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Real determinacy and real Blackwell determinacy

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This article is dedicated to Hiroshi Sakai, the organizer of this conference, who just got married and started his new life. Congratulations!

1 Introduction

In this paper, we discuss the connection between Gale-Stewart games and Blackwell games, especially the relationship between the strong versions of determinacy axioms of Gale-Stewart games and Blackwell games, i.e., the Axiom of Real Determinacy $\mathrm{AD}_{\mathbb{R}}$ and the Axiom of Real Blackwell Determinacy $\mathrm{Bl}\text{-}\mathrm{AD}_{\mathbb{R}}$. We put a stress on motivation & ideas of the proofs and the details of the arguments for the proofs are often omitted while one can find them in the references we will mention.

In 1953, Gale and Stewart [7] developed the general theory of infinite games, so-called *Gale-Stewart games*, which are two-player zero-sum infinite games with perfect information. The theory of Gale-Stewart games has been investigated by many logicians and now it is one of the main topics in set theory and it has connections with other topics in set theory as well as model theory and computer science.

In 1928, John von Neumann proved his famous *minimax theorem* which is about finite games with imperfect information. Infinite versions of von Neumann's games were introduced by David Blackwell [3] where he proved the analogue of von Neumann's theorem for G_δ sets of reals (i.e., $\boldsymbol{\Pi}^0_2$ sets of reals). The games he introduced are called *Blackwell games* and they were called by him "games with slightly imperfect information" in his paper [4].

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In 1998, Martin [16] proved that in most cases, Blackwell determinacy axioms follow from the corresponding determinacy axiom. Martin conjectured that they are equivalent, and many instances of equivalence have been shown (e.g., [17] and Martin's proof of $\mathbf{\Pi}^1_1$ determinacy presented in [15, Corollary 3.9]). However, the general question, and in particular the most intriguing instance, viz. whether the Axiom of Determinacy (AD) and the Axiom of Blackwell determinacy (Bl-AD) are equivalent, remains open.

In this paper, we summarize the development of the research on the relationship between AD and Bl-AD and look at the other mentioned determinacy axioms, the stronger $\mathrm{AD}_{\mathbb{R}}$ and its Blackwell analogue Bl-$\mathrm{AD}_{\mathbb{R}}$. We mainly discuss the relationship between $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$ and relate it to the one between AD and Bl-AD.

In § 2, we introduce Gale-Stewart games & Blackwell games and discuss their background. In § 3, we discuss the relationship between AD and Bl-AD especially their equiconsistency. In § 4, we turn our attention to the consequences of Bl-$\mathrm{AD}_{\mathbb{R}}$ especially the two among them, that $\mathbb{R}^{\#}$ exists and that every set of reals is ∞-Borel. From them one can derive that Bl-$\mathrm{AD}_{\mathbb{R}}$ implies the consistency of AD (so the consistency of Bl-$\mathrm{AD}_{\mathbb{R}}$ is strictly stronger than that of AD by Gödel's Second Incompleteness Theorem), and that Bl-$\mathrm{AD}_{\mathbb{R}}$ implies that every set of reals has almost all the known regularity properties. In § 5, we discuss the possibility of the equivalence between $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$. In § 6, we discuss the possibility of the equiconsistency between $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$.

Throughout this paper, we work in ZF+$\mathrm{DC}_{\mathbb{R}}$. Here by $\mathrm{DC}_{\mathbb{R}}$, we mean the axiom of dependent choice for relations on the reals.[1] This small fragment of the axiom of choice is necessary for the definition of axioms of Blackwell determinacy. Using $\mathrm{DC}_{\mathbb{R}}$, we can develop the basics of measure theory. If we need more than ZF+$\mathrm{DC}_{\mathbb{R}}$ for some definitions and statements, we explicitly mention the additional axioms. We use standard notations from set theory and assume familiarity with descriptive set theory. Apart from the standard notations, by *reals*, we mean elements of the Cantor space and we use $\mathbb{R}$ to denote the Cantor space.

[1] In other words, for any relation R on the reals such that $(\forall x \in \mathbb{R})\ (\exists y \in \mathbb{R})\ (x, y) \in R$, there is a function $f\colon \omega \to \mathbb{R}$ such that $\big(f(n), f(n+1)\big) \in R$ for every $n \in \omega$.

2 Definitions & Background

In this section, we introduce Gale-Stewart games & Blackwell games and discuss their background. We start with Gale-Stewart games.

In 1913, Ernst Zermelo [24] investigated finite games with perfect information as a formalization of the game of chess and proved the determinacy of these games, i.e., in chess, exactly one of the following holds: 1) The first player has a winning strategy, 2) the second player has a winning strategy, or 3) both players have strategies forcing to be a draw. In 1953, Gale and Stewart [7] extended Zermelo's work to a large class of infinite games so-called *Gale-Stewart games*, which are two-player zero-sum infinite games with perfect information.

Let us describe what Gale-Stewart games are: Let X be a nonempty set and A be a subset of ${}^{\omega}X$. The *Gale-Stewart game* $G_X(A)$ is played by two players, player I and player II. They play elements of X ω-many times in turn, i.e., player I starts with choosing an element x_0 of X, then player II responds with $x_1 \in X$, then player I moves with $x_2 \in X$ and player II chooses x_3 and so on. After ω moves, they have produced an ω-sequence $x = \langle x_n \mid n \in \omega \rangle \in {}^{\omega}X$. Player I wins if x is in A and player II wins if x is not in A.

This type of games are zero-sum in the sense that one of the players always wins the games and when one wins, the other loses. They are games with perfect information in the sense that both players know what they have previously played and they can decide the next move considering their previous moves as in chess.

We are interested in whether one of the players has a winning strategy in the game $G_X(A)$, i.e., whether one of the players has a way to play this game such that no matter her opponent moves, she will always win this game. Let us formulate the notion of winning strategies.

Definition 2.1. Let X be a nonempty set and A be a subset of ${}^{\omega}X$. A *strategy for player I* is a function $\sigma\colon X^{\mathrm{Even}} \to X$, where X^{Even} is the set of finite sequences of elements in X with even length. A *strategy for player II* is a function $\tau\colon X^{\mathrm{Odd}} \to X$, where X^{Odd} is the set of finite sequences of elements in X with odd length. Given a strategy σ for player I and a strategy τ for player II, one can produce the run $\sigma * \tau$ of the game $G_X(A)$ according to σ and τ by letting player I follow σ and player II follow τ, more precisely, the run $\sigma * \tau$ of the game $G_X(A)$ is a unique ω-sequence of elements in X

with the following property: For any natural number n,

$$(\sigma * \tau)(n) = \nu_{\sigma,\tau}\big((\sigma * \tau)\restriction n\big),$$

where for a finite sequence s of elements in X, $\nu_{\sigma,\tau}(s) = \sigma(s)$ if the length of s is even and $\nu_{\sigma,\tau}(s) = \tau(s)$ if the length of s is odd. A strategy σ for player I is *winning in the game* $G_X(A)$ if for any strategy τ for player II, $\sigma * \tau$ is in A. A strategy τ for player II is *winning in the game* $G_X(A)$ if for any strategy σ for player I, $\sigma * \tau$ is not in A. A subset A of ${}^{\omega}X$ is *determined* if one of the players has a winning strategy in the game $G_X(A)$.

We are interested in what kind of sets A are determined. Let us list some results concerning this question. From now on, we see ${}^{\omega}X$ as a topological space by the product topology where each coordinate (i.e., X) is seen as the discrete space. By AC, we mean the Axiom of Choice.

Theorem 2.2 (Gale and Stewart). (AC) Let X be a nonempty set.

1. Every closed subset of ${}^{\omega}X$ and every open subset of ${}^{\omega}X$ are determined. If X is well-ordered, one does not need AC.

2. There is a subset of ${}^{\omega}\omega$ which is not determined.

Proof. See, e.g., [12, Lemma 33.1, Lemma 33.17]. □

Theorem 2.3 (Martin). (AC) Let X be a nonempty set. Then every Borel subset of ${}^{\omega}X$ is determined.

Proof. See, e.g., [13, Theorem 20.5]. □

Theorem 2.4 (Davis; Gödel and Addison). ZFC cannot prove that every $\boldsymbol{\Sigma}^1_1$ subset of the Baire space is determined.

Proof. The statement follows from the combination of the following two results: The first is that every $\boldsymbol{\Sigma}^1_1$ subset of the Baire space is determined, then every $\boldsymbol{\Pi}^1_1$ subset of the Baire space has the perfect set property and the second one is that ZFC cannot prove that every $\boldsymbol{\Pi}^1_1$ subset of the Baire space has the perfect set property. The first result is due to Davis [5] and the second result was announced by Gödel [8] and the details of the proof were given by Addison [1]. For the proofs, see, e.g., [18, p. 224 & 225] and [12, Corollary 25.37]. □

Gale-Stewart games are general enough that they can be used to simulate several kinds of infinite games in mathematics, e.g., Banach-Mazur games. In particular, the determinacy of Gale-Stewart games implies that of several other kinds of games. From this, one can prove several properties of sets of reals assuming the determinacy of Gale-Stewart games such as Lebesgue measurability, the Baire property, and the perfect set property.

Despite the fact that it contradicts the Axiom of Choice by the second item of Theorem 2.2, Mycielski and Steinhaus [20] introduced the *Axiom of Determinacy* (AD), which states that every subset of the Baire space is determined and investigated the consequences of this axiom. They proved that AD implies that every set of reals is Lebesgue measurable and that every subset of the Baire space has the Baire property and the perfect set property where each of the statements contradict the Axiom of Choice. Beside such properties for sets of reals, AD supplies a beautiful structural theory. Moreover, models of AD (or AD^+) have been investigated for a long time and they are essential for the research on inner models with large cardinals. In this way, the study of AD has been one of the central topics in set theory despite the fact that AD contradicts AC.

One can define AD_X for a nonempty set X as follows: Every subset of ${}^\omega X$ is determined. Let us list some known observations on AD_X:

Proposition 2.5.

1. Let X, Y be nonempty sets and assume that there is an injection from X to Y. Then AD_Y implies AD_X. In particular, $\mathrm{AD}_{\mathbb{R}}$ implies $\mathrm{AD}_\omega = \mathrm{AD}$.

2. The axioms AD_{ω_1} and $\mathrm{AD}_{\mathcal{P}(\mathbb{R})}$ are inconsistent.

Proof. The first statement is a folklore and it is easy (one can simulate the games on X by those on Y by using a given injection from X to Y). For the second statement, the inconsistency of AD_{ω_1} is due to Mycielski [19] and that of $\mathrm{AD}_{\mathcal{P}(\mathbb{R})}$ follows from the inconsistency of AD_{ω_1}, the fact that there is an injection from ω_1 into $\mathcal{P}(\mathbb{R})$, and the first item of this proposition. (One can send a countable ordinal α to the set of all reals coding the structure $(\alpha, \in)$ and this is an injection from ω_1 into $\mathcal{P}(\mathbb{R})$.) □

We investigate AD and $\mathrm{AD}_{\mathbb{R}}$ in later sections.

Next, we introduce Blackwell games which are infinite games with imperfect information. Although one can simulate many kinds of games with Gale-Stewart games, players often do not know about the opponent's moves in many games in real life and game theory such as rock-paper-scissors. It is certainly the case that none of the players has a winning strategy in rock-paper-scissors (otherwise people would not want to decide things by playing rock-paper-scissors). In rock-paper-scissors, instead of having a winning strategy, one can demand them having an *optimal strategy* in the sense that they can maximize the probability of their winning if they are asked to assign probabilities on each hand, namely both players assign $1/3$ to each hand and having at least $1/2$ as the probability of their winning.

In 1928, John von Neumann extended this phenomenon to zero-sum games with finite choices and proved so-called *minimax theorem*. Blackwell games are one version of infinite two-player zero-sum games with imperfect information and their determinacy is an extension of von Neumann's minimax theorem.

Let us describe what Blackwell games are: Let X be a nonempty finite set. As with Gale-Stewart games, Blackwell games are played by two players, player I and player II. Players choose probabilities on X (i.e., functions $p\colon X \to [0,1]$ such that $\sum_{x \in X} p(x) = 1$) instead of elements of X ω times one by one in turn. With those probabilities, one can deduce a Borel probability μ on ${}^{\omega}X$, i.e., a measure assigning probabilities to each Borel subset of ${}^{\omega}X$. Then given a payoff set $A \subseteq {}^{\omega}X$, $\mu^{-}(A)$ is the probability of winning for player I and $1 - \mu^{+}(A)$ is the probability of winning for player II, where μ^{-} is the inner measure of μ and μ^{+} is the outer measure of μ. Blackwell determinacy for A asserts that

$$\sup_{\sigma} \inf_{\tau} \mu^{-}_{\sigma,\tau}(A) = \inf_{\tau} \sup_{\sigma} \mu^{+}_{\sigma,\tau}(A),$$

where σ (τ) ranges over strategies for player I (player II resp.) in this game and $\mu_{\sigma,\tau}$ is the Borel probability derived from σ and τ. Here is the precise definition:

Definition 2.6. Let X be a nonempty finite set. A *Blackwell strategy for player I* is a function $\sigma\colon X^{\mathrm{Even}} \to \mathrm{Prob}(X)$, where $\mathrm{Prob}(X)$ is the set of functions $\mu\colon X \to [0,1]$ with $\sum_{x \in X} \mu(x) = 1$. A *Blackwell strategy for player II* is a function $\sigma\colon X^{\mathrm{Odd}} \to \mathrm{Prob}(X)$ such that for any $s \in X^{\mathrm{Even}}$ and $x, y \in X$, $\tau(s^\frown\langle x\rangle) = \tau(s^\frown\langle y\rangle)$.

