数理解析研究所講究録
第 1754 巻 2011 年 21-31



# Borel approximation of coanalytic sets with Borel sections and the regularity properties for $\mathbf{\Sigma}_2^1$ sets of reals

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In 2008, Fujita showed the following:

**Theorem 1** (Fujita [5]). The following are equivalent:

1. If $A \subseteq \mathbb{R} \times \mathbb{R}$ is $\mathbf{\Pi}_1^1$ and for any real $x$, $A_x$ is Borel where $A_x = \{y \in \mathbb{R} | (x,y) \in A\}$, then there is a comeager Borel set $D \subseteq \mathbb{R}$ such that $A \cap (D \times \mathbb{R})$ is Borel, and

2. every $\mathbf{\Sigma}_2^1$ set of reals has the Baire property.

We show that one can generalize the above theorem to a wide class of tree-type ccc forcings. More precisely:

**Theorem 2.** Let $\mathbb{P}$ be a strongly arboreal, $\mathbf{\Sigma}_1^1$, provably ccc forcing. Then the following are equivalent:

1. If $A \subseteq \mathbb{R} \times \mathbb{R}$ is $\mathbf{\Pi}_1^1$ and for any real $x$, $A_x$ is Borel, then there is a Borel set $D \subseteq \mathbb{R}$ such that $D$ is of $\mathbb{P}$-measure one and $A \cap (D \times \mathbb{R})$ is Borel, and

2. every $\mathbf{\Sigma}_2^1$ set of reals is $\mathbb{P}$-measurable.

We also show that this equivalence fails for non-ccc forcings. In fact, for Sacks forcing, the corresponding statement to the first fails in ZFC while the one for the second is consistent with ZFC.

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Throughout this paper, we assume the basic knowledge of descriptive set theory and forcing which can be obtained from e.g., [9], [8], and [2]. By *reals*, we mean elements of the Cantor space or those of the Baire space and we use $\mathbb{R}$ to denote the set of reals.

To prove Theorem 2, let us start with basic definitions. Let $n$ be a natural number with $n \geq 1$. A partial order $\mathbb{P}$ is $\boldsymbol{\Sigma}^1_n$ if the sets $P$, $\leq_{\mathbb{P}}$, and $\perp_{\mathbb{P}}$ are $\boldsymbol{\Sigma}^1_n$, where $\mathbb{P} = (P, \leq_{\mathbb{P}})$ and $\perp_{\mathbb{P}}$ is the incompatibility relation in $\mathbb{P}$. A partial order $\mathbb{P}$ is *provably ccc* if there is a formula $\phi$ defining $\mathbb{P}$ and the statement "$\phi$ defines a ccc partial order" is provable in ZFC. A partial order $\mathbb{P}$ is *arboreal* if its conditions are perfect trees on $\omega$ (or on 2) ordered by inclusion. But this class of forcings contains some trivial forcings such as $\mathbb{P} = \{{}^{<\omega}\omega\}$. We need the following stronger notion:

**Definition 3.** A partial order $\mathbb{P}$ is *strongly arboreal* if it is arboreal and the following holds:

$$(\forall T \in \mathbb{P})\ (\forall t \in T)\ T_t \in \mathbb{P},$$

where $T_t = \{s \in T \mid \text{either } s \subseteq t \text{ or } s \supseteq t\}$.

With strongly arboreal forcings, one can code generic objects by reals in the standard way: Let $\mathbb{P}$ be strongly arboreal and $G$ be $\mathbb{P}$-generic over $V$. Let $x_G = \bigcup\{\mathrm{stem}(T) \mid T \in G\}$, where $\mathrm{stem}(T)$ is the longest $t \in T$ such that $T_t = T$. Then $x_G$ is a real and $G = \{T \in \mathbb{P} \mid x_G \in [T]\}$, where $[T]$ is the set of all infinite paths through $T$. Hence $V[x_G] = V[G]$. We call such real $x_G$ a *$\mathbb{P}$-generic real over $V$*.

