xi$ is countable.
For each $\xi < \omega_1$, let

$$X_\xi = \{ T \in \mathbb{P} \mid (T, 1_{\mathbb{P}_\xi}) \not\Vdash_{\mathbb{P} \times \mathbb{P}_\xi} \tilde{\phi}(\dot{x}, \check{\xi}, \check{r}) \}^{L[r]},$$

where $\mathbb{P}_\xi$ is Coll($\omega, \xi$) and $\dot{x}$ is a canonical $\mathbb{P}$-name for a generic real.

We show that $\bigcup_{\xi<\omega_1} X_\xi$ is a dense subset of $\mathbb{P}^{L(\mathbb{R})}$ in $L[r]$. Let $T$ be any element of $\mathbb{P}^{L[r]}$. Take a $\mathbb{P}$-generic real $x$ over $L[r]$ in $V$ with $x \in [T]$. Then by the assumption, there is a $\xi < \omega_1$ such that $(x, \xi) \in X$. Take a function $g: \omega \to \xi$ generic over $L[r, x]$. Then $L[r, x, g] \Vdash \tilde{\phi}(x, \xi, r)$. Hence there is a $T' \leq T$ and a condition $p$ in $\mathbb{P}_\xi$ such that $L[r] \Vdash "(T', p) \Vdash \tilde{\phi}(\dot{x}, \check{\xi}, \check{r})"$. Since $\mathbb{P}_\xi$ is homogeneous, it follows that $L[r] \models "(T', 1_{\mathbb{P}_\xi}) \not\Vdash \tilde{\phi}(\dot{x}, \check{\xi}, \check{r})", \text{ so } T' \leq T \text{ and } T \in \bigcup_{\xi<\omega_1} X_\xi$, as desired.

Since $\mathbb{P}$ is provably ccc, $L[r] \models "\mathbb{P} \text{ is ccc}"$, so there is a $\delta < \omega_1$ such that $\bigcup_{\xi<\delta} X_\xi$ is a predense subset of $\mathbb{P}$ in $L[r]$. We show that this $\delta$ is the desired countable ordinal. Take any $\mathbb{P}$-generic real $x$ over $L[r]$. Then since $L[r]$ thinks $\bigcup_{\xi<\delta} X_\xi$ is a predense subset of $\mathbb{P}$, the generic filter $G_x$ meets $\bigcup_{\xi<\delta} X_\xi$ and hence there is a $\xi < \delta$ such that $G_x \cap X_\xi \neq \emptyset$. By the definition of $X_\xi$, for a function $g: \omega \to \xi$ generic over $L[r, x]$, $L[r, x, g] \Vdash \tilde{\phi}(x, \xi, r)$, hence $\tilde{\phi}(x, \xi, r)$ holds also in $V$ and $(x, \xi) \in X$, as desired. $\square$ (Lemma 14)

We now finish showing the implication from 2. to 1. Let $A \subseteq \mathbb{R} \times \mathbb{R}$ be $\Pi^1_1$ such that for any real $x$, $A_x$ is Borel. Let $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f^{-1}(WO) = A$. Take any real $x$. Since $A_x$ is Borel, the set $f^{-1}(\{x\} \times A_x)$ is $\Sigma^1_1$, hence by boundedness theorem, it is bounded in $WO$, i.e.,

$$(\forall x) (\exists \xi) (\forall y) \text{ if } (x, y) \in A, \text{ then } |f(x, y)| < \xi.$$ 

Set

$$X = \{(x, \xi) \mid f^{-1}(\{x\} \times A_x) \subseteq \omega_0 \},$$

where $WO_\xi = \{w \in \omega_0 \mid |w| < \xi\}$ for each $\xi < \omega_1$.

Then for any $x$ there is a $\xi$ with $(x, \xi) \in X$. It is also easy to see that $X$ is $\Pi^1_2(r)$ in the codes for some real $r$. By Lemma 14, there is a $\delta < \omega_1$ such that for any $\mathbb{P}$-generic real $x$ over $L[r]$ there is a $\xi < \delta$ such that $(x, \xi) \in X$. Hence $A$ is the same as the Borel set $f^{-1}(WO_\delta)$ on $G(L[r]) \times \mathbb{R}$, where $G(L[r])$ is the set of $\mathbb{P}$-generic reals over $L[r]$. By 2. and Theorem 12, the set $G(L[r])$ is of $\mathbb{P}$-measure one. Since $\mathbb{P}$ is ccc, $I_\mathbb{P}$ is Borel generated, so there is a Borel set $D \subseteq G(L[r])$ of $\mathbb{P}$-measure one and $A \cap (D \times \mathbb{R})$ is Borel, as desired. $\blacksquare$ (Theorem 2)
After the RIMS set theory conference in 2008, Fujita asked if one could take $\delta$ in Lemma 14 below $\gamma_{2}^{1}$ if $X$ is $\Pi_{2}^{1}$ (lightface) in the codes and if $\mathbb{P}$ is Cohen forcing, where $\gamma_{2}^{1}$ is the least countable ordinal that meets every set $A \subseteq \omega_{1}$ which is $\Pi_{2}^{1}$ (lightface) in the codes.\(^3\) We show that this is generally the case for each strongly arboreal, $\Sigma_{1}^{1}$, provably ccc forcing $\mathbb{P}$:

**Proposition 15.** Let $\mathbb{P}$ be a strongly arboreal, $\Sigma_{1}^{1}$ (lightface), ccc forcing and $X \subseteq \mathbb{R} \times \omega_{1}$ be $\Pi_{2}^{1}$ (lightface) in the codes such that for any real $x$ there is a $\xi < \omega_{1}$ with $(x, \xi) \in X$. Then there is a $\delta < \gamma_{2}^{1}$ such that for any $\mathbb{P}$-generic real $x$ over $L$, there is a $\xi < \delta$ with $(x, \xi) \in X$.

**Proof.** Let $X \subseteq \mathbb{R} \times \omega_{1}$ be $\Pi_{2}^{1}$ in the codes such that for any real $x$ there is a $\xi$ with $(x, \xi) \in X$.

Let $A$ be as follows:

$$A = \{ \gamma < \omega_{1} \mid (\forall x \colon \mathbb{P}\text{-generic over } L) (\exists \xi < \gamma) (x, \xi) \in X \}.$$  

By Lemma 14, $A$ is nonempty. Hence it suffices to show that $A$ is $\Pi_{2}^{1}$ in the codes.

Since $X$ is $\Pi_{2}^{1}$ in the codes, pick a $\Pi_{2}^{1}$-formula $\phi$ such that

$$(\forall x, w) (\phi(x, w) \iff w \in WO \text{ and } (x, |w|) \in X).$$

Let $\tilde{\phi}$ be the following:

$$\tilde{\phi}(x, \xi) \iff (\forall w \in WO) |w| = \xi \rightarrow \phi(x, w).$$

Then

$$A = \{ \gamma < \omega_{1} \mid (\forall x \colon \mathbb{P}\text{-generic over } L) (\exists \xi < \gamma) \tilde{\phi}(x, \xi) \}$$

$$= \{ \gamma < \omega_{1} \mid L \models \text{"}(1_{\mathbb{P}}, 1_{\mathbb{P}_{\gamma}}) \Vdash_{\mathbb{P} \times \mathbb{P}_{\gamma}} (\exists \xi < \check{\gamma}) \tilde{\phi}(\dot{x}, \xi)\text{"} \} ,$$

where $\mathbb{P}_{\gamma}$ is Coll$(\omega, \gamma)$ and $\dot{x}$ is a canonical $\mathbb{P}$-name for a generic real.

**Claim 16.** For $\gamma < \omega_{1}$,

$L \models \text{"}(1_{\mathbb{P}}, 1_{\mathbb{P}_{\gamma}}) \Vdash_{\mathbb{P} \times \mathbb{P}_{\gamma}} (\exists \xi < \check{\gamma}) \tilde{\phi}(\dot{x}, \xi)\text{"} \iff V \models \text{"}(1_{\mathbb{P}}, 1_{\mathbb{P}_{\gamma}}) \Vdash_{\mathbb{P} \times \mathbb{P}_{\gamma}} (\exists \xi < \check{\gamma}) \tilde{\phi}(\dot{x}, \xi)\text{"} $\(^2\)

\(^{2}\gamma_{2}^{1}\) is also the least ordinal such that every $\Pi_{1}^{1}$ (lightface) Borel set is $\Pi_{\alpha}^{0}$ (boldface) for some $\alpha < \omega_{1}$. For the details, see [7].
Proof of Claim 16. The direction from left to right follows from the fact that if \((x, g)\) is \(P \times P_\gamma\)-generic over \(V\), then so is over \(L\) by Lemma 5.

For right to left, suppose \(L \models \text{"}(1_{P}, 1_{P_\gamma}) \Vdash_{P \times P_\gamma} (\exists \xi < \check{\gamma}) \phi(\dot{x}, \xi)\)" fails. Then there is a \((p, q) \in P \times P_\gamma\) in \(L\) such that \(L \models \text{"}(p, q) \Vdash_{P \times P_\gamma} (\forall \xi < \check{\gamma}) \neg \phi(\dot{x}, \xi)\)". Take a \(P \times P_\gamma\)-generic \((x, g)\) over \(V\) with \(x \in [p]\) and \(g \supseteq q\).

By the assumption, there exists a \(\xi < \gamma\) such that \(V[x, g] \models \tilde{\phi}(x, \xi)\). But \((x, g)\) is also \(P \times P_\gamma\)-generic over \(L\) and \(L[x, g] \models \tilde{\phi}(x, \xi)\), contradicting \(L \models \text{"}(p, q) \Vdash_{P \times P_\gamma} (\forall \xi < \check{\gamma}) \neg \phi(\dot{x}, \xi)\)".