Given Blackwell strategies σ, τ for player I and II respectively, let $\nu(\sigma,\tau)\colon {}^{<\omega}X \to$

$\mathrm{Prob}(X)$ be as follows: For each finite sequence s of elements of X,

$$\nu(\sigma,\tau)(s) = \begin{cases} \sigma(s) & \text{if } s \in X^{\mathrm{Even}}, \\ \tau(s) & \text{if } s \in X^{\mathrm{Odd}}. \end{cases}$$

For each finite sequence s of elements of X, define

$$\mu_{\sigma,\tau}([s]) = \prod_{i=0}^{\mathrm{lh}(s)-1} \nu(\sigma,\tau)(s \restriction i)\ (s(i)),$$

where $[s]$ denotes the set of $x \in {}^{\omega}X$ such that $x \supseteq s$ and these sets are basic open sets in the topological space ${}^{\omega}X$. With the help of $\mathrm{DC}_{{}^{\omega}X}$, one can uniquely extend $\mu_{\sigma,\tau}$ to a Borel probability on ${}^{\omega}X$, i.e., the probability whose domain is the set of all Borel sets in the space ${}^{\omega}X$. Let us also use $\mu_{\sigma,\tau}$ for denoting this Borel probability.

A subset A of ${}^{\omega}X$ *has a value* if

$$\sup_{\sigma}\inf_{\tau} \mu^{-}_{\sigma,\tau}(A) = \inf_{\tau}\sup_{\sigma} \mu^{+}_{\sigma,\tau}(A),$$

where σ (τ) ranges over Blackwell strategies for I (II resp.) and $\mu^{-}_{\sigma,\tau}$ ($\mu^{+}_{\sigma,\tau}$) denotes the inner (outer resp.) measure of $\mu_{\sigma,\tau}$.

The *Axiom of Blackwell Determinacy* (Bl-AD) asserts that for any nonempty finite set X, every subset A of ${}^{\omega}X$ has a value.

Note that one can formulate rock-paper-scissors with Blackwell games by setting $X = \{\text{rock}, \text{paper}, \text{scissors}\}$ and $A = \{x \in {}^{\omega}X \mid x(0)\,\text{"wins" against } x(1)\}$. This is why we demand the additional condition to Blackwell strategies τ for II, namely for any $s \in X^{\mathrm{Even}}$ and $x, y \in X$, $\tau(s^\frown\langle x\rangle) = \tau(s^\frown\langle y\rangle)$. (Otherwise, player II could decide her hand (i.e., $x(1)$) after looking at the player I's hand (i.e., $x(0)$).)

The following theorem explains how Blackwell determinacy affects the properties for sets of reals:

Theorem 2.7 (Vervoort). Let $\mathbf{\Gamma} \subseteq \mathcal{P}(\mathbb{R})$ be a pointclass closed under continuous preimages. Then if every set in $\mathbf{\Gamma}$ has a value, then every set in $\mathbf{\Gamma}$ is Lebesgue measurable.[2] In particular, Bl-AD implies that every set of reals is Lebesgue measurable and it contradicts the Axiom of Choice.

[2] Here by Lebesgue measure, we mean the product measure μ_{L} on the Cantor space, i.e., $\mu_{\mathrm{L}}([s]) = 2^{-\mathrm{lh}(s)}$ for every $s \in {}^{<\omega}2$, where $\mathrm{lh}(s)$ denotes the length of s.

Proof. Set $X = 2$ and let σ_0 be the randomized Blackwell strategy for I, i.e., $\sigma_0(s)(0) = \sigma_0(s)(1) = 1/2$ for any $s \in 2^{\mathrm{Even}}$. Let τ_0 be the randomized Blackwell strategy for II, i.e., $\sigma_0(s)(0) = \sigma_0(s)(1) = 1/2$ for any $s \in 2^{\mathrm{Odd}}$. The idea is to realize the Lebesgue measure μ_{L} by taking a pull-back measure of $\mu_{\sigma_0,\tau}(\mu_{\sigma,\tau_0})$ for any τ (σ resp.) via a certain continuous function.

Let $\pi\colon {}^{\omega}2 \to {}^{\omega}2$ be the following function: For each x in ${}^{\omega}2$ and n in ω, $\pi(x)(n) = |x(2n) - x(2n+1)|$. Then it is easy to see that π is continuous.

Claim 2.8. For any Blackwell strategy τ for II, μ_{L} is the pull-back measure of $\mu_{\sigma_0,\tau}$ via π, i.e., for any Borel subset B of the Cantor space, $\mu_{\mathrm{L}}(B) = \mu_{\sigma_0,\tau}(\pi^{-1}(B))$. The corresponding statement holds for τ_0.

Proof of Claim 2.8. It is enough to show that $\mu_{\mathrm{L}}([s]) = \mu_{\sigma_0,\tau}(\pi^{-1}([s]))$ for any $s \in {}^{<\omega}2$ and this can be done by induction on the length of s. Here the additional condition for Blackwell strategies for player II is essential. Since this is a routine calculation, we omit the details. □ (Claim 2.8)

Let A be in $\mathbf{\Gamma}$. We show that A is Lebesgue measurable. It is enough to show that $\mu_{\mathrm{L}}^{+}(A) \leq \mu_{\mathrm{L}}^{-}(A)$. Since π is continuous, $\pi^{-1}(A) \in \mathbf{\Gamma}$, hence by the assumption, $\pi^{-1}(A)$ has a value, i.e.,

$$\sup_{\sigma} \inf_{\tau} \mu_{\sigma,\tau}^{-}(\pi^{-1}(A)) = \inf_{\tau} \sup_{\sigma} \mu_{\sigma,\tau}^{+}(\pi^{-1}(A)). \tag{1}$$

Then by Claim 2.8, for any τ,

$$\mu_{\mathrm{L}}^{+}(A) = \mu_{\sigma_0,\tau}^{+}(\pi^{-1}(A)) \leq \sup_{\sigma} \mu_{\sigma,\tau}^{+}(\pi^{-1}(A))$$

and hence

$$\mu_{\mathrm{L}}^{+}(A) \leq \inf_{\tau} \sup_{\sigma} \mu_{\sigma,\tau}^{+}(\pi^{-1}(A)).$$

By the same argument, one can prove that

$$\sup_{\sigma} \inf_{\tau} \mu_{\sigma,\tau}^{-}(\pi^{-1}(A)) \leq \mu_{\mathrm{L}}^{-}(A).$$

Therefore, by the formula (1),

$$\begin{aligned} \mu_{\mathrm{L}}^{+}(A) &\leq \inf_{\tau} \sup_{\sigma} \mu_{\sigma,\tau}^{+}(\pi^{-1}(A)) \\ &= \sup_{\sigma} \inf_{\tau} \mu_{\sigma,\tau}^{-}(\pi^{-1}(A)) \\ &\leq \mu_{\mathrm{L}}^{-}(A), \end{aligned}$$

as desired. ■

We discuss the connection between Blackwell determinacy and other regularity properties such as the Baire property in § 4.

As for Gale-Stewart games, one could ask what kind of subsets of ${}^{\omega}X$ have a value for a nonempty finite X. After proving that every G_δ set of reals has a value, Blackwell asked whether every Borel set of reals has a value. In 1998, Martin proved the following:

Theorem 2.9 (Martin). Let $\mathbf{\Gamma}$ be a pointclass closed under continuous preimages. Then if every set in $\mathbf{\Gamma}$ is determined, then every set in $\mathbf{\Gamma}$ has a value. In particular, every Borel set of reals has a value and AD implies Bl-AD.

Proof. See [16]. □

Hence determinacy implies Blackwell determinacy. After proving the above theorem, Martin conjectured the following:

Conjecture 2.10 (Martin). AD and Bl-AD are equivalent.

This conjecture is still open to be true or false. We discuss this conjecture in § 3 and § 5.

It is a natural question whether one can introduce Blackwell determinacy when the base set X is infinite. It is possible exactly in the same way. But when $X = \omega$, a simple subset of ${}^{\omega}X$ already does not have a value as follows:

Proposition 2.11. Let $X = \omega$ and $A = \{x \in {}^{\omega}\omega \mid x(0) > x(1)\}$. Then A does not have a value.

The above game is namely about "Which player takes a larger natural number on their first moves?", which is a game of just one move for each player.

Proof of Proposition 2.11. We show that $\sup_\sigma \inf_\tau \mu^-_{\sigma,\tau}(A) = 0$ while $\inf_\tau \sup_\sigma \mu^+_{\sigma,\tau}(A) = 1$.

To prove $\inf_\tau \sup_\sigma \mu^+_{\sigma,\tau}(A) = 1$, let $\epsilon > 0$ and τ be any Blackwell strategy for player II. Then by the condition for a Blackwell strategy for player II, for all natural numbers m, n, $\tau(\langle m\rangle) = \tau(\langle n\rangle)$. Let us call it τ_0. Then since $\sum_{n\in\omega} \tau_0(n) = 1$, there is a natural number n_0 such that $\sum_{n\geq n_0} \tau_0(n) < \epsilon$. Let σ be any Blackwell strategy for player I such that $\sigma(\emptyset)(n_0) = 1$. Then by the definition of $\mu_{\sigma,\tau}$, $\mu_{\sigma,\tau}(A) < \epsilon$. Since ϵ and τ are arbitrary, $\inf_\tau \sup_\sigma \mu^+_{\sigma,\tau}(A) = 1$, as desired.

The argument for $\sup_\sigma \inf_\tau \mu^-_{\sigma,\tau}(A) = 0$ is similar. ■

Note that for the above proof, we used the additional condition for Blackwell strategies for player II essentially.

Instead of giving up generalizing Blackwell determinacy to arbitrary nonempty set X, we will look at another formulation of Blackwell games, which is with less imperfect information in some sense. As mentioned, the reason why the payoff set $A = \{x \in {}^{\omega}\omega \mid x(0) > x(1)\}$ does not have a value is that we demand player II to choose probabilities independent of the last positions she is at, i.e., $\tau(\langle m \rangle) = \tau(\langle n \rangle)$ for each m and n. The natural question is: What if we do not demand such additional condition to player II? In such a case, $\inf_\tau \sup_\sigma \mu^+_{\sigma,\tau}(A) = 0$ and A has a value.[3]

This observation lead Martin, Neeman, and Vervoort to introduce the following formulation of Blackwell determinacy and prove the following theorem:

Definition 2.12. Let X be a nonempty finite set. A *mixed strategy for player I* is a function $\sigma\colon X^{\mathrm{Even}} \to \mathrm{Prob}(X)$. A *mixed strategy for player II* is a function $\sigma\colon X^{\mathrm{Odd}} \to \mathrm{Prob}(X)$. Given mixed strategies σ, and τ for player I and II respectively, one can introduce the Borel probability $\mu_{\sigma,\tau}$ on ${}^{\omega}X$ in the same way as in Definition 2.6.

A subset A of ${}^{\omega}X$ *has a mixed value* if

$$\sup_\sigma \inf_\tau \mu^-_{\sigma,\tau}(A) = \inf_\tau \sup_\sigma \mu^+_{\sigma,\tau}(A),$$

where σ (τ) ranges over **mixed** strategies for I (II resp.).

Theorem 2.13 (Martin, Neeman, and Vervoort). Let $\mathbf{\Gamma}$ be a pointclass closed under continuous preimages.

1. Assume every set in $\mathbf{\Gamma}$ has a mixed value. Then for any A in $\mathbf{\Gamma}$, its mixed value is either 0 or 1 and there is a mixed strategy witnessing it, i.e., either there is a mixed strategy τ for player II such that for any mixed strategy σ for I, $\mu_{\sigma,\tau}(A) = 0$ or there is a mixed strategy σ for player I such that for any mixed strategy τ for player II, $\mu_{\sigma,\tau}(A) = 1$.

2. Every set in $\mathbf{\Gamma}$ has a value if and only if every set in $\mathbf{\Gamma}$ has a mixed value.

[3] Indeed, given σ, let τ be any such that $\tau(\langle n \rangle)(n+1) = 1$, i.e., II plays $n+1$ with probability 1 on her first move if I plays n. Then $\mu_{\sigma,\tau}(A) = 0$.

Proof. For the first item, see [17, Lemma 3.7 & 3.10].

For the second item, the direction from left to right is easy because one can simulate the latter formulation of Blackwell games with the former of them by looking at the function $\pi\colon {}^\omega X \to {}^\omega X$ with $\pi(x)(2n) = x(4n)$ and $\pi(x)(2n+1) = x(4n+3)$. For right to left, one should check that the arguments in the proof of Theorem 2.9 go through by using mixed strategies witnessing the mixed value to be 0 or 1 instead of using winning strategies. $\square$

Following the lead by Theorem 2.13, we now introduce Blackwell determinacy for arbitrary nonempty set X:

Definition 2.14. Let X be a nonempty set. We assume the axiom $\mathrm{DC}_{{}^\omega X}$[4]. A *mixed strategy for player I* is a function $\sigma\colon X^{\mathrm{Even}} \to \mathrm{Prob}_\omega(X)$, where $\mathrm{Prob}_\omega(X)$ is the set of functions $\mu\colon X \to [0,1]$ with $\sum_{x\in X}\mu(x) = 1$.[5] A *mixed strategy for player II* is a function $\tau\colon X^{\mathrm{Odd}} \to \mathrm{Prob}_\omega(X)$.

Given mixed strategies σ, τ for player I and II respectively, with the help of $\mathrm{DC}_{{}^\omega X}$, one can introduce the Borel probability $\mu_{\sigma,\tau}$ on ${}^\omega X$ in the same way as in Definition 2.6.

Let A be a subset of ${}^\omega X$. A mixed strategy σ for player I is *optimal in A* if for any mixed strategy τ for player II, A is $\mu_{\sigma,\tau}$-measurable and $\mu_{\sigma,\tau}(A) = 1$. A mixed strategy τ for player II is *optimal in A* if for any mixed strategy σ for player I, A is $\mu_{\sigma,\tau}$-measurable and $\mu_{\sigma,\tau}(A) = 0$. A set A is *Blackwell-determined* if one of the players has an optimal strategy in A. The axiom $\mathrm{Bl\text{-}AD}_X$ states that every subset of ${}^\omega X$ is Blackwell-determined.