Almost all typical forcings related to regularity properties are strongly arboreal:

**Example 4.**

1. Cohen forcing $\mathbb{C}$: Let $T_0$ be ${}^{<\omega}\omega$. Consider the partial order $\big(\{(T_0)_s \mid s \in {}^{<\omega}\omega\}, \subseteq\big)$. Then this is strongly arboreal and equivalent to Cohen forcing.
2. Random forcing $\mathbb{B}$: Consider the set of all perfect trees $T$ on 2 such that for any $t \in T$, $[T_t]$ has a positive Lebesgue measure, ordered by inclusion. Then this forcing is strongly arboreal and equivalent to random forcing.
3. Hechler forcing $\mathbb{D}$: For $(n, f) \in \mathbb{D}$, let

$$T_{(n,f)} = \Big\{t \in {}^{<\omega}\omega \mid \text{either } t \subseteq f{\restriction}n \text{ or } \Big(t \supseteq f{\restriction}n \text{ and } \big(\forall m \in \mathrm{dom}(t)\big)\ t(m) \geq f(m)\Big)\Big\}.$$

Then the partial order $(\{T_{(n,f)} \mid (n,f) \in \mathbb{D}\}, \subseteq)$ is strongly arboreal and equivalent to Hechler forcing.

4. Mathias forcing $\mathbb{R}_M$: For a condition $(s, A)$ in $\mathbb{R}_M$, let

$$T_{(s,A)} = \{t \in {}^{<\omega}\omega \mid t \text{ is strictly increasing and } s \subseteq \operatorname{ran}(t) \subseteq s \cup A\}.$$

Then $\{T_{(s,A)} \mid (s,A) \in \mathbb{R}_M\}$ is a strongly arboreal forcing equivalent to Mathias forcing.

5. Sacks forcing $\mathbb{S}$, Silver forcing $\mathbb{V}$, Miller forcing $\mathbb{M}$, Laver forcing $\mathbb{L}$: These forcings can be naturally seen as strongly arboreal forcings.

The following is as expected:

**Lemma 5.** Let $\mathbb{P}$ be a strongly arboreal, $\mathbf{\Sigma}^1_1$, provably ccc forcing and $M$ be an inner model of ZFC containing parameter defining $\mathbb{P}$ with a $\Sigma^1_1$-formula. Then if $x$ is $\mathbb{P}$-generic over $V$, then $x$ is $\mathbb{P}^M$-generic over $M$.

*Proof.* Since $\mathbb{P}$ is $\mathbf{\Sigma}^1_1$, $\mathbb{P}^M = \mathbb{P} \cap M$. So it suffices to show that if $A \subseteq \mathbb{P}^M$ is a maximal antichain in $M$, so is in $V$. Let $A \subseteq \mathbb{P}^M$ be a maximal antichain in $M$. Since $\mathbb{P}$ is provably ccc, $M$ thinks $\mathbb{P}^M$ is ccc. So $A$ is countable in $M$ and there is a real $r$ coding $A$ in $M$. Since $\mathbb{P}$ is $\mathbf{\Sigma}^1_1$, the statement "a real $r$ codes a maximal antichain in $\mathbb{P}$" is $\mathbf{\Pi}^1_2$. So the real $r$ also codes the maximal antichain $A$ in $V$, as desired. ■

We now introduce a $\sigma$-ideal $I_{\mathbb{P}}$ on the reals expressing "smallness" for each strongly arboreal forcing $\mathbb{P}$.

**Definition 6.** Let $\mathbb{P}$ be a strongly arboreal forcing. A set of reals $A$ is $\mathbb{P}$-*null* if for any $T$ in $\mathbb{P}$ there is a $T' \leq T$ such that $[T'] \cap A = \emptyset$. Let $N_{\mathbb{P}}$ denote the set of all $\mathbb{P}$-null sets and $I_{\mathbb{P}}$ denote the $\sigma$-ideal generated by $\mathbb{P}$-null sets, i.e., the set of all countable unions of $\mathbb{P}$-null sets. A set of reals $A$ is of $\mathbb{P}$-*measure one* if $\mathbb{R} \setminus A$ is in $I_{\mathbb{P}}$.

**Example 7.**

1. Cohen forcing $\mathbb{C}$: $\mathbb{C}$-null sets are the same as nowhere dense sets of reals and $I_{\mathbb{C}}$ is the ideal of meager sets of reals.

2. Random forcing $\mathbb{B}$: $\mathbb{B}$-null sets are the same as Lebesgue null sets in the Baire space and $I_{\mathbb{B}}$ is the Lebesgue null ideal.