Therefore,

\[ A = \{ \gamma < \omega_1 \mid "(1_{P}, 1_{P_\gamma}) \Vdash_{P \times P_\gamma} (\exists \xi < \check{\gamma}) \phi(\dot{x}, \xi)" \} . \]

Let \(\psi\) be the following:

\[ \psi(w) \iff w \in \text{WO} \text{ and } (1_{P}, 1_{P_{|w|}}) \Vdash_{P \times P_{|w|}} "(\exists n \in \omega) \phi(\dot{x}, w \upharpoonright n)" , \]

where \(w \upharpoonright n\) is the real coding the well-order \(\leq_w\) below \(n\), i.e. \(\leq_{w|n} = \{(l, m) \mid l \leq_w m <_w n\}\). Then

\[ (\forall w)(\psi(w) \iff w \in \text{WO} \text{ and } |w| \in A) . \]

Hence it suffices to show that \(\psi\) is equivalent to a \(\Pi^1_2\)-formula. Since \(P_{|w|}\) is ccc in \(V^P\), \(P \times P_{|w|}\) is also ccc. Moreover, it is easy to see that \(P \times P_{|w|}\) is \(\Sigma^1_2(w)\) uniformly in \(w \in \text{WO}\). Hence, by the same argument as in Theorem 2.7 (1) in Bagaria and Bosch [1], since \((\exists n \in \omega) \phi(x, w \upharpoonright n)\) is \(\Pi^1_2\) in \(x\) and \(w\), so is \((1_{P}, 1_{P_{|w|}}) \Vdash_{P \times P_{|w|}} "(\exists n \in \omega) \phi(\dot{x}, w \upharpoonright n)" \) in \(w\). Therefore, \(\psi\) is equivalent to a \(\Pi^1_2\)-formula.

\[ \blacksquare \] (Proposition 15)

As announced in the beginning of this paper, we now show that the first item in Theorem 2 fails in ZFC for \(P = S\) (Sacks forcing):

**Proposition 17.** There is a \(\Pi^1_1\) set \(A \subseteq \mathbb{R} \times \mathbb{R}\) such that for every \(x\), \(A_x\) is Borel and there is no set \(D\) of \(S\)-measure one such that \(A \cap (D \times \mathbb{R})\) is Borel.

**Proof.** Let \(A\) be the following:

\[ A = \{(x, y) \mid x, y \in \text{WO} \text{ and } |x| = |y|\} . \]

It is easy to see that \(A\) is \(\Pi^1_1\) and \(A_x\) is Borel for every \(x\).
To derive a contradiction, let $D$ be a set of $S$-measure one such that $A \cap (D \times \mathbb{R})$ is Borel. Let $B$ be the projection of $A \cap (D \times \mathbb{R})$ to the first coordinate. Then $B$ is analytic and by boundedness lemma, there is a $\delta < \omega_1$ such that the length of any element of $B$ is less than $\delta$.

But this means that the set $C = \{y \mid |y| = \delta\}$ is disjoint from $B$. Since $C$ is a subset of the projection of $A$ to the first coordinate, it is disjoint from $D$ and it clearly contains a perfect set, contradicting the choice of $D$. \hfill\Box

It is also notable that Lemma 14 can consistently fail for Sacks forcing:

**Proposition 18.** Let $s$ be a Sacks real over $L$. Then in $L[s]$, there is an $X \subseteq \mathbb{R} \times \omega_1$ which is $\Pi^1_2$ in the codes such that for every real $x$, there is a $\xi < \omega_1$ with $(x, \xi) \in X$ and that there is no $\delta < \omega_1$ such that for any Sacks real $x$ over $L$, there is a $\xi < \delta$ with $(x, \xi) \in X$.

**Proof.** We work in $L[s]$. Let $X$ be the following:

$$X = \{(x, \xi) \mid x \in \text{WO} \text{ and } |x| = \xi\} \cup \{(x, 0) \mid x \notin \text{WO}\}.$$  

It is easy to see that $A$ is $\Pi^1_2$ in the codes and that for any every $x$ there is an ordinal $\xi$ with $(x, \xi) \in A$.

To derive a contradiction, suppose there is a $\delta < \omega_1$ such that for any Sacks real $x$ over $L$, there is a $\xi < \delta$ with $(x, \xi) \in A$. It is easy to find a non-constructible surjection from $\omega$ to $\delta$. Code that real as a relation on $\omega$ and make it a real in $\text{WO}$. Call it $x$. Then $(x, \delta) \in A$. But since $x$ is non-constructible, $x$ is also a Sacks real over $L$, contradicting the choice of $\delta$. \hfill\Box

Finally note that the second item of Theorem 2 for Sacks forcing is consistent with ZFC: In fact, it is equivalent to the statement that for any real $r$ there is a real $x$ which is not in $L[r]^4$, which is easily seen to be consistent with ZFC.

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4For the proof, see [3, Theorem 7.1].
References