Note that Bl-AD is equivalent to $\mathrm{Bl\text{-}AD}_2$ by the second item of Theorem 2.13 and that one can state $\mathrm{Bl\text{-}AD}_{\mathbb{R}}$ within $\mathrm{ZF}+\mathrm{DC}_{\mathbb{R}}$. The following is an analogue with Proposition 2.5:

Proposition 2.15. Let X, Y be nonempty sets and suppose that there is an injection from X to Y and assume $\mathrm{DC}_{{}^\omega Y}$. Then $\mathrm{Bl\text{-}AD}_Y$ implies $\mathrm{Bl\text{-}AD}_X$. In particular, $\mathrm{Bl\text{-}AD}_{\mathbb{R}}$ implies Bl-AD.

Proof. This is easy and routine. We omit the details. $\square$

[4] In other words, for any relation R on ${}^\omega X$ such that $(\forall x)\ (\exists y)\ (x,y) \in R$, there is a function $f\colon \omega \to {}^\omega X$ such that $(f(n), f(n+1)) \in R$ for every $n \in \omega$.

[5] We use $\mathrm{Prob}_\omega(X)$ to denote such functions because they are the same as Borel probabilities μ on X with countable support, i.e., there is a countable subset A of X with $\mu(A) = 1$.

As suspected, determinacy implies Blackwell determinacy even with the latter formulation:

Theorem 2.16 (Martin). Let X be a nonempty set and assume $\mathrm{DC}_{{}^{\omega}X}$. If there is a winning strategy for player I (resp., II) in a subset A of ${}^{\omega}X$, then there is an optimal strategy for player I (resp., II) in A. In particular, $\mathrm{AD}_{\mathbb{R}}$ implies that $\mathrm{Bl}\text{-}\mathrm{AD}_{\mathbb{R}}$.

Proof. Given a strategy σ for player I (resp., II), one can naturally translate σ into a mixed strategy $\hat{\sigma}$ for player I (resp., II) by setting $\hat{\sigma}(s)$ to be the Dirac measure concentrating on $\sigma(s)$, i.e., $\hat{\sigma}(s)\big(\sigma(s)\big) = 1$. It is easy to see that if σ is winning in A, then $\hat{\sigma}$ is optimal in A. ■

We discuss the relationship between $\mathrm{AD}_{\mathbb{R}}$ and $\mathrm{Bl}\text{-}\mathrm{AD}_{\mathbb{R}}$ in § 5 and § 6.

3 AD and Bl-AD

In the last section, we saw the definitions of Gale-Stewart games and Blackwell games and learned that determinacy implies Blackwell determinacy. Let us discuss the other implication: Does Blackwell determinacy imply determinacy? In fact, this is the case if we focus on finite games:

Definition 3.1. Let X be a nonempty set and A be a subset of ${}^{\omega}X$. The set A *describes a finite game* if there are a natural number $n < \omega$ and a subset B of X^n such that $A = \bigcup_{s \in B}[s]$.

Theorem 3.2 (Löwe). Let X be a nonempty set which is totally ordered and assume $\mathrm{DC}_{{}^{\omega}X}$. Let A be a subset of ${}^{\omega}X$ which describes a finite game. Then if A is Blackwell determined, then it is determined.

Proof. The idea is to choose a path (i.e., a run of the game) with positive probability without using the Axiom of Choice.[6] We need that X is totally ordered to avoid using the Axiom of Choice. Here are the details: Let $<_X$ be a total order on X and n, B witness that A describes a finite game. Assume σ is an optimal strategy for I in A. We show that there is a winning strategy for I in A. (The case player II having an optimal strategy in A can be treated in the same way.) We will define a strategy $\tilde{\sigma}$ for I and prove that it is winning for I in A.

[6]Indeed, the determinacy of closed sets follow from the Axiom of Choice easily.

Let us first look at the probability $\sigma(\emptyset)$ on X. Since $\sum_{x \in X} \sigma(\emptyset)(x) = 1$, there is a natural number $m > 0$ such that

$$\{x \in X \mid \sigma(\emptyset)(x) > 1/m\} \neq \emptyset.$$

Take the least such m and call it m_0. Then the set $C_0 = \{x \in X \mid \sigma(\emptyset)(x) > 1/m_0\}$ is a nonempty finite subset of X (the number of elements of C_0 is at most m_0). Since any total order on a finite set is well-ordered, C_0 is well-ordered by $<_X$. So take the $<_X$-least element x_0 of C_0 and define $\tilde{\sigma}(\emptyset) = x_0$. We have now specified the first move of $\tilde{\sigma}$.

We continue this process and define $\tilde{\sigma}$ in the same way: More precisely, for $s \in X^{\text{Even}}$, we define $\tilde{\sigma}(s)$ as follows: Let m be the least positive natural number such that $C = \{x \in X \mid \sigma(s)(x) > 1/m\} \neq \emptyset$. Then let $\tilde{\sigma}(s)$ be the $<_X$-least element x of C.

We show that $\tilde{\sigma}$ is a winning strategy for I in A. Let τ be any strategy for II. We prove that the run $\tilde{\sigma} * \tau$ of the game belongs to A. First, it is easy to observe that for any natural number $\ell \leq n$, $\mu_{\sigma,\hat{\tau}}\big([(\tilde{\sigma} * \tau) \restriction \ell]\big) > 0$ by induction on ℓ, where $\hat{\tau}$ is the derived mixed strategy for II from τ, i.e., $\hat{\tau}(s)\big(\tau(s)\big) = 1$ for any $s \in X^{\text{Odd}}$. In particular, $\mu_{\sigma,\hat{\tau}}\big([(\tilde{\sigma} * \tau) \restriction n]\big) > 0$. But since σ is optimal, this means that $(\tilde{\sigma} * \tau) \restriction n$ is in B, which implies that $\tilde{\sigma} * \tau \in A$, as desired. ■

If we look at the argument above carefully, one could notice that it works also for every clopen subset A of ${}^{\omega}X$: Suppose σ is an optimal strategy for I in A. Since A is closed, there is a tree T on X such that $A = [T]$, where $[T] = \{x \in {}^{\omega}X \mid (\forall n \in \omega)\ x \restriction n \in T\}$. Then if $\tilde{\sigma}$ and τ are as above, for any natural number n, $(\tilde{\sigma} * \tau) \restriction n \in T$ because $\mu_{\sigma,\hat{\tau}}\big([(\tilde{\sigma} * \tau) \restriction n]\big) > 0$ and σ is optimal for I in A. Hence $\tilde{\sigma} * \tau \in [T] = A$, as desired. We need to assume that A is also open because we need to do the same argument for the complement of A in case player II has an optimal strategy in A.

Moreover, if we look at this argument more carefully, we could notice that this argument is based on some kind of "continuity" of A, i.e., when player I (or II) deals with the conditions at any finite stage, she will win the game. This "continuity" is realized by the fact that A has a tree representation ($A = [T]$ in this case). One might wonder if one could arrange this kind of "continuity" for even more complicated sets and that is in fact possible: A set A of reals is *Suslin* if there are an ordinal γ and a tree T on $2 \times \gamma$ such that $A = \mathrm{p}[T]$, where $\mathrm{p}[T] = \{x \in \mathbb{R} \mid (\exists y \in {}^{\omega}\gamma)\ (x, y) \in [T]\}$. A set A of reals is *co-Suslin* if the complement of A is Suslin. Suslin sets are generalization of closed sets (consider $\gamma = 1$ for closed sets) and Suslin & co-Suslin sets are generalization of clopen sets.

Theorem 3.3 (Martin, Neeman, and Vervoort)**.** Assume Bl-AD. Then every Suslin & co-Suslin set of reals is determined.

Proof. See [17, Lemma 4.1]. □

By using Theorem 3.3, one can prove that much more complicated sets of reals are determined assuming Bl-AD: It is a basic fact that every $\mathbf{\Sigma}^1_2$ set of reals is Suslin. Hence every $\mathbf{\Delta}^1_2$ set of reals is Suslin & co-Suslin, hence determined by Theorem 3.3. Then by Moschovakis' Second Periodicity Theorem, every $\mathbf{\Sigma}^1_4$ set of reals is Suslin and so every $\mathbf{\Delta}^1_4$ set of reals is Suslin & co-Suslin, and again determined by Theorem 3.3 and so on... By looking at the scale analysis in $\mathrm{L}(\mathbb{R})$ by Steel, one can go up to the fact that every set of reals in $\mathrm{L}_{\delta^2_1}(\mathbb{R})$ is determined, where δ^2_1 is the supremum of ordinals which can be obtained by the length of $\mathbf{\Delta}^2_1$ prewellorderings on the reals. Since $\mathrm{L}_{\delta^2_1}(\mathbb{R}) \prec_{\Sigma_1} \mathrm{L}(\mathbb{R})$, if there is a non-determined set of reals in $\mathrm{L}(\mathbb{R})$, then there is one in $\mathrm{L}_{\delta^2_1}(\mathbb{R})$. Hence AD holds in $\mathrm{L}(\mathbb{R})$. This brief description is summarized by the following theorem:

Theorem 3.4 (Kechris and Woodin)**.** Assume every Suslin & co-Suslin set of reals is determined. Then AD holds in $\mathrm{L}(\mathbb{R})$.

Proof. See [14]. □

Corollary 3.5 (Martin, Neeman, and Vervoort)**.** In $\mathrm{L}(\mathbb{R})$, AD and Bl-AD are equivalent. Hence they are equiconsistent.

Proof. It follows from Theorem 3.3 and Theorem 3.4. □

Hence AD and Bl-AD are equiconsistent while the equivalence between them is still not known to be true or false as mentioned in § 2.

4 Consequences of Bl-AD$_{\mathbb{R}}$

In this section, we focus on what one can deduce from Bl-AD$_{\mathbb{R}}$. The following is an easy corollary obtained from Theorem 3.2:

Corollary 4.1 (Löwe). Assume Bl-AD$_{\mathbb{R}}$. Then every relation on the reals can be uniformized by a function with the same domain, i.e., $(\forall R \subseteq \mathbb{R} \times \mathbb{R})$ $(\exists f \subseteq R)$ such that f is a function and $\mathrm{dom}(f) = \mathrm{dom}(R)$.

Proof. Let R be a relation on the reals. Consider the following game with length 2: Player I first chooses a real x and player II replies by choosing a real y. Player II wins if $(x, y) \in R$ or x is not in the domain of R. This game be easily finitely described by a payoff set $A \subseteq {}^{\omega}\mathbb{R}$. By Bl-AD$_{\mathbb{R}}$ and Theorem 3.2, this game is determined. So either player I or II has a winning strategy.

We show that player II has a winning strategy by arguing that player I cannot have a winning strategy. Let σ be any strategy for I. If $\sigma(\emptyset)$ is not in the domain of R, II wins. If it is in the domain, there is a real y such that $(\sigma(\emptyset), y) \in R$. Let II play this y. Then II wins the game as well. So II can defeat the strategy σ in either case, as desired.

Let τ be a winning strategy for II in this game. Then the function $f\colon \mathrm{dom}(R) \to \mathbb{R}$ can be obtained by $f(x) = \tau(x)$ for each $x \in R$, which is the desired function uniformizing R. ■

In the last section, we saw that the determinacy of every Suslin & co-Suslin set follows from Bl-AD. One can ask what kind of other sets can be determined assuming Bl-AD$_{\mathbb{R}}$. Here is a useful example:

Definition 4.2. A subset A of ${}^{\omega}\mathbb{R}$ is *range-invariant* if for any $\vec{x}, \vec{y} \in {}^{\omega}\mathbb{R}$ with $\mathrm{ran}(\vec{x}) = \mathrm{ran}(\vec{y})$, $\vec{x} \in A$ if and only if $\vec{y} \in A$.

Theorem 4.3 (I.). Assume Bl-AD$_{\mathbb{R}}$. Then every range-invariant set is determined.

Proof. Let A be range-invariant and σ be an optimal strategy for player I in A. We show that there is a winning strategy for player I in A. (The case for player II is similar.) The following is the key-point: Let $\mathcal{P}_{\omega_1}(\mathbb{R})$ be the set of all countable sets of reals. A countable set a of reals is *closed under a function* $F\colon {}^{<\omega}\mathbb{R} \to \mathcal{P}_{\omega_1}(\mathbb{R})$ if for any finite sequence s of elements in a, $F(s) \subseteq a$.

Claim 4.4. Player I has a winning strategy in A if and only if there is a function $F\colon {}^{<\omega}\mathbb{R} \to \mathcal{P}_{\omega_1}(\mathbb{R})$ such that if a is closed under F, then any enumeration of a belongs to A.

Proof of Claim 4.4. The direction from left to right is easy: Given a winning strategy $\sigma'\colon \mathbb{R}^{\mathrm{Even}} \to \mathbb{R}$ for player I in A, let $\hat{\sigma}'\colon {}^{\omega}\mathbb{R} \to \mathbb{R}$ be any extension of σ' and F be such that $F(s) = \{\hat{\sigma}'(s)\}$. This F works for our purpose using the range-invariance of A.

For the direction from right to left, given such F, we uniformize the relation $\{(s, y) \in {}^{<\omega}\mathbb{R}\times\mathbb{R} \mid y \text{ codes } F(s)\}$ by a function g. This is possible by Corollary 4.1. Then by a standard book-keeping argument, one can arrange a strategy σ' for I in such a way that any run $\vec{x}$ of the game following σ' gives its domain to be closed under g, hence closed under F, so $\vec{x} \in A$ by the range-invariance of A, as desired.

For the details, see [11, Claim 3.2 & 3.3]. □ (Claim 4.4)

By Claim 4.4, it suffices to find an $F\colon {}^{<\omega}\mathbb{R} \to \mathcal{P}_{\omega_1}(\mathbb{R})$ such that if a is closed under F, then any enumeration of a belongs to A.