3. Hechler forcing $\mathbb{D}$: $\mathbb{D}$-null sets are the same as nowhere dense sets in the dominating topology, i.e., the topology generated by $\{[s,f] \mid (s,f) \in \mathbb{D}\}$ where

$$[s,f] = \{x \in {}^{\omega}\omega \mid s \subseteq x \text{ and } (\forall n \geq \mathrm{dom}(s))\ x(n) \geq f(n)\}.$$

Hence $I_{\mathbb{D}}$ is the meager ideal in the dominating topology.

4. Mathias forcing $\mathbb{R}_M$: A set of reals $A$ is $\mathbb{R}_M$-null if and only if $\{\mathrm{ran}(x) \mid x \in A \cap A_0\}$ is Ramsey null or meager in the Ellentuck topology, where $A_0$ is the set of strictly increasing infinite sequences of natural numbers. Hence $I_{\mathbb{R}_M} = N_{\mathbb{R}_M}$.

5. Sacks forcing $\mathbb{S}$: In this case, $I_{\mathbb{S}} = N_{\mathbb{S}}$ by a standard fusion argument. The ideal $I_{\mathbb{S}}$ is called the Marczewski ideal and often denoted by $s_0$.

As with Sacks forcing, all the typical non-ccc tree-type forcings admitting a fusion argument satisfy the equation $I_{\mathbb{P}} = N_{\mathbb{P}}$. In the case of ccc forcings, $I_{\mathbb{P}}$ is often different from $N_{\mathbb{P}}$ (e.g., Cohen forcing and Hechler forcing).

We now introduce $\mathbb{P}$-measurability:

**Definition 8.** Let $\mathbb{P}$ be strongly arboreal. A set of reals $A$ is $\mathbb{P}$-*measurable* if for any $T$ in $\mathbb{P}$ there is a $T' \leq T$ such that either $[T'] \cap A \in I_{\mathbb{P}}$ or $[T'] \setminus A \in I_{\mathbb{P}}$.

As is expected, $\mathbb{P}$-measurability coincides with a known regularity property for $\mathbb{P}$ when $\mathbb{P}$ is ccc:

**Proposition 9.** Let $\mathbb{P}$ be a strongly arboreal, ccc forcing and let $A$ be a set of reals. Then $A$ is $\mathbb{P}$-measurable if and only if there is a Borel set $B$ such that $A \triangle B \in I_{\mathbb{P}}$, where $A \triangle B$ is the symmetric difference between $A$ and $B$.

*Proof.* See Proposition 2.9 in [6]. □

Proposition 9 does not hold for non-ccc forcings such as Sacks forcing.[1]

But $\mathbb{P}$-measurability is almost the same as the regularity properties for non-ccc forcings $\mathbb{P}$, e.g., for Mathias forcing, a set of reals $A$ is $\mathbb{R}_M$-measurable if and only if $\{\mathrm{ran}(x) \mid x \in A \cap A_0\}$ is completely Ramsey (or has the

[1]For example, assuming every $\mathbf{\Pi}^1_1$ set has the perfect set property (i.e., either the set is countable or contains a perfect subset), there is no $\mathbf{\Sigma}^1_1$ Bernstein set (i.e., a set where neither it nor its complement contains a perfect subset) but for a $\mathbf{\Sigma}^1_1$ set of reals $A$, $A$ is approximated by a Borel set modulo $I_{\mathbb{S}}$ if and only if $A$ is Borel. This is because $I_{\mathbb{S}}$ restricted to analytic sets (or co-analytic sets) is the set of all countable sets of reals by the assumption that every $\mathbf{\Pi}^1_1$ set has the perfect set property.

Baire property in the Ellentuck topology), where $A_0$ is the set of all strictly increasing infinite sequences of natural numbers. Also, for Sacks forcing, the following holds:

**Proposition 10** (Brendle, Löwe). Let $\mathbf{\Gamma}$ be a topologically reasonable pointclass, i.e., it is a set of sets of reals closed under continuous preimages and any intersection between a set in $\mathbf{\Gamma}$ and a closed set of reals. Then every set in $\mathbf{\Gamma}$ is $\mathbb{S}$-measurable if and only if there is no Bernstein set in $\mathbf{\Gamma}$.[2]

*Proof.* See [3, Lemma 2.1]. □

As expected, every $\mathbf{\Sigma}^1_1$ set of reals is $\mathbb{P}$-measurable:

**Theorem 11.** Let $\mathbb{P}$ be a strongly arboreal, proper forcing. Then every $\mathbf{\Sigma}^1_1$ set of reals is $\mathbb{P}$-measurable.