Let F be as follows:

$$F(s) = \begin{cases} \emptyset & \text{if } \mathrm{lh}(s) \text{ is odd,} \\ \{y \in \mathbb{R} \mid \sigma(s)(y) \neq 0\} & \text{otherwise.} \end{cases}$$

Then F is as desired: If a is closed under F, then enumerate a to be $\langle a_n \mid n \in \omega\rangle$ and let player I follow σ and let player II play the Dirac measure for a_n at her nth move. Then by the closure of a under F, the probability of the set $\{x \in {}^{\omega}\mathbb{R} \mid \mathrm{ran}(x) = a\}$ is 1 and since σ is optimal for player I in A, there is an x such that the range of x is a and x is in A. But by the range-invariance of A, any enumeration of a belongs to A. ■

The determinacy of range-invariant sets can be used to show the following theorem: Let X be a set and κ be an uncountable cardinal. Let $\mathcal{P}_{\omega_1}(\mathbb{R})$ be the set of all countable sets of reals. Let U be a set of subsets of $\mathcal{P}_{\omega_1}(\mathbb{R})$. We say that U is *σ-complete* if U is closed under countable intersections; we say it is *fine* if for any $x \in X$, $\{a \in \mathcal{P}_{\omega_1}(\mathbb{R}) \mid x \in a\} \in U$; we say that U is *normal* if for any family $\{A_x \in U \mid x \in \mathbb{R}\}$, the diagonal intersection $\triangle_{x\in\mathbb{R}} A_x$ is in U (where $\triangle_{x\in X} A_x = \{a \in \mathcal{P}_{\omega_1}(\mathbb{R}) \mid (\forall x \in a)\ a \in A_x\}$); we say that it is a *fine normal measure* if it is a non-principal, fine, normal, σ-complete ultrafilter.

Theorem 4.5 (de Kloet, Löwe, I.). Assume $\mathrm{Bl\text{-}AD}_{\mathbb{R}}$. Then there is a fine normal measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$.

Sketch of proof. We define a family $U \subseteq \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ as follows: Fix $A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$ and consider the following game $\tilde{G}_A$: Players alternately play reals;

say that they produce an infinite sequence $\vec{x} = (x_i \mid i \in \omega)$. Then player II wins the game $\tilde{G}_A$ if $\operatorname{ran}(\vec{x}) \in A$, otherwise player I wins. Since the payoff set of this game is range-invariant as a Gale-Stewart game, by Theorem 4.3, it is determined.

We say that $A \in U$ if and only if player II has a winning strategy in the game $\tilde{G}_A$. We show that U is a fine normal measure under the assumption of Bl-AD$_{\mathbb{R}}$. The determinacy of games of the form $\tilde{G}_A$ can be used to show that U is an ultrafilter. Except the normality, the argument is exactly the same as the one in [22, Lemma 3.1] and it is not difficult. So we will omit it. For the normality of U, one needs to use Bl-AD$_{\mathbb{R}}$. For the details, see Section 3 in [11]. □

Solovay proved the following interesting theorem:

Theorem 4.6 (Solovay). Assume there is a fine normal measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. Then $\mathbb{R}^{\#}$ exists.[7]

Sketch of proof. Fix a fine normal measure U on $\mathcal{P}_{\omega_1}(\mathbb{R})$. It is easy to construct a surjection $\pi \colon \mathcal{P}_{\omega_1}(\mathbb{R}) \to \omega_1$ such that $\pi^{-1}(\{\alpha\}) \notin U$.[8] Then from U, π derives a measure on ω_1 and hence ω_1 is measurable. From this, it follows that every real has a sharp and that every $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ has a sharp (because every a can be coded by a real).

We now define X as follows: For a sentence ϕ in the language for $\mathbb{R}^{\#}$,

$$\phi \in X \iff \{a \in \mathcal{P}_{\omega_1}(\mathbb{R}) \mid \phi \in a^{\#}\} \in U.$$

It is a routine to check that X satisfies the conditions for $\mathbb{R}^{\#}$. For the details, see [22, Lemma 4.1 & Theorem 4.4]. □

Corollary 4.7 (de Kloet, Löwe, I.).

1. Bl-AD$_{\mathbb{R}}$ implies that $\mathbb{R}^{\#}$ exists.

2. The consistency of Bl-AD$_{\mathbb{R}}$ is strictly stronger than that of AD.

Proof. The first item follows from Theorem 4.5 and Theorem 4.6.

For the second item, first note that Bl-AD$_{\mathbb{R}}$ implies $\mathrm{AD}^{\mathrm{L}(\mathbb{R})}$ by Proposition 2.15 and Corollary 3.5. Since we have $\mathbb{R}^{\#}$ by the first item of this

[7] For the definition of $\mathbb{R}^{\#}$, see [22].

[8] In fact, let $\rho \colon \mathbb{R} \to \omega_1$ be any surjection and set $\pi(a) = \sup_{x \in a} \rho(x)$.

theorem, by the property of $\mathbb{R}^{\#}$, one can construct a set-size elementary substructure of $L(\mathbb{R})$ and that structure knows that $AD^{L(\mathbb{R})}$ holds. Therefore, Bl-$AD_{\mathbb{R}}$ implies the consistency of AD and by Gödel's Second Incompleteness Theorem, the consistency of Bl-$AD_{\mathbb{R}}$ is strictly stronger than that of AD. ■

Next, we show that Bl-$AD_{\mathbb{R}}$ implies that every set of reals has almost all the known regularity properties and is ∞-Borel. We start with proving the perfect set property for every set of reals. Recall that a set of reals A has the *perfect set property* if either A is countable or A contains a perfect subset, where a perfect set of reals is a closed set without isolated points.

Theorem 4.8 (I.). Assume Bl-$AD_{\mathbb{R}}$. Then every set of reals has the perfect set property.

Proof. The theorem follows from the following two lemmas:

Lemma 4.9. Assume Bl-$AD_{\mathbb{R}}$. Then every relation on the reals can be uniformized by a Borel function modulo a Lebesgue null set, i.e., for any relation R on the reals, there is a Borel function f such that the set $\{x \mid (x, f(x)) \in R$ or there is no real y with $(x, y) \in R\}$ is of Lebesgue measure one.

Proof of Lemma 4.9. The conclusion follows by a folklore argument from Lebesgue measurability and uniformization for any relation on the reals both of which are consequences of Bl-$AD_{\mathbb{R}}$ by Theorem 2.7 and Corollary 4.1.

Let R be an arbitrary relation on the reals. We may assume the domain of R is the whole space, i.e., for any real x, there is a real y such that $(x, y) \in R$. We will find a Borel function uniformizing R almost everywhere.

By the uniformization principle, there is a function g uniformizing R. For each finite binary sequence s, the set $g^{-1}([s])$ is Lebesgue measurable by Theorem 2.7. Hence for each s there is a Borel set B_s such that $g^{-1}([s]) \triangle B_s$ is Lebesgue null. Now define f so that the following holds: For each finite binary sequence s,

$$f(x) \in [s] \iff x \in B_s.$$

Then by the property of B_s, f is defined almost everywhere, Borel, and is equal to g almost everywhere. Hence any Borel extension of f will be the one we desired. □ (Lemma 4.9)

Lemma 4.10 (Raisonnier and Stern). Suppose every relation on the reals can be uniformized by a Borel function modulo a Lebesgue null set. Then every set of reals has the perfect set property.

Proof of Lemma 4.10. See [21, Theorem 5]. □

□ (Theorem 4.8)

We now prove that every set of reals has the Baire property assuming Bl-$\mathrm{AD}_{\mathbb{R}}$. Let us first prepare some notions for the proof. We first introduce the Blackwell meager ideal as an analogue of the meager ideal. A set A of reals is *Blackwell meager* if player II has an optimal strategy in the Banach-Mazur game $G^{**}(A)$. Let I_{BM} denote the set of all Blackwell meager sets of reals.

Lemma 4.11 (I.). Assume Bl-AD. Then any meager set is in I_{BM}, $[s] \notin I_{\mathrm{BM}}$ for each finite binary sequence s, and I_{BM} is a σ-ideal. Moreover, every set of reals is measurable via I_{BM}, i.e., for any set A of reals and finite binary sequence s, there is a finite binary sequence t extending s such that either $[t] \cap A$ or $[t] \setminus A$ is in I_{BM}.[9]

Proof. See [11, Lemma 4.4]. Although it is not difficult, the argument comes with several codings and it is tedious. □

Next, we introduce the Stone space $\mathrm{St}(\mathbb{P})$ and $\mathbb{P}$-Baireness for a partial order $\mathbb{P}$. For a partial order $\mathbb{P}$, the *Stone space* of $\mathbb{P}$ (denoted by $\mathrm{St}(\mathbb{P})$) is the set of all ultrafilters on $\mathbb{P}$ equipped with the topology generated by $\{O_p \mid p \in \mathbb{P}\}$, where $O_p = \{u \in \mathrm{St}(\mathbb{P}) \mid u \ni p\}$. For example, if $\mathbb{P}$ is Cohen forcing $\mathbb{C}$, then $\mathrm{St}(\mathbb{C})$ is homeomorphic to the Baire space ${}^{\omega}\omega$.

Before defining $\mathbb{P}$-Baireness, let us see the connection between Baire measurable functions from $\mathrm{St}(\mathbb{P})$ to the reals and $\mathbb{P}$-names for reals. Let X, Y be topological spaces. Then a function $f\colon X \to Y$ is *Baire measurable* if for any open set U in Y, $f^{-1}(U)$ has the Baire property in X. Baire measurable functions are the same as continuous functions modulo meager sets: Let X, Y be topological spaces and assume Y is second countable. Then it is fairly easy to see that a function $f\colon X \to Y$ is Baire measurable if and only if there is a comeager set D in X such that $f \restriction D$ is continuous.

[9] This formulation of measurability is equivalent to the existence of an open set U such that the symmetric difference $A \triangle U$ is Blackwell meager.

There is a natural correspondence between Baire measurable functions from St($\mathbb{P}$) to the reals and $\mathbb{P}$-names for reals:

Lemma 4.12 (Feng, Magidor, and Woodin). Let $\mathbb{P}$ be a partial order.

1. If $f\colon \mathrm{St}(\mathbb{P}) \to {}^{\omega}\omega$ is a Baire measurable function, then

$$\tau_f = \{(m,n)\check{}, p) \mid O_p \setminus \{u \in \mathrm{St}(\mathbb{P}) \mid f(u)(m) = n\} \text{ is meager}\}$$

is a $\mathbb{P}$-name for a real.

2. Let τ be a $\mathbb{P}$-name for a real. Define f_τ as follows: For $u \in \mathrm{St}(\mathbb{P})$ and $m, n \in \omega$,

$$f_\tau(u)(m) = n \iff (\exists p \in u)\ p \Vdash \tau(\check{m}) = \check{n}.$$

Then the domain of f_τ is comeager in St($\mathbb{P}$) and f_τ is continuous on the domain. Hence it can be uniquely extended to a Baire measurable function from St($\mathbb{P}$) to the reals modulo meager sets.

3. If $f\colon \mathrm{St}(\mathbb{P}) \to {}^{\omega}\omega$ is a Baire measurable function, then f_{τ_f} and f agree on a comeager set in St($\mathbb{P}$). Also, if τ is a $\mathbb{P}$-name for a real, then $\Vdash \tau_{f_\tau} = \tau$.

Proof. See [6, Theorem 3.2]. □

We now define the property $\mathbb{P}$-Baireness. Let $\mathbb{P}$ be a partial order and A be a set of reals. Then A is *$\mathbb{P}$-Baire* if for any Baire measurable function $f\colon \mathrm{St}(\mathbb{P}) \to {}^{\omega}\omega$, $f^{-1}(A)$ has the Baire property in St($\mathbb{P}$). It is easy to see that every Borel set of reals is $\mathbb{P}$-Baire for any $\mathbb{P}$.

If $\mathbb{P}$ is an element of $\mathcal{H}_{\omega_1}$, then St($\mathbb{P}$) is essentially the same as St($\mathbb{C}$) where $\mathbb{C}$ is Cohen forcing, hence the Baire space ${}^{\omega}\omega$ by the following lemma:

Lemma 4.13. If $i\colon \mathbb{P} \to \mathbb{Q}$ is dense for partial orders $\mathbb{P}$ and $\mathbb{Q}$ in $\mathcal{H}_{\omega_1}$, then $St(\mathbb{P})$ and $St(\mathbb{Q})$ are isomorphic as Baire spaces, i.e., there is a topological homeomorphism between a comeager set in $St(\mathbb{P})$ and a comeager set in $St(\mathbb{Q})$. In particular, if every set of reals has the Baire property, then every set of reals is $\mathbb{P}$-Baire for any $\mathbb{P}$ with $\mathbb{P} \in \mathcal{H}_{\omega_1}$.

Proof. Since $\mathbb{P}$ and $\mathbb{Q}$ are in $\mathcal{H}_{\omega_1}$, so is i. Pick a real x coding $(\mathbb{P}, \mathbb{Q}, i)$ and take a countable transitive model M of ZF$-$P with $x \in M$, where P denotes the Power Set Axiom. (E.g., consider $\mathrm{L}_\alpha[x]$ for a suitable countable α.) Then i induces a natural bijection between $\mathbb{P}$-generic filters over M and $\mathbb{Q}$-generic filters over M. Since M is countable, the set of all $\mathbb{P}$-generic filters over M is comeager in St($\mathbb{P}$) and the same holds for $\mathbb{Q}$. This natural bijection witnesses the conclusion. ■

Since every meager set is Blackwell meager as we have seen in Lemma 4.11, if $\mathbb{P}$ is in $\mathcal{H}_{\omega_1}$, then one can define Blackwell meagerness for subsets of $\mathrm{St}(\mathbb{P})$ via an isomorphism between the Baire space and $\mathrm{St}(\mathbb{P})$ as Baire spaces and identify them as structures of topological spaces together with Blackwell meager ideals. From now on, we will use this identification without any notice.

We are now ready to prove the Baire property for every set of reals from $\mathrm{Bl}\text{-}\mathrm{AD}_{\mathbb{R}}$.

Theorem 4.14. Assume $\mathrm{Bl}\text{-}\mathrm{AD}_{\mathbb{R}}$. Then every set of reals has the Baire property.

Proof. Take any set A of reals. We show that A has the Baire property. Let $\mathcal{A}^2_A$ be the second-order arithmetic structure with A as a unary predicate. Since any relation on the reals can be uniformized by a function by Corollary 4.1, we can construct a Skolem function F for $\mathcal{A}^2_A$ and by a simple coding of finite sequences of reals and formulas via reals, we regard it as a function from the reals to themselves. Let Γ_F be the graph of F, i.e., $\Gamma_F = \{(x, s) \in \mathbb{R} \times {}^{<\omega}2 \mid F(x) \supseteq s\}$.

The following are the key objects for the proof (they are called *term relations*): Recall from Lemma 4.12 that for a $\mathbb{P}$-name τ for a real, f_τ is the Baire measurable function (which is continuous on a comeager set) corresponding to τ.

$$\begin{aligned}\tau_A = \{(\mathbb{P}, p, \sigma) \in \mathcal{H}_{\omega_1} \mid{}& \sigma \text{ is a } \mathbb{P}\text{-name for a real and} \\ & (\forall^\infty G \in \mathrm{St}(\mathbb{P}))\ p \in G \implies f_\sigma(G) \in A\},\end{aligned}$$

where $\big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big)$ means "for all G modulo a Blackwell meager set in $\mathrm{St}(\mathbb{P})\ldots$". One can define $\tau_{A^c}, \tau_{\Gamma_F}$, and $\tau_{(\Gamma_F)^c}$ in the same way.

Let $M = \mathrm{HOD}^{\mathrm{L}[X]}_X$ where $X = \big(\tau_A, \tau_{A^c}, \tau_{\Gamma_F}, \tau_{(\Gamma_F)^c}\big)$ and for $G \in \mathrm{St}(\mathbb{P})$, let $A_G = \{f_\sigma(G) \mid (\exists p \in G)\ (\mathbb{P}, p, \sigma) \in \tau_A \cap M\}$.

The following are the key claims:

Claim 4.15.

1. Let $\mathbb{P}$ be a partial order in M. Then $\big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big)$ $A_G = A \cap M[G] \in M[G]$ and $M[G]$ is closed under F.

2. Let $\mathbb{P} = \mathrm{Coll}(\omega, 2^\omega)^M$, where $\mathrm{Coll}(\omega, 2^\omega)$ is the forcing collapsing the cardinal 2^ω into countable with finite conditions. Then $\big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big)$ A_G has the Baire property in $M[G]$.

Idea of Claim 4.15. The intuition for the first item is that since M knows τ_A and τ_{T_F}, $M[G]$ knows A_G and is closed under F for Blackwell comeager many G.

The point for the second item is that G is $\mathbb{P}$-generic over M for a comeager many G[10] and that in $M[G]$, the set of Cohen reals over M is comeager while A_G is captured by M in some sense.

For the details, see [11, Claim 4.8]. □ (Claim 4.15)

We now finish the proof of Theorem 4.14 by showing that A has the Baire property. Let G be such that the conclusions of Claim 4.15 hold. By the first item of Claim 4.15, the structure $(\omega, {}^{\omega}\omega \cap M[G], \mathrm{app}, +, \cdot, =, 0, 1, A_G)$ is an elementary substructure of $\mathcal{A}^2_A$. Since the Baire property for A can be described in the structure $\mathcal{A}^2_A$ in this language and A_G has the Baire property in $M[G]$, A also has the Baire property, as desired. ■ (Theorem 4.14)

We now prove that every set of reals is ∞-Borel assuming Bl-AD$_{\mathbb{R}}$. Let us first introduce infinitary Borel codes and discuss their basic properties. Infinitary Borel codes (∞-Borel codes) are a transfinite generalization of Borel codes: Let $\mathcal{L}_{\infty,0}(\{\mathbf{a}_n\}_{n\in\omega})$ be the language allowing arbitrary many conjunctions and disjunctions and no quantifiers with atomic sentences $\mathbf{a}_n$ for each $n \in \omega$. The ∞-*Borel codes* are the sentences in $\mathcal{L}_{\infty,0}(\{\mathbf{a}_n\}_{n\in\omega})$ belonging to any Γ such that

- the atomic sentence $\mathbf{a}_n$ is in Γ for each $n \in \omega$,
- if ϕ is in Γ, then so is $\neg\phi$, and
- if α is an ordinal and $\langle \phi_\beta \mid \beta < \alpha \rangle$ is a sequence of sentences each of which is in Γ, then $\bigvee_{\beta<\alpha} \phi_\beta$ is also in Γ.

To each ∞-Borel code ϕ, we assign a set of reals B_ϕ in the same way as decoding Borel codes:

- if $\phi = \mathbf{a}_n$, then $B_\phi = \{x \in {}^{\omega}2 \mid x(n) = 1\}$,
- if $\phi = \neg\psi$, then $B_\phi = {}^{\omega}2 \setminus B_\psi$, and

[10]This is because $\mathcal{P}(\alpha) \cap M$ is countable for any ordinal α in M: Indeed, since M is a transitive model of ZFC, if $\mathcal{P}(\alpha) \cap M$ was uncountable, then there would be an uncountable sequence of distinct reals from which one can construct a set of reals without the perfect set property, contradicting Theorem 4.8.

- if $\phi = \bigvee_{\beta<\alpha} \psi_\beta$, then $B_\phi = \bigcup_{\beta<\alpha} B_{\psi_\beta}$.

A set of reals A is called ∞-*Borel* if there is an ∞-Borel code ϕ such that $A = B_\phi$.

As Borel codes, one can regard ∞-Borel codes as wellfounded trees with atomic sentences $\mathbf{a}_n$ on terminal nodes and decode them by assigning sets of reals on each node recursively from terminal nodes. (If a node has only one successor, then it means "negation" and if a node has more than one successors, then it means "disjunction".) The only difference between Borel codes and ∞-Borel codes is that trees are on ω for Borel codes while trees are on ordinals for ∞-Borel codes. From this visualization, it is easy to see that the statement "ϕ is an ∞-Borel code" is absolute between any transitive models of ZF.

Given an ∞-Borel code ϕ and a real x, the problem whether x is in B_ϕ can be easily translated into the following kind of satisfaction game using the above visualization of ∞-Borel codes via wellfounded trees: Let us regard ϕ as a wellfounded tree T_ϕ on ordinals with terminal nodes labeled by atomic sentences. In the game $G_c(T_\phi)$, there are two players, Spoiler and Duplicator, and a counter designating which player should move next.

We start with the top node (the empty sequence) with the counter designating Duplicator. If the node has only one successor, no player is supposed to decide anything and they move to the unique successor and exchange the name in the counter. (This is for the negation.) If the node has more than one successor, then the player designated by the counter chooses one of the successors and keeps the name of the counter. (This is for the disjunction.) If the node is a terminal node, then look at the atomic sentence labeled at the node, say $\mathbf{a}_n$. If the real x satisfies that $x(n) = 1$, then the player designated by the counter wins, otherwise the other player wins.

It is fairly easy to see that a real x is in B_ϕ if and only if Duplicator has a winning strategy in the game $G_c(T_\phi)$. By the fact that the payoff set of this game is a clopen subset of ${}^\omega\gamma$ for some ordinal γ, being a winning strategy in this game is absolute in any transitive model of ZF. Hence the statement "a real x is in B_ϕ" is absolute between transitive models of ZF.

The following characterization of ∞-Borel sets is very useful:

Fact 4.16 (Folklore). Let A be a set of reals. Then the following are equivalent:

1. A is ∞-Borel, and

2. there is a formula ϕ in the language of set theory and a set S of ordinals such that for each real x,

$$x \in A \iff \mathrm{L}[S, x] \models \phi(x).$$

Proof. See [23]. □

Standard examples of ∞-Borel sets are Suslin sets. Recall that a set of reals A is *Suslin* if there are an ordinal γ and a tree T on $2 \times \gamma$ such that $A = \mathrm{p}[T]$, where $\mathrm{p}[T]$ is the projection of $[T]$ to the first coordinate, i.e.,

$$\mathrm{p}[T] = \{x \in {}^{\omega}2 \mid (\exists f \in {}^{\omega}\gamma)\ (x, f) \in [T]\}.$$

By the above fact, every Suslin set is ∞-Borel. Assuming the Axiom of Choice, it is easy to see that every set of reals is Suslin, in particular ∞-Borel. Hence the property ∞-Borelness is trivial in the ZFC context while it is nontrivial and powerful in a determinacy world, as we will see later.

Theorem 4.17 (Woodin, I.). Assume Bl-AD$_{\mathbb{R}}$. Then every set of reals is ∞-Borel.

Idea of proof. [11] Let A be an arbitrary set of reals. We show that A is ∞-Borel. By Fact 4.16, it is enough to find a set of ordinals S and ϕ such that for each real x,

$$x \in A \iff \mathrm{L}[S, x] \models \phi(x).$$

The idea is that to each $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$, we assign a transitive model M_a of ZFC such that for each real x in a, $x \in A \iff M_a[x] \models \phi(x)$, where ϕ is uniform in a. By taking M_a to be the HOD of some structure and using a variant of Vopěnka algebra, one can make sense of $M_a[x]$ because any real is generic over HOD with respect to a Vopěnka algebra. Using a fine normal measure U on $\mathcal{P}_{\omega_1}(\mathbb{R})$ (whose existence was ensured by Theorem 4.5), we take the ultraproduct M_∞ of M_a $\big(a \in \mathcal{P}_{\omega_1}(\mathbb{R})\big)$. Then for any real x,

$$x \in A \iff M_\infty[x] \models \phi(x).$$

[11] For the details, see [11, Theorem 4.10].

This is enough to see the second item of Fact 4.16 for A since M_∞ is a transitive model of ZFC and its large fragment can be coded by a set of ordinals.[12] By Fact 4.16, we can conclude that A is ∞-Borel.

Here are more descriptions for M_a: Let τ'_A be the term relation as in the proof of Theorem 4.14 except that this time we use the meager ideal instead of Blackwell meager ideal. Let R_A be as follows:

$$\begin{aligned} R_A = \{(x,y) \mid {} & \text{if } x \text{ codes a } (\mathbb{P},p,\sigma) \in \tau_A, \text{ then } y \text{ codes a } (D_i \mid i < \omega) \\ & \text{such that } (\forall i)\ D_i \text{ is dense in } \mathbb{P} \text{ and} \\ & \big(\forall G \in \mathrm{St}(\mathbb{P})\big)\ \big(p \in G, (\forall i)\ G \cap D_i \neq \emptyset \implies f_\sigma(G) \in A\big)\}. \end{aligned}$$

Let F_A uniformize R_A and Γ_A be the graph of F_A, i.e., $\Gamma_A = \{(x,s) \mid s \in {}^{<\omega}2, F_A(x) \supseteq s\}$. One can define $\tau'_{A^c}, \tau'_{\Gamma_F}$, and $\tau'_{(\Gamma_F)^c}$ in the same way. Then for each $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$, M_a would be $\mathrm{HOD}_X^{\mathrm{L}_{\omega_1}[X](a)}$, where $X = (A, \tau_A, \tau_{\Gamma_A}, \tau_{A^c}, \tau_{\Gamma_{A^c}})$. The following is the key-point:

Claim 4.18 (Woodin). Let M be a transitive subset of $\mathcal{H}_{\omega_1}$ and $(M, \in, \tau_A, \tau_{\Gamma_A})$ is a model of ZFC.[13] Let $(\mathbb{P},p,\sigma) \in M \cap \tau_A$. Then for every $\mathbb{P}$-generic filter G over M, if p is in G, then $\sigma^G \in A$. The same holds for A^c.

Proof of Claim 4.18. See [11, Claim 4.12]. □ (Claim 4.18)

It is easy to check that M_a satisfies the condition for M in Claim 4.18. Then one can prove that for any a and x in a,

$$x \in A \iff M_a[x] \vDash b_a \in G_x,$$

where $b_a = \sup\{q \in \mathbb{Q}_a^* \mid (\mathbb{Q}_a^*, q, \dot{y}_G) \in \tau_A\}$ in M_a, $\mathbb{Q}_a^*$ is a variant of Vopěnka algebra, $\dot{y}_G$ is a name for a generic real, and G_x is the generic filter induced by x. This is what we desired. □ (Theorem 4.17)

Together with the non-existence of uncountable sequences of distinct reals, the strong ∞-Borelness for every set of reals gives us almost all the

[12]Here we used Łoś's theorem for M_a $(a \in \mathcal{P}_{\omega_1}(\mathbb{R}))$. It is not in general the case that Łoś's theorem holds in a choiceless world. This is possible in this case because we have a uniform way of defining M_a and its well-order.

[13]Here it satisfies Comprehension scheme and Replacement scheme for formulas in the language of set theory with predicates for τ_A and τ_{Γ_A}.

regularity properties. For that, we introduce $\mathbb{P}$-measurability for a wide class of tree-type forcings, which covers almost all the known regularity properties.

We start with defining a class of tree-type forcings we will work on. A partial order $\mathbb{P}$ is *arboreal* if its conditions are perfect trees on ω (or on 2) ordered by inclusion. But this class of forcings contains some trivial forcings such as $\mathbb{P} = \{{}^{<\omega}\omega\}$. We need the following stronger notion:

Definition 4.19. A partial order $\mathbb{P}$ is *strongly arboreal* if it is arboreal and the following holds:

$$(\forall T \in \mathbb{P})\ (\forall t \in T)\ T_t \in \mathbb{P},$$

where $T_t = \{s \in T \mid \text{either } s \subseteq t \text{ or } s \supseteq t\}$.

With strongly arboreal forcings, one can code generic objects by reals in the standard way: Let $\mathbb{P}$ be strongly arboreal and G be $\mathbb{P}$-generic over V. Let $x_G = \bigcup\{\mathrm{stem}(T) \mid T \in G\}$, where stem$(T)$ is the longest $t \in T$ such that $T_t = T$. Then x_G is a real and $G = \{T \in \mathbb{P} \mid x_G \in [T]\}$, where $[T]$ is the set of all infinite paths through T. Hence $V[x_G] = V[G]$. We call such real x_G a $\mathbb{P}$-*generic real over* V.

Almost all typical forcings related to regularity properties are strongly arboreal. For the details, see [10, Example 2.5].

We now introduce a σ-ideal $I_{\mathbb{P}}$ on the reals expressing "smallness" for each strongly arboreal forcing $\mathbb{P}$.