*Proof.* It follows from the fact that every $\mathbf{\Sigma}^1_1$ set of reals is universally Baire, that every universally Baire set of reals is $\mathbb{P}$-Baire, and that every $\mathbb{P}$-Baire set of reals is $\mathbb{P}$-measurable. For the details, see [4] and Section 3 in [6]. □

We are now ready to state the theorem characterizing the regularity properties for $\mathbf{\Sigma}^1_2$ sets of reals in terms of the existence of many generic reals over $\mathrm{L}[r]$ for a real $r$, which we will use for the proof of Theorem 2:

**Theorem 12.** Let $\mathbb{P}$ be a strongly arboreal, $\Sigma^1_1$, provably ccc forcing. Then the following are equivalent:

1. Every $\mathbf{\Sigma}^1_2$ set of reals is $\mathbb{P}$-measurable, and
2. for any real $r$, the set of $\mathbb{P}$-generic reals over $\mathrm{L}[r]$ is of $\mathbb{P}$-measure one.

*Proof.* See Definition 2.11, Lemma 2.13 (3), Definition 2.15, Proposition 2.17 (3), and Theorem 4.4 in [6]. □

We are now ready to prove Theorem 2:

[2]In general, the property not being a Bernstein set does not imply $\mathbb{S}$-measurability while the converse is true. By using the axiom of choice, one can construct a set of reals which is not $\mathbb{S}$-measurable but is not a Bernstein set.

*Proof of Theorem 2.* The argument is exactly the same as the one in Theorem 1 in [5]. For the sake of completeness, we will give the proof.

We first show the implication from 1. to 2. Let $P$ be a $\mathbf{\Sigma}^1_2$ set of reals. We will show that $P$ is $\mathbb{P}$-measurable. Since $P$ is $\mathbf{\Sigma}^1_2$, there is a $\mathbf{\Pi}^1_1$ set $A \subseteq \mathbb{R}\times\mathbb{R}$ such that $P = \{x \in \mathbb{R} \mid (\exists y)\ (x,y) \in A\}$. By Kondô's uniformization theorem, there is a $\mathbf{\Pi}^1_1$ function $f\colon P \to \mathbb{R}$ uniformizing $A$. Then for any real $x$, $f_x = \{y \mid (x,y) \in f\} = \{f(x)\}$ is Borel, so by applying the assumption for $f$, there is a Borel set $D$ of reals such that $D$ is of $\mathbb{P}$-measure one and $f \cap (D \times \mathbb{R})$ is Borel. Hence $P \cap D = \{x \mid (\exists y)\ (x,y) \in f \cap (D \times \mathbb{R})\}$ is $\mathbf{\Sigma}^1_1$ and is $\mathbb{P}$-measurable by Theorem 11. So by Proposition 9, there is a Borel set $B$ such that $(P \cap D)\triangle B$ is in $I_{\mathbb{P}}$. Since $D$ is of $\mathbb{P}$-measure one, $P\triangle B$ is also in $I_{\mathbb{P}}$. Again by Proposition 9, $P$ is $\mathbb{P}$-measurable, as desired.

We now show the implication from 2. to 1. Let WO be the set of reals coding a well-order on $\omega$. It is well-known that WO is a complete $\mathbf{\Pi}^1_1$ set of reals. For an element $w$ of WO, $|w|$ denotes the countable ordinal that $w$ codes. We need the following notion and lemma for the proof:

**Definition 13.** Let $r$ be a real. A set $X \subseteq \mathbb{R} \times \omega_1$ is $\Pi^1_2(r)$ *in the codes* if the set

$$\{(x,w) \in \mathbb{R}\times\mathbb{R} \mid w \in \mathrm{WO} \text{ and } (x,|w|) \in X\}$$

is $\Pi^1_2(r)$.