Definition 4.20. Let $\mathbb{P}$ be a strongly arboreal forcing. A set of reals A is $\mathbb{P}$-*null* if for any T in $\mathbb{P}$ there is a $T' \leq T$ such that $[T'] \cap A = \emptyset$. Let $N_{\mathbb{P}}$ denote the set of all $\mathbb{P}$-null sets and $I_{\mathbb{P}}$ denote the σ-ideal generated by $\mathbb{P}$-null sets, i.e., the set of all countable unions of $\mathbb{P}$-null sets.

Most typical σ-ideals related to regularity properties are the same as $I_{\mathbb{P}}$. For the details, see [10, Example 2.7].

We now introduce $\mathbb{P}$-measurability:

Definition 4.21. Let $\mathbb{P}$ be strongly arboreal. A set of reals A is $\mathbb{P}$-*measurable* if for any T in $\mathbb{P}$ there is a $T' \leq T$ such that either $[T'] \cap A \in I_{\mathbb{P}}$ or $[T'] \setminus A \in I_{\mathbb{P}}$.

Before proving regularity properties from Bl-AD$_{\mathbb{R}}$, let us introduce a variant of proper forcing. Let $\mathbb{P}$ be a partial order. We say $\mathbb{P}$ is *strongly proper* if for any countable transitive model M of a finite fragment of ZFC, if $\mathbb{P}$, $\leq_{\mathbb{P}}$, $\perp_{\mathbb{P}}$

are absolute between M and V respectively, (i.e., $P^M, \leq_{\mathbb{P}}^M, \perp_{\mathbb{P}}^M$ are the same as $P \cap M, \leq_{\mathbb{P}} \cap (M \times M), \perp_{\mathbb{P}} \cap (M \times M)$ respectively), then for any condition p in P^M (or $\mathbb{P} \cap M$), there is an $(M, \mathbb{P})$-generic condition q below p, i.e., if $M \models$ "A is a maximal antichain in $\mathbb{P}$", then $A \cap M$ is predense below q. Every strongly proper, projective forcing is proper and all the typical examples of proper, provably Δ^1_2 forcings are strongly proper.

Proposition 4.22. Assume that there is no uncountable sequence of distinct reals and every set of reals is ∞-Borel. Then every set of reals is $\mathbb{P}$-measurable for any strongly arboreal, strongly proper forcing $\mathbb{P}$.

Proof. The results for Cohen forcing, random forcing, and Mathias forcing are well-known. Let A be a set of reals and $\mathbb{P}$ be a strongly arboreal, strongly proper forcing. Take any condition T in $\mathbb{P}$. We show that there is an extension $T'' \leq T$ such that either $[T''] \cap A$ or $[T''] \setminus A$ is in $I_{\mathbb{P}}$.

Pick ∞-Borel codes S_1 and S_2 coding A and $\mathbb{P}$ with formulas ϕ and ψ, respectively. Then $\mathrm{L}[S_1, S_2, T]$ correctly computes $\mathbb{P}$ and $\mathbb{P}^{\mathrm{L}[S_1,S_2,T]}$ is strongly arboreal in $\mathrm{L}[S_1, S_2, T]$. Also, there is an extension $T' \leq T$ in $\mathbb{P}^{\mathrm{L}[S_1,S_2,T]}$ such that either $\mathrm{L}[S_1, S_2, T] \models$ "$T' \Vdash \mathrm{L}[S_1, \dot{x}_G] \models \phi(\dot{x}_G, \check{S}_1)$" or $\mathrm{L}[S_1, S_2, T] \models$ "$T' \Vdash \mathrm{L}[S_1, \dot{x}_G] \models \neg\phi(\dot{x}_G, \check{S}_1)$", where $\dot{x}_G$ is a canonical $\mathbb{P}$-name for a generic real. We may assume that $\mathrm{L}[S_1, S_2, T] \models$ "$T' \Vdash \mathrm{L}[S_1, \dot{x}_G] \models \phi(\dot{x}_G, \check{S}_1)$". (The other case is similar.)

Since there is no uncountable sequence of distinct reals, the set of all dense sets of $\mathbb{P}^{\mathrm{L}[S_1,S_2,T]}$ in $\mathrm{L}[S_1, S_2, T]$ is countable. Take any countable transitive model $M \subseteq \mathrm{L}[S_1, S_2, T]$ such that M contains all the reals and all the dense subsets of $\mathbb{P}^{\mathrm{L}[S_1,S_2,T]}$ in $\mathrm{L}[S_1, S_2, T]$. Since $\mathrm{L}[S_1, S_2, T]$ computes $\mathbb{P}$ correctly, M also computes $\mathbb{P}$ correctly.

Now we apply the strong properness of $\mathbb{P}$ and get an extension $T'' \leq T'$ such that T'' is $(M, \mathbb{P})$-generic condition and hence also $(\mathrm{L}[S_1, S_2, T], \mathbb{P})$-generic. Therefore maximal antichains in $\mathbb{P}^{\mathrm{L}[S_1,S_2,T]}$ stay maximal in V below T''. Together with the condition that the set of all dense sets in $\mathrm{L}[S_1, S_2, T]$ is countable, we can conclude that almost all the reals are $\mathbb{P}$-generic over $\mathrm{L}[S_1, S_2, T]$ below T'' in V. Since we have $\mathrm{L}[S_1, S_2, T] \models$ "$T' \Vdash \mathrm{L}[S_1, \dot{x}_G] \models \phi(\dot{x}_G, \check{S}_1)$", almost all the reals below T'' belong to A, as desired. ∎

Corollary 4.23. Assume Bl-AD$_{\mathbb{R}}$. Then every set of reals is $\mathbb{P}$-measurable for any strongly arboreal, strongly proper forcing $\mathbb{P}$.

5 Toward the equivalence between $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$

In this section, we discuss the following conjecture:

Conjecture 5.1. Assume DC.[14] $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$ are equivalent.

Since $\mathrm{AD}_{\mathbb{R}}$ implies Bl-$\mathrm{AD}_{\mathbb{R}}$ by Theorem 2.16, the question is whether Bl-$\mathrm{AD}_{\mathbb{R}}$ implies $\mathrm{AD}_{\mathbb{R}}$ in ZF+DC. Woodin proved the following:

Theorem 5.2 (Woodin)**.** Assume AD and DC. Then the following are equivalent:

1. every set of reals is Suslin,

2. the axiom $\mathrm{AD}_{\mathbb{R}}$ holds, and

3. every relation on the reals can be uniformized.

Hence, to prove Conjecture 5.1, it suffices to show that every set of reals is Suslin from Bl-$\mathrm{AD}_{\mathbb{R}}$: If every set of reals is Suslin, then by Theorem 3.3, AD holds. Now by Theorem 5.2 and Corollary 4.1, $\mathrm{AD}_{\mathbb{R}}$ holds assuming Bl-$\mathrm{AD}_{\mathbb{R}}$ and DC.

It is important to note that Martin's conjecture (i.e., that AD and Bl-AD are equivalent) implies Conjecture 5.1 by Theorem 5.2. Hence proving Conjecture 5.1 would be a partial answer to the question whether AD and Bl-AD are equivalent.

We try to mimic the arguments for the implication from uniformization to Suslinness in Theorem 5.2 and reduce Conjecture 5.1 to a small conjecture. For the implication, we need the following three steps:

Step

1. Prove that every set of reals is strongly ∞-Borel.

[14] Here DC stands for the *Axiom of Dependent Choice*, i.e., for any nonempty set X and a relation R on X with $(\forall x)$ $(\exists y)$ $(x, y) \in R$, there exists a function $f\colon \omega \to X$ such that $(f(n), f(n+1)) \in R$ for every natural number n.

2. Prove the statement so-called "Becker's bound", i.e., let A be a subset of $\mathbb{R}^3$ and assume $\exists^{\mathbb{R}} A$ is a strict well-founded relation on a set of reals where $\exists^{\mathbb{R}} A = \{(x, y) \mid (\exists z)\ (x, y, z) \in A\}$. Suppose A has a strong ∞-Borel code S and let γ be an ordinal less than Θ such that the tree of S is on γ. Then the length of $\exists^{\mathbb{R}} A$ is less than γ^+, the least cardinal above γ.

3. Prove that every set of reals is Suslin using Becker's bound.

We will achieve the first item & the second item from Bl-AD$_{\mathbb{R}}$ and make a small conjecture toward the third item. For the rest of this section, we assume Bl-AD$_{\mathbb{R}}$ & DC, and fix a fine normal measure U on $\mathcal{P}_{\omega_1}(\mathbb{R})$, which exists by Theorem 4.5.

We start with the definition of strong ∞-Borel codes. An ∞-Borel code S is *strong* if the tree of S is a tree on γ for some $\gamma < \Theta$ and for any $f\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ and surjection $\pi\colon \mathbb{R} \to \gamma$, there is an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that a is closed under f, $S\restriction(\pi``a)$ is an $\infty$-Borel code, and $B_{S\restriction(\pi``a)} \subseteq B_S$. A set of reals A is *strong ∞-Borel* if $A = B_S$ for some strong ∞-Borel code S.

Intuitively, an ∞-Borel code S is strong if and only if for stationary many $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$, a behaves nicely to S.

There is a finer version of Fact 4.16 as follows:

Fact 5.3.

1. Let S be a strong ∞-Borel code and $\gamma < \Theta$ be such that S is a tree on β for some $\beta < \gamma$ and $\mathrm{L}_\gamma[S, x] \models$ "KP + Σ_1-Separation" for any real x. Let $\phi(S, x)$ be a Σ_1-formula expressing "$x \in B_S$". Then for any function $f\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ and surjection $\pi\colon \mathbb{R} \to \gamma$, there is an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that a is closed under f and for any real x, if $\mathrm{L}_{\bar\gamma}[\bar S, x] \models \phi(\bar S, x)$, then $\mathrm{L}_\gamma[S, x] \models \phi(S, x)$, where $\mathrm{L}_{\bar\gamma}[\bar S]$ is the transitive collapse of the Skolem hull of $(\pi``a) \cup \{S\}$ in $\mathrm{L}_\gamma[S]$.

2. Let γ be an ordinal with $\gamma < \Theta$, ϕ be a Σ_1-formula, and S be a bounded subset of γ such that $\mathrm{L}_\gamma[S, x] \models$ "KP + Σ_1-Separation" for any real x. Set $A = \{x \in \mathbb{R} \mid \mathrm{L}_\gamma[S, x] \models \phi(S, x)\}$. Assume that for any function $f\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ and surjection $\pi\colon \mathbb{R} \to \gamma$, there is an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that a is closed under f and for any real x, if $\mathrm{L}_{\bar\gamma}[\bar S, x] \models \phi(\bar S, x)$, then $\mathrm{L}_\gamma[S, x] \models \phi(S, x)$, where $\mathrm{L}_{\bar\gamma}[\bar S]$ is the transitive collapse of the Skolem hull of $(\pi``a) \cup \{S\}$ in $\mathrm{L}_\gamma[S]$. Then A is strong ∞-Borel.

Proof. This can be done by closely looking at the argument for Fact 4.16 in [23]. □

We now proceed Step 1:

Theorem 5.4 (Woodin, I.). Assume Bl-AD$_{\mathbb{R}}$ and DC. Then every set of reals is strong ∞-Borel.

Idea of proof. [15]

Let A be a set of reals. We show that A is strongly ∞-Borel. For $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$, let $(M_a, \mathbb{Q}^*_a, b_a)$ be as in the proof of Theorem 4.17. Since each M_a is a transitive model of ZFC, one can code an essential part of $(M_a, \mathbb{Q}^*_a, b_a)$ by a bounded subset S_a of a countable ordinal α_a such that $\mathrm{L}_{\alpha_a}[S_a, x]$ is a model of KP + Σ_1-Separation for any real x in a. Let $S_\infty = \prod_a S_a, \alpha_\infty = \prod_a \alpha_a$ via U, where U was the fine normal measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ we fixed at the beginning of this section. By the argument in the proof of Theorem 4.17, we have that there is a formula ϕ such that for any a and x,

$$x \in A \iff \mathrm{L}_{\alpha_a}[S_a, x] \models \phi\ (S_a, x) \tag{2}$$
$$x \in A \iff \mathrm{L}_{\alpha_\infty}[S_\infty, x] \models \phi\ (S_\infty, x). \tag{3}$$

The first point is as follows:

Claim 5.5. $\alpha_\infty < \Theta$.

Proof. If follows from $j(\omega_1) = \Theta$, where $j\colon V \to \mathrm{Ult}(V, U)$ is the ultrapower embedding because each α_a is countable. For the details, see [11, Lemma 5.3].

□ (Claim 5.5)

By Fact 5.3, it is enough to show that $\alpha_\infty < \Theta$ and the following: For any function $f\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ and surjection $\pi\colon \mathbb{R} \to \alpha_\infty$, there is an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that a is closed under f and for any real x, if $\mathrm{L}_{\bar{\alpha_\infty}}[\bar{S_\infty}, x] \models \phi(\bar{S_\infty}, x)$, then $\mathrm{L}_{\alpha_\infty}[S_\infty, x] \models \phi(S_\infty, x)$, where $\mathrm{L}_{\bar{\alpha_\infty}}[\bar{S_\infty}]$ is the transitive collapse of the Skolem hull of $(\pi``a) \cup \{S_\infty\}$ in $\mathrm{L}_{\alpha_\infty}[S_\infty]$.

Let us fix $f\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ and $\pi\colon \mathbb{R} \to \alpha_\infty$. By the formulas (2) and (3) above, the following claim will complete the proof:

Claim 5.6. There are a and b in $\mathcal{P}_{\omega_1}(\mathbb{R})$ such that a is closed under f and $(X_a, \in)$ is isomorphic to $(\mathrm{L}_{\alpha_b}[S_b], \in)$, where X_a is the Skolem hull of $(\pi``a) \cup \{S_\infty\}$ in $\mathrm{L}_{\alpha_\infty}[S_\infty]$.

[15] For the details, see [11, Theorem 5.7]

Idea for proof of Claim 5.6. Let Γ_f be the graph of f, i.e., $\Gamma_f = \{(x,s) \in \mathbb{R} \times {}^{<\omega}2 \mid f(x) \supseteq s\}$. For each b, consider the following game $\hat{G}_b$ in $\mathrm{L}[S_b, S_\infty, \Gamma_f, \pi]$: In ω rounds,

1. Player I and II produce a countable elementary substructure X of $\mathrm{L}_{\alpha_b}[S_b]$,
2. Player II produces an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ which is closed under f, and
3. Player II tries to construct an isomorphism between $(X, \in)$ and $(X_a, \in)$, where X_a is the Skolem hull of $(\pi``a) \cup \{S_\infty\}$ in $\mathrm{L}_{\alpha_\infty}[S_\infty]$.

Player II wins if she succeeds to construct an isomorphism between $(X, \in)$ and $(X_a, \in)$. This is an open game on $T \times \mathbb{R}$ for some wellorderable set T. Hence by $\mathrm{DC}_{\mathbb{R}}$, it is determined.

The main point is as follows:

Subclaim 5.7. There is a $b \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that player II has a winning strategy in the game $\hat{G}_b$.

Proof. See [11, Subclaim 5.9] □ (Subclaim 5.7)

Hence there is a $b \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that player II has a winning strategy τ in the game $\hat{G}_b$ in $\mathrm{L}[S_b, S_\infty, \Gamma_f, \pi]$. Since the game is open, τ is also winning in V. Since $\mathrm{L}_{\alpha_b}[S_b]$ is countable in V, we can let player I move in such a way that $X = \mathrm{L}_{\alpha_b}[S_b]$ and let player II follow τ. Since τ is winning in V, there is an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that a is closed under f and $\mathrm{L}_{\alpha_b}[S_b] = X$ is isomorphic to X_a, as desired. □ (Claim 5.6)

□ (Theorem 5.4)

We now proceed Step 2: For a natural number n with $n \geq 1$ and a subset A of $\mathbb{R}^{n+1}$, $\exists^{\mathbb{R}} A = \{x \in \mathbb{R}^n \mid (\exists y \in \mathbb{R})\ (x,y) \in A\}$.

Theorem 5.8. Assume Bl-$\mathrm{AD}_{\mathbb{R}}$ and DC. Let A be a subset of $\mathbb{R}^3$ and assume $\exists^{\mathbb{R}} A$ is a strict well-founded relation on a set of reals. Suppose A has a strong ∞-Borel code S and let γ be an ordinal less than Θ such that the tree of S is on γ. Then the length of $\exists^{\mathbb{R}} A$ is less than γ^+.

Sketch of proof. Let A, S, and γ be as in the assumptions. We show that the length of $\exists^{\mathbb{R}} A$ is less than γ^+. Fix a surjection $\pi \colon \mathbb{R} \to \gamma$. Let us start with the following lemma As mentioned, if S is strong, then for stationary $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$, a behaves nicely. With the help of range-invariant determinacy, one can prove that the set of such as contains a club:

Lemma 5.9. There is a function $f \colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ such that if a is closed under f, then $S\restriction(\pi``a)$ is an $\infty$-Borel code and $B_{S\restriction(\pi``a)} \subseteq B_S$.

Proof of Lemma 5.9. Let us consider the following game: Player I and II choose reals one by one and produce an ω-sequence $\vec{x}$ of reals. Setting $a = \mathrm{ran}(\vec{x})$, player I wins if $S\restriction(\pi``a)$ is an $\infty$-Borel code and $B_{S\restriction(\pi``a)} \subseteq B_S$. Since S is a strong ∞-Borel code, player I can defeat any strategy for player II because strategies can be seen as functions from ${}^{<\omega}\mathbb{R}$ to $\mathbb{R}$ as in the proof of Claim 4.4. Since the payoff set of this game is range-invariant, by Theorem 4.3, this game is determined. Hence player I has a winning strategy and by Claim 4.4, there is a function f as desired. $\square$ (Lemma 5.9)

We fix an f_0 satisfying the conclusion of Lemma 5.9 for the rest of this proof. Here is the key-point:

Lemma 5.10. Let G be $\mathrm{Coll}(\omega, \gamma)$-generic over V. Then in $V[G]$, there is an elementary embedding $j \colon \mathrm{L}(\mathbb{R}, S, f_0, \pi) \to \mathrm{L}(j(\mathbb{R}), j(S), j(f_0), j(\pi))$ such that $j(\mathbb{R}) = \mathbb{R}^{V[G]}$.

Idea for proof of Lemma 5.10. [16] We briefly discuss the construction of j. Given a surjection $\pi \colon \mathbb{R} \to \gamma$ and a fine normal measure U on $\mathcal{P}_{\omega_1}(\mathbb{R})$, let U_π be the fine normal measure on $\mathcal{P}_{\omega_1}(\gamma)$ induced from π and U.

A subset A of ${}^{\omega}\gamma$ is *weakly meager* if there is an $X \in U_\pi$ such that $(\forall b \in X)\, {}^{\omega}b \cap B$ is meager in the space ${}^{\omega}b$.[17] Let I be the set of all weakly meager subsets of ${}^{\omega}\gamma$.

Let $i \colon \mathrm{Coll}(\omega, \gamma) \to {}^{\omega}\gamma$ be as follows: $i(s) = [s]/I$. Then one can prove that i is a dense embedding. Let G be a $\mathrm{Coll}(\omega, \gamma)$-generic filter over V. Then one can induce a V-ultrafilter U' on ${}^{\omega}\gamma$ extending the complements of weakly meager sets.

[16] For the details, see [11, Lemma 5.12].

[17] Note that if A is meager in the topological space ${}^{\omega}\gamma$, then it is weakly meager. The converse is not true in general.

We take the generic ultrapower $\left({}^{({}^{\omega}\gamma)^V}\mathrm{L}(\mathbb{R},S,f_0,\pi)\cap V\right)/U'$ and let j be the ultrapower map.[18] One can prove that Łoś's Theorem holds for this ultrapower and one can also check the desired conditions for j: One of the issues is the well-foundedness of the target model. This can be dealt with by observing that $j\restriction \mathrm{Ord} = j_{U_\pi}\restriction \mathrm{Ord}$, where U_π is the fine normal measure on $\mathcal{P}_{\omega_1}(\gamma)$ in V induced by U and π. Since the latter ultrapower is taken in V, its target is well-founded. Hence the target of j is also well-founded.

We omit the rest of the arguments. □ (Lemma 5.10)

We now finish the proof of Theorem 5.8. Let $M = \mathrm{L}\big(j(\mathbb{R}), j(S), j(f_0), j(\pi)\big)$. We first claim that S and j"S are in M. Since γ is countable in $V[G]$, there is a real x coding S in $V[G]$. But by Lemma 5.10, such an x is in M. Hence S is also in M. Since γ is countable in $V[G]$, there is an $a\in\mathcal{P}_{\omega_1}(\mathbb{R})$ such that π"$a = S$ and hence $j(\pi)$"$a = j$"S in $V[G]$. But since $j(\pi)\in M$ and $a\in M$ by Lemma 5.10, j"$S = j(\pi)$"a is also in M, as desired.

By Lemma 5.9 and elementarity of j, the following is true in M: For any a closed under $j(f)$, $j(S)\restriction a$ is an ∞-Borel code and $B_{j(S)\restriction a}\subseteq B_{j(S)}$. Also, by elementarity of j, $\exists^{\mathbb{R}}B_{j(S)}$ is a well-founded relation on a set of reals in M.

Set $a = j$"S. Since a is closed under $j(f)$, in M, $j(S)\restriction a$ is an ∞-Borel code, $B_{j(S)\restriction a}\subseteq B_{j(S)}$, and $\exists^{\mathbb{R}}B_{j\text{``}S}$ is also a wellfounded relation on a set of reals in $M$. Since $j$"$S$ is countable in $M$, the relation $\exists^{\mathbb{R}}B_{j\text{``}S}$ is $\boldsymbol{\Sigma}^1_1$ and hence by Kunen-Martin Theorem (see [18, 2G.2]), its rank is less than ω_1 in M which is the same as γ^+ in V.

Finally, since S and j"S are equivalent as Borel codes, $\exists^{\mathbb{R}}B_S$ has length less than ω_1 in M and since M has more reals than V, $\left(\exists^{\mathbb{R}}B_S\right)^V\subseteq\left(\exists^{\mathbb{R}}B_S\right)^M$. Therefore, the length of $\left(\exists^{\mathbb{R}}B_S\right)^V$ is less than $\omega_1^M = (\gamma^+)^V$, as desired.

□ (Theorem 5.8)

We now discuss Step 3. Becker proved the following:

Theorem 5.11 (Becker). Assume AD, DC, and the uniformization for every relation on the reals. Suppose that the conclusion of Theorem 5.8 holds, i.e., let A be a subset of $\mathbb{R}^3$ and assume $\exists^{\mathbb{R}}A$ is a well-founded relation on a set of reals. Suppose A has a strong ∞-Borel code S and let γ be an ordinal less than Θ such that the tree of S is on γ. Then the length of $\exists^{\mathbb{R}}A$ is less than γ^+. Then every set of reals is Suslin.

[18] When we take the ultrapower, we consider all the $\mathrm{L}(\mathbb{R},S,f_0,\pi)$-valued functions in V which are not necessarily in $\mathrm{L}(\mathbb{R},S,f_0,\pi)$.

Proof. See [2]. □

We try to simulate Becker's argument, make a small conjecture, and reduce Conjecture 5.1 to the small conjecture.

For the rest of this section, we assume Bl-AD$_{\mathbb{R}}$ and DC. We fix a set of reals A and give a scenario to prove that A is Suslin. We fix a simple surjection ρ from the reals to $\{0,1\}$, e.g., $x \mapsto x(0)$.

Claim 5.12. There is a sequence $\big((\Gamma_n, <_n, \gamma_n,) \mid n < \omega\big)$ such that for all n,

1. Γ_n is a Spector pointclass closed under $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$, $\Gamma_n \subsetneq \Gamma_{n+1}$, and $A \in \Gamma_0$,

2. every relation on the reals which is projective in a set in $\boldsymbol{\Gamma}_n$ can be uniformized by a function in $\boldsymbol{\Gamma}_{n+1}$,

3. $<_n$ is in $\boldsymbol{\Gamma}_n$ and a strict wellfounded relation on the reals with length γ_n and every set of reals which is projective in a set in $\boldsymbol{\Gamma}_n$ has a strong ∞-Borel code whose tree is on γ_{n+1}.

Proof. One can construct such a sequence inductively assuming that every set of reals is strong ∞-Borel, that every relation on the reals can be uniformized, and some weak version of Wadge's Lemma (see Lemma 6.4 in the next section) ensured by Bl-AD. For the details, see [11, Claim 5.22]. □

We fix $\big((\Gamma_n, <_n, \gamma_n) \mid n < \omega\big)$ as above and let $\Gamma^{\mathrm{I}}_n = \Gamma_{2n}, \Gamma^{\mathrm{II}}_n = \Gamma_{2n+1}, <^{\mathrm{I}}_n$ be induced by ρ, $<^{\mathrm{II}}_n = <_{2n+1}$, $\gamma^{\mathrm{I}}_n = \omega$ and $\gamma^{\mathrm{II}}_n = \gamma_{2n+1}$, Let $\rho^{\mathrm{I}}_n = \rho$ and ρ^{II}_n be the surjection between the reals onto ${}^n\gamma_{2n+1}$ induced by $<_{2n+1}$.

Let π^{II}_n be the function $a \mapsto \rho^{\mathrm{II}}_n[G^n_a]$ where G^n is a universal set for $\boldsymbol{\Gamma}^{\mathrm{II}}_n$ sets of reals (we do not use π^{I}_n). Then by some weak Moschovakis' Lemma[19], π^{II}_n is a surjection from the reals onto ${}^n\gamma^{\mathrm{II}}_n$.

Consider the following game $\hat{G}_A$: Player I plays 0 or 1 and player II plays reals one by one in turn and they produce a real z and a sequence $t \in {}^\omega\mathbb{R}$, respectively. Setting $T_n = \pi^{\mathrm{II}}_n\big(t(n)\big)$, player II wins if for all $n < m$, $T_{n+1}\restriction n \subseteq T_n$, $T_{n+1}\restriction n = T_m\restriction n$, and $z \in A \iff \bigcup_{n\in\omega} T_{n+1}\restriction n$ is illfounded, where $T_m\restriction n = \{s\restriction n \mid s \in T_m\}$. This is an integer-real game in the sense player I chooses integers and player II chooses reals.

[19]For the details of such weak Moschovakis' Lemma, see [11, Theorem 5.18].

We introduce an integer-integer game $\tilde{G}_A$ simulating the game $\hat{G}_A$. In the game $\tilde{G}_A$, players choose pairs of 0 or 1 one by one and produce a pair of reals (x_0, y_0) and (a_0, b_0) in ω rounds respectively.

From (x_0, y_0) and (a_0, b_0), we "decode" a real z and an ω-sequence of reals t respectively as follows: For each pointclass Γ above, we fix a set U^{Γ} universal for relations in $\mathbf{\Gamma}$. Setting $F_0 = U^{\Gamma_0^{\mathrm{I}}}_{x_0}$, F_0 is a function from the reals to perfect sets of reals (or codes of perfect sets) (otherwise player I loses). Let $P_{x_0} = F(x_0)$. Then y_0 is an element of P_{x_0} (otherwise player I loses) and is identified with a triple (u_0, x_1, y_1) of reals by looking at a canonical homeomorphism between P_{x_0} and $\mathbb{R}^3$.

Then setting $F_1 = U^{\Gamma_1^{\mathrm{I}}}_{x_1}$, F_1 is a function from the reals to perfect trees on 2 (or codes of trees) (otherwise player I loses). Let $P_{x_1} = F(x_1)$. Then y_1 is an element of P_{x_1} (otherwise player I loses) and is identified with a triple (u_1, x_2, y_2) of reals by looking at a canonical homeomorphism between P_{x_1} and $\mathbb{R}^3$.