**Lemma 14.** Let $r$ be a real and $X \subseteq \mathbb{R}\times\omega_1$ be $\Pi^1_2(r)$ in the codes. Suppose that for any real $x$ there is a $\xi < \omega_1$ such that $(x,\xi) \in X$. Then there is a countable ordinal $\delta$ such that for any $\mathbb{P}$-generic real $x$ over $\mathrm{L}[r]$, there is a $\xi < \delta$ such that $(x,\xi) \in X$.

*Proof of Lemma 14.* Since $X$ is $\Pi^1_2(r)$ in the codes, pick a $\Pi^1_2$-formula $\phi(x,w,v)$ such that

$$(\forall x, w)\ \big(\phi(x,w,r) \iff w \in \mathrm{WO} \text{ and } (x,|w|) \in X\big).$$

Let $\tilde{\phi}(x,\xi,r)$ be the following:

$$\tilde{\phi}(x,\xi) \iff (\forall w \in \mathrm{WO})\ |w| = \xi \to \phi(x,w,r).$$

Then $\tilde{\phi}$ is absolute among all the transitive proper class models of ZFC in which $\xi$ is countable.

For each $\xi < \omega_1$, let

$$X_\xi = \{T \in \mathbb{P} \mid (T, \mathbf{1}_{\mathbb{P}_\xi}) \Vdash_{\mathbb{P}\times\mathbb{P}_\xi} \tilde{\phi}(\dot{x}, \check{\xi}, \check{r})\}^{\mathrm{L}[r]},$$

where $\mathbb{P}_\xi$ is $\mathrm{Coll}(\omega, \xi)$ and $\dot{x}$ is a canonical $\mathbb{P}$-name for a generic real.

We show that $\bigcup_{\xi<\omega_1} X_\xi$ is a dense subset of $\mathbb{P}^{\mathrm{L}(\mathbb{R})}$ in $\mathrm{L}[r]$. Let $T$ be any element of $\mathbb{P}^{\mathrm{L}[r]}$. Take a $\mathbb{P}$-generic real $x$ over $\mathrm{L}[r]$ in $V$ with $x \in [T]$. Then by the assumption, there is a $\xi < \omega_1$ such that $(x, \xi) \in X$. Take a function $g \colon \omega \to \xi$ generic over $\mathrm{L}[r, x]$. Then $\mathrm{L}[r, x, g] \models \tilde{\phi}(x, \xi, r)$. Hence there is a $T' \leq T$ and a condition $p$ in $\mathbb{P}_\xi$ such that $\mathrm{L}[r] \models$ "$(T', p) \Vdash \tilde{\phi}(\dot{x}, \check{\xi}, \check{r})$". Since $\mathbb{P}_\xi$ is homogeneous, it follows that $\mathrm{L}[r] \models$ "$(T', \mathbf{1}_{\mathbb{P}_\xi} \Vdash \tilde{\phi}(\dot{x}, \check{\xi}, \check{r})$", so $T' \leq T$ and $T \in \bigcup_{\xi<\omega_1} X_\xi$, as desired.

Since $\mathbb{P}$ is provably ccc, $\mathrm{L}[r] \models$ "$\mathbb{P}$ is ccc", so there is a $\delta < \omega_1$ such that $\bigcup_{\xi<\delta} X_\xi$ is a predense subset of $\mathbb{P}$ in $\mathrm{L}[r]$. We show that this $\delta$ is the desired countable ordinal. Take any $\mathbb{P}$-generic real $x$ over $\mathrm{L}[r]$. Then since $\mathrm{L}[r]$ thinks $\bigcup_{\xi<\delta} X_\xi$ is a predense subset of $\mathbb{P}$, the generic filter $G_x$ meets $\bigcup_{\xi<\delta} X_\xi$ and hence there is a $\xi < \delta$ such that $G_x \cap X_\xi \neq \emptyset$. By the definition of $X_\xi$, for a function $g \colon \omega \to \xi$ generic over $\mathrm{L}[r, x]$, $\mathrm{L}[r, x, g] \models \tilde{\phi}(x, \xi, r)$, hence $\tilde{\phi}(x, \xi, r)$ holds also in $V$ and $(x, \xi) \in X$, as desired. $\square$ (Lemma 14)

We now finish showing the implication from 2. to 1. Let $A \subseteq \mathbb{R} \times \mathbb{R}$ be $\mathbf{\Pi}^1_1$ such that for any real $x$, $A_x$ is Borel. Let $f \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f^{-1}(\mathrm{WO}) = A$. Take any real $x$. Since $A_x$ is Borel, the set $f``(\{x\} \times A_x)$ is $\mathbf{\Sigma}^1_1$, hence by boundedness theorem, it is bounded in WO, i.e.,

$$(\forall x)\ (\exists \xi)\ (\forall y) \text{ if } (x, y) \in A, \text{ then } |f(x, y)| < \xi.$$

Set

$$X = \{(x, \xi) \mid f``(\{x\} \times A_x) \subseteq \mathrm{WO}_\xi\},$$

where $\mathrm{WO}_\xi = \{w \in \mathrm{WO} \mid |w| < \xi\}$ for each $\xi < \omega_1$.

Then for any $x$ there is a $\xi$ with $(x, \xi) \in X$. It is also easy to see that $X$ is $\Pi^1_2(r)$ in the codes for some real $r$. By Lemma 14, there is a $\delta < \omega_1$ such that for any $\mathbb{P}$-generic real $x$ over $\mathrm{L}[r]$ there is a $\xi < \delta$ such that $(x, \xi) \in X$. Hence $A$ is the same as the Borel set $f^{-1}(\mathrm{WO}_\delta)$ on $G(\mathrm{L}[r]) \times \mathbb{R}$, where $G(\mathrm{L}[r])$ is the set of $\mathbb{P}$-generic reals over $\mathrm{L}[r]$. By 2. and Theorem 12, the set $G(\mathrm{L}[r])$ is of $\mathbb{P}$-measure one. Since $\mathbb{P}$ is ccc, $I_{\mathbb{P}}$ is Borel generated, so there is a Borel set $D \subseteq G(\mathrm{L}[r])$ of $\mathbb{P}$-measure one and $A \cap (D \times \mathbb{R})$ is Borel, as desired.

$\blacksquare$ (Theorem 2)

After the RIMS set theory conference in 2008, Fujita asked if one could take $\delta$ in Lemma 14 below $\gamma^1_2$ if $X$ is $\Pi^1_2$ (lightface) in the codes and if $\mathbb{P}$ is Cohen forcing, where $\gamma^1_2$ is the least countable ordinal that meets every set $A \subseteq \omega_1$ which is $\Pi^1_2$ (lightface) in the codes.[3] We show that this is generally the case for each strongly arboreal, $\Sigma^1_1$, provably ccc forcing $\mathbb{P}$:

**Proposition 15.** Let $\mathbb{P}$ be a strongly arboreal, $\Sigma^1_1$ (lightface), ccc forcing and $X \subseteq \mathbb{R} \times \omega_1$ be $\Pi^1_2$ (lightface) in the codes such that for any real $x$ there is a $\xi < \omega_1$ with $(x, \xi) \in X$. Then there is a $\delta < \gamma^1_2$ such that for any $\mathbb{P}$-generic real $x$ over L, there is a $\xi < \delta$ with $(x, \xi) \in X$.

*Proof.* Let $X \subseteq \mathbb{R} \times \omega_1$ be $\Pi^1_2$ in the codes such that for any real $x$ there is a $\xi$ with $(x, \xi) \in X$.

Let $A$ be as follows:

$$A = \{\gamma < \omega_1 \mid (\forall x\colon \mathbb{P}\text{-generic over L})\ (\exists \xi < \gamma)\ (x, \xi) \in X\}.$$

By Lemma 14, $A$ is nonempty. Hence it suffices to show that $A$ is $\Pi^1_2$ in the codes.

Since $X$ is $\Pi^1_2$ in the codes, pick a $\Pi^1_2$-formula $\phi$ such that

$$(\forall x, w)\ \big(\phi(x, w) \iff w \in \mathrm{WO} \text{ and } (x, |w|) \in X\big).$$

Let $\tilde{\phi}$ be the following:

$$\tilde{\phi}(x, \xi) \iff (\forall w \in \mathrm{WO})\ |w| = \xi \to \phi(x, w).$$

Then

$$\begin{aligned} A &= \{\gamma < \omega_1 \mid (\forall x\colon \mathbb{P}\text{-generic over L})\ (\exists \xi < \gamma)\ \tilde{\phi}(x, \xi)\} \\ &= \{\gamma < \omega_1 \mid \mathrm{L} \models \text{“}(1_{\mathbb{P}}, 1_{\mathbb{P}_\gamma}) \Vdash_{\mathbb{P} \times \mathbb{P}_\gamma} (\exists \xi < \check{\gamma})\ \tilde{\phi}(\dot{x}, \xi)\text{”}\}, \end{aligned}$$

where $\mathbb{P}_\gamma$ is $\mathrm{Coll}(\omega, \gamma)$ and $\dot{x}$ is a canonical $\mathbb{P}$-name for a generic real.

**Claim 16.** For $\gamma < \omega_1$,
$\mathrm{L} \models \text{“}(1_{\mathbb{P}}, 1_{\mathbb{P}_\gamma}) \Vdash_{\mathbb{P} \times \mathbb{P}_\gamma} (\exists \xi < \check{\gamma})\ \tilde{\phi}(\dot{x}, \xi)\text{”} \iff V \models \text{“}(1_{\mathbb{P}}, 1_{\mathbb{P}_\gamma}) \Vdash_{\mathbb{P} \times \mathbb{P}_\gamma} (\exists \xi < \check{\gamma})\ \tilde{\phi}(\dot{x}, \xi)\text{”}$

[3] $\gamma^1_2$ is also the least ordinal such that every $\Pi^1_1$ (lightface) Borel set is $\mathbf{\Pi}^0_\alpha$ (boldface) for some $\alpha < \omega_1$. For the details, see [7].

*Proof of Claim 16.* The direction from left to right follows from the fact that if $(x, g)$ is $\mathbb{P} \times \mathbb{P}_\gamma$-generic over $V$, then so is over L by Lemma 5.

For right to left, suppose $\mathrm{L} \models$ "$(1_\mathbb{P}, 1_{\mathbb{P}_\gamma}) \Vdash_{\mathbb{P}\times\mathbb{P}_\gamma} (\exists \xi < \check{\gamma})\ \tilde{\phi}(\dot{x}, \xi)$" fails. Then there is a $(p, q) \in \mathbb{P} \times \mathbb{P}_\gamma$ in L such that $\mathrm{L} \models$ "$(p, q) \Vdash_{\mathbb{P}\times\mathbb{P}_\gamma} (\forall \xi < \check{\gamma})\ \neg\tilde{\phi}(\dot{x}, \xi)$". Take a $\mathbb{P} \times \mathbb{P}_\gamma$-generic $(x, g)$ over $V$ with $x \in [p]$ and $g \supseteq q$. By the assumption, there exists a $\xi < \gamma$ such that $V[x, g] \models \tilde{\phi}(x, \xi)$. But $(x, g)$ is also $\mathbb{P} \times \mathbb{P}_\gamma$-generic over L and $\mathrm{L}[x, g] \models \tilde{\phi}(x, \xi)$, contradicting $\mathrm{L} \models$ "$(p, q) \Vdash_{\mathbb{P}\times\mathbb{P}_\gamma} (\forall \xi < \check{\gamma})\ \neg\tilde{\phi}(\dot{x}, \xi)$". □ (Claim 16)

Therefore,

$$A = \{\gamma < \omega_1 \mid \text{“}(1_\mathbb{P}, 1_{\mathbb{P}_\gamma}) \Vdash_{\mathbb{P}\times\mathbb{P}_\gamma} (\exists \xi < \check{\gamma})\ \tilde{\phi}(\dot{x}, \xi)\text{”}\}.$$

Let $\psi$ be the following:

$$\psi(w) \iff w \in \mathrm{WO} \text{ and } (1_\mathbb{P}, 1_{\mathbb{P}_{|w|}}) \Vdash_{\mathbb{P}\times\mathbb{P}_{|w|}} \text{“}(\exists n \in \omega)\ \phi(\dot{x}, w \restriction n)\text{”},$$

where $w \restriction n$ is the real coding the well-order $\leq_w$ below $n$, i.e. $\leq_{w \restriction n} = \{(l, m) \mid l \leq_w m <_w n\}$. Then

$$(\forall w)\big(\psi(w) \iff w \in \mathrm{WO} \text{ and } |w| \in A\big).$$

Hence it suffices to show that $\psi$ is equivalent to a $\Pi^1_2$-formula. Since $\mathbb{P}_{|w|}$ is ccc in $V^\mathbb{P}$, $\mathbb{P} \times \mathbb{P}_{|w|}$ is also ccc. Moreover, it is easy to see that $\mathbb{P} \times \mathbb{P}_{|w|}$ is $\Sigma^1_1(w)$ uniformly in $w \in \mathrm{WO}$. Hence, by the same argument as in Theorem 2.7 (1) in Bagaria and Bosch [1], since $(\exists n \in \omega)\ \phi(x, w \restriction n)$ is $\Pi^1_2$ in $x$ and $w$, so is $(1_\mathbb{P}, 1_{\mathbb{P}_{|w|}}) \Vdash_{\mathbb{P}\times\mathbb{P}_{|w|}}$ "$(\exists n \in \omega)\ \phi(\dot{x}, w \restriction n)$" in $w$. Therefore, $\psi$ is equivalent to a $\Pi^1_2$-formula. ■ (Proposition 15)

As announced in the beginning of this paper, we now show that the first item in Theorem 2 fails in ZFC for $\mathbb{P} = \mathbb{S}$ (Sacks forcing):

**Proposition 17.** There is a $\Pi^1_1$ set $A \subseteq \mathbb{R} \times \mathbb{R}$ such that for every $x$, $A_x$ is Borel and there is no set $D$ of $\mathbb{S}$-measure one such that $A \cap (D \times \mathbb{R})$ is Borel.

*Proof.* Let $A$ be the following:

$$A = \{(x, y) \mid x, y \in \mathrm{WO} \text{ and } |x| = |y|\}.$$

It is easy to see that $A$ is $\Pi^1_1$ and $A_x$ is Borel for every $x$.

To derive a contradiction, let $D$ be a set of $\mathbb{S}$-measure one such that $A \cap (D \times \mathbb{R})$ is Borel. Let $B$ be the projection of $A \cap (D \times \mathbb{R})$ to the first coordinate. Then $B$ is analytic and by boundedness lemma, there is a $\delta < \omega_1$ such that the length of any element of $B$ is less than $\delta$.

But this means that the set $C = \{y \mid |y| = \delta\}$ is disjoint from $B$. Since $C$ is a subset of the projection of $A$ to the first coordinate, it is disjoint from $D$ and it clearly contains a perfect set, contradicting the choice of $D$. ∎

It is also notable that Lemma 14 can consistently fail for Sacks forcing:

**Proposition 18.** Let $s$ be a Sacks real over L. Then in L$[s]$, there is an $X \subseteq \mathbb{R} \times \omega_1$ which is $\Pi^1_2$ in the codes such that for every real $x$, there is a $\xi < \omega_1$ with $(x, \xi) \in X$ and that there is no $\delta < \omega_1$ such that for any Sacks real $x$ over L, there is a $\xi < \delta$ with $(x, \xi) \in X$.

*Proof.* We work in L$[s]$. Let $X$ be the following:

$$X = \{(x, \xi) \mid x \in \mathrm{WO} \text{ and } |x| = \xi\} \cup \{(x, 0) \mid x \notin \mathrm{WO}\}.$$

It is easy to see that $A$ is $\Pi^1_2$ in the codes and that for any every $x$ there is an ordinal $\xi$ with $(x, \xi) \in A$.

To derive a contradiction, suppose there is a $\delta < \omega_1$ such that for any Sacks real $x$ over L, there is a $\xi < \delta$ with $(x, \xi) \in A$. It is easy to find a non-constructible surjection from $\omega$ to $\delta$. Code that real as a relation on $\omega$ and make it a real in WO. Call it $x$. Then $(x, \delta) \in A$. But since $x$ is non-constructible, $x$ is also a Sacks real over L, contradicting the choice of $\delta$. ∎

Finally note that the second item of Theorem 2 for Sacks forcing is consistent with ZFC: In fact, it is equivalent to the statement that for any real $r$ there is a real $x$ which is not in L$[r]$,[4] which is easily seen to be consistent with ZFC.

## Acknowledgment

The author would like to thank Hiroshi Fujita for raising this topic and discussing it with him. He is also grateful to Hiroshi Sakai for organizing this great conference.

[4] For the proof, see [3, Theorem 7.1].

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