Continuing this process, one can unwrap (x_n, y_n) and obtain (u_n, x_{n+1}, y_{n+1}) for each n and get an ω-sequence $(u_n \mid n < \omega)$. Let $z_0(n) = \rho(u_n)$. In the same way, one can obtain an ω-sequence $(t_n \mid n < \omega)$ of reals from (a_0, b_0). Setting $T_n = \pi_n^{\mathrm{II}}(t(n))$, player II wins if for all $n < m$, $T_{n+1}\restriction n \subseteq T_n$, $T_{n+1}\restriction n = T_m\restriction n$, and $z \in A \iff \bigcup_{n\in\omega} T_{n+1}\restriction n$ is illfounded.

Becker proved the following:

Lemma 5.13.

1. If player I has a winning strategy in the game $\tilde{G}_A$, then player I has a winning strategy σ in the game $\hat{G}_A$ such that σ is a countable union of sets in $\mathbf{\Gamma}_n^{\mathrm{II}}$ for some n as a set of reals.

2. If player II has a winning strategy in the game $\tilde{G}_A$, then player II has a winning strategy in the game $\hat{G}_A$.

Proof. See [2, Lemma A & B]. □

We show and conjecture the following: Let $B \subseteq {}^{\omega}\mathbb{R}$. A mixed strategy σ for player I is *weakly optimal in B* if for any $s \in \mathbb{R}^{\mathrm{Even}}$, the set $\{x \mid \sigma(s)(x) \neq 0\}$ is finite and for any ω-sequence y of reals, $\mu_{\sigma,\tau_y}(B) > 1/2$. One can introduce the weak optimality for mixed strategies for player II in the same way. Note that if player I has an optimal strategy in some payoff set, then player I has a weakly optimal strategy in the same payoff set. The same holds for player II.

Lemma 5.14. If player I has an optimal strategy in the game $\tilde{G}_A$, then player I has a weakly optimal strategy σ in the game $\hat{G}_A$ such that σ is a countable union of sets in $\mathbf{\Gamma}^{\mathrm{II}}_n$ for some n as a set of reals.

Proof. See [11, Lemma 5.24]. $\square$

Conjecture 5.15. If player II has an optimal strategy in the game $\tilde{G}_A$, then player II has a weakly optimal strategy in the game $\hat{G}_A$.

From Lemma 5.14 together with Theorem 5.8, one can conclude the following:

Lemma 5.16. There is no optimal strategy for player I in the game $\tilde{G}_A$.

Proof. To derive a contradiction, suppose player I has an optimal strategy in the game $\tilde{G}_A$. Then by Lemma 5.14, player I has a weakly optimal strategy σ in the game $\hat{G}_A$ such that σ is in a countable union of sets in $\mathbf{\Gamma}^{\mathrm{I}}_n$ for some n as a set of reals.

Consider the following set:

$$X = \{(t,s) \in {}^{\omega}\mathbb{R} \times {}^{<\omega}\mathbb{R} \mid \mu_{\sigma,\tau_t}\big(\{(z,t') \mid t' = t \text{ and } z \in A\}\big) > 1/2 \text{ and } \\ (\forall i < \mathrm{s})\ \big(|s(0)|_{<^{\mathrm{II}}_0}, \dots, |s(i)|_{<^{\mathrm{II}}_i}\big) \in T_{i+1}\restriction i\},$$

where $|s(i)|_{<^{\mathrm{II}}_i}$ is the rank of $s(i)$ with respect to the wellfounded relation $<^{\mathrm{II}}_i$ and $T_i = \rho^{\mathrm{II}}_i\big(t(i)\big)$. For (t,s) and (t',s') in X, $(t,s) < (t',s')$ if t and t' code the same tree T and s codes a node in T extending a node coded by s'. Note that for any (t,s) in X, if T is the tree coded by t, T is wellfounded because σ is weakly optimal in the game $\hat{G}_A$. Hence $(X,<)$ is a strict wellfounded relation on X. Let $\gamma_\omega = \sup\{\gamma^{\mathrm{II}}_n \mid n \in \omega\}$. By DC, the cofinality of Θ is greater than ω. Hence $\gamma_\omega < \Theta$. Note that for any ordinal $\alpha < \gamma^+_\omega$, there is a wellfounded tree T coded by some real t as in the definition of X such that the length of T is α. Hence the length of $(X,<)$ is γ^+_ω.

Since σ is a countable union of sets in $\mathbf{\Gamma}^{\mathrm{II}}_n$ for some n as a set of reals, the set $<$ on X is in $\exists^{\mathbb{R}} \bigwedge^{\omega} \bigvee^{\omega} \bigcup_{n\in\omega} \mathbf{\Gamma}^{\mathrm{II}}_n$, i.e., it is a projection of a countable intersection of countable unions of sets in $\mathbf{\Gamma}^{\mathrm{II}}_n$ for some n. Since every set in $\mathbf{\Gamma}^{\mathrm{II}}_n$ has a strong ∞-Borel code whose tree is on γ^{II}_n for every n, every set in $\bigwedge^{\omega} \bigvee^{\omega} \bigcup_{n\in\omega} \mathbf{\Gamma}^{\mathrm{II}}_n$ has a strong ∞-Borel code whose tree is on γ^+_ω. By Theorem 5.8, the length of $<$ on X must be less than γ^+_ω, which is not possible because it was equal to γ^+_ω. Contradiction! $\blacksquare$

We close this section by proving that Conjecture 5.15 implies Conjecture 5.1.

Theorem 5.17. Conjecture 5.15 implies Conjecture 5.1.

Proof. By Lemma 5.16, player I does not have an optimal strategy in the game $\tilde{G}_A$. Hence by Bl-AD, player II has an optimal strategy in the game $\tilde{G}_A$. By Conjecture 5.15, player II has a weakly optimal strategy τ in the game $\tilde{G}_A$. Note that τ can be seen as a real because each measure on the reals given by τ is with finite support by the weak optimality of τ.

For each finite binary sequence s with length n, let $t_s = \{u \in {}^n\mathbb{R} \mid (\forall i < n)\ \tau\Big(\big((s\restriction i) * (u\restriction(i-1))\big)\Big)\big(s(i)\big) \neq 0\}$, where $(s\restriction i) * \big(u\restriction(i-1)\big)$ is the concatenation of $s\restriction i$ and $u\restriction(i-1)$ bit by bit. For each finite binary sequence s, we identify t_s with a set of n-tuples of natural numbers via a map π_s by using the isomorphisms between $(a, <_{\mathbb{R}})$ and $(n, \in)$ for a finite set of reals a and a natural number, where $<_{\mathbb{R}}$ is a standard total order on the reals.

For any real x, $t_x = \bigcup_{n\in\omega} t_{x\restriction n}$ is a tree on natural numbers and $(\pi_s \mid s \in {}^{<\omega}\omega)$ induces a homeomorphism π_x between $[t_x]$ and $[\{t' \in {}^{<\omega}\mathbb{R} \mid \mu_{\sigma_x,\tau}([t']) \neq 0\}]$. Consider the following tree:

$$T = \{(s,t,u) \in \bigcup_{n\in\omega}({}^n2 \times {}^n\omega \times {}^n\gamma_\omega) \mid\ t \in \pi_s(t_s) \text{ and } \big(\forall i < \mathrm{lh}(s)\big)\ u(i) = |x_i|_{<^{\mathrm{II}}_i}\},$$

where x_i is the $t(i)$th real of the set of successors of $(x_j \mid j < i)$ in $t_s\restriction i$. Then by the weak optimality of τ, the following holds: Setting $B = \{(x,y) \in \mathbb{R} \times {}^\omega\omega \mid (\exists f \in {}^\omega\gamma_\omega)\ (x,y,f) \in [T]\}$, for any real x,

$$\begin{aligned} x \in A &\iff \mu_{\sigma_x,\tau}\big(\pi_x[B_x]\big) > 1/2 \\ &\iff (\exists T' : \text{ a tree on } 2)\ [T'] \subseteq B_x \text{ and } \mu_{\sigma_x,\tau}\big(\pi_x\big[[T']\big]\big) > 1/2. \end{aligned}$$

Since B is Suslin, the set $\{(x,T') \mid [T'] \subseteq B_x\}$ is also Suslin. Hence A is Suslin, as desired.

We have shown that every set of reals is Suslin. Then by Theorem 3.3, AD holds. Now by Theorem 5.2 and Theorem 4.1, $\mathrm{AD}_{\mathbb{R}}$ holds. ■

6 Toward the equiconsistency between $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$

In the last section, we discussed the possibility of the equivalence between $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$ under ZF+DC. Solovay proved the following:

Theorem 6.1 (Solovay). If we have $\mathrm{AD}_{\mathbb{R}}$ and DC, then we can prove the consistency of $\mathrm{AD}_{\mathbb{R}}$. Hence the consistency of $\mathrm{AD}_{\mathbb{R}}$+DC is strictly stronger than that of $\mathrm{AD}_{\mathbb{R}}$.

Proof. See [22]. □

Hence assuming DC to see the equivalence between $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$ is not optimal. One can ask whether they are equivalent without DC. So far we do not have any scenario to answer this question. Instead, one could ask the equiconsistency between $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$. In this section, we discuss the following conjecture:

Conjecture 6.2. $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$ are equiconsistent.

Woodin conjectured the following:

Conjecture 6.3 (Woodin). Assume the following:

1. every Suslin & co-Suslin set of reals is determined, and
2. there is a fine normal measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$.

Then either there is an inner model of $\mathrm{AD}_{\mathbb{R}}$ or there is an inner model M of AD^+ such that M contains all the reals and $\Theta^M = \Theta^V$.

Here AD^+ is conjunction of the statements that $\mathrm{DC}_{\mathbb{R}}$ holds, that every set of reals is ∞-Borel, and that Ordinal Determinacy holds, i.e., for any ordinal $\gamma < \Theta$, a continuous function $\pi\colon {}^{\omega}\gamma \to \mathbb{R}$, and a set of reals A, $\pi^{-1}(A)$ is determined.

We show that Conjecture 6.3 implies Conjecture 6.2. For that, we need a weak Wadge's Lemma: Let A be a set of reals. For a natural number $n \geq 1$, a set of reals B is $\boldsymbol{\Sigma}^1_n$ *in* A if B is definable by a Σ^1_n formula in the structure $\mathcal{A}^2_A$ that is the second order structure with A as an unary predicate with a parameter x for some real x.

Lemma 6.4 (Weak version of Wadge's Lemma). Assume Bl-AD. Then for any two sets of reals A and B, either A is $\boldsymbol{\Sigma}^1_2$ in B or B is $\boldsymbol{\Sigma}^1_2$ in A.

Proof. We consider the Wadge game $G_{\mathrm{W}}(A, B)$. By Bl-AD, one of the players has an optimal strategy in $G_{\mathrm{W}}(A, B)$. Assume player II has an optimal strategy τ in $G_{\mathrm{W}}(A, B)$. Then for any real x,

$$x \in A \iff \mu_{\sigma_x,\tau}\big(\{(x', y) \mid x' = x \text{ and } y \in B\}\big) = 1.$$

It is easy to see that the right hand side of the equivalence is Σ^1_2 in B. If player I has an optimal strategy in $G_{\mathrm{W}}(A, B)$, then one can prove that B is Σ^1_2 in A^c in the same way and hence B is Σ^1_2 in A. ■

Theorem 6.5. Suppose Conjecture 6.3 holds. Then Conjecture 6.2 holds, i.e., $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$ are equiconsistent.

Proof. First note that the assumptions in Conjecture 6.3 hold if we assume Bl-$\mathrm{AD}_{\mathbb{R}}$. Hence by Conjecture 6.3, there is an inner model of $\mathrm{AD}_{\mathbb{R}}$ or there is an inner model M of AD^+ such that M contains all the reals and $\Theta^M = \Theta^V$. If there is an inner model of $\mathrm{AD}_{\mathbb{R}}$, then we are done. Hence we assume that there is an inner model M of AD^+ such that M contains all the reals and $\Theta^M = \Theta^V$.

We show that $\mathrm{AD}_{\mathbb{R}}$ holds in V. First we claim that M contains all the sets of reals. Suppose not. Then there is a set of reals A which is not in M. Then by Lemma 6.4, every set of reals in M is $\Sigma^1_2(A)$. Then Θ^M must be less than Θ^V because one can code all the prewellorderings by reals using A in V, which contradicts the condition of M. Hence every set of reals is in M. Since we have uniformization for every relation on the reals in V, it is also true in M. We use the following fact:

Fact 6.6. Assume AD^+. Then the following are equivalent:

1. the axiom $\mathrm{AD}_{\mathbb{R}}$ holds, and

2. every relation on the reals can be uniformized.

Proof. See [23]. □ (Fact 6.6)

By Fact 6.6, since every relation on the reals can be uniformized in M, M satisfies $\mathrm{AD}_{\mathbb{R}}$. Since $\mathcal{P}(\mathbb{R}) \cap M = \mathcal{P}(\mathbb{R})$, $\mathrm{AD}_{\mathbb{R}}$ holds in V, as desired. ■ (Theorem 6.5)

At the end, we briefly discuss the connection between Conjecture 6.3 and the recent technique **Core Model Induction** in inner model theory.

Core Model Induction is a powerful technique to establish consistency of large cardinals from that of combinatorial principles. In core model theory, one usually first assumes some "smallness" of V with respect to having inner models with large cardinals and using the properties of K and its closeness to V, one derives a contradiction from some combinatorial principle, thus establishing the consistency of some targeted large cardinals.

In Core Model Induction, one repeats this process by induction on the complexity of sets of reals coding sharp operations for large cardinals (such as $x \mapsto M_n^{\#}(x)$), mouse operations, or nice iteration strategies for a countable mouse. In this process, one proves that more and more complex sets of reals are determined, produces larger models of AD^+ more and more, and translates those models of AD^+ to mice with large cardinals.

Sargsyan proved the following striking theorem describing the connection between strong generic embeddings and models of $\mathrm{AD}_{\mathbb{R}}$:

Theorem 6.7 (Sargsyan). Assume CH and that there is a generic embedding $j\colon V \to M$ such that

1. M is transitive and ${}^{\omega}M \cap V[G] \subseteq M$,
2. G is a generic filter of a homogeneous forcing, and
3. $j \restriction \mathrm{Ord}$ is definable in V.

Then there is a model of $\mathrm{ZF}+\mathrm{AD}_{\mathbb{R}}+$"$\Theta$ is regular"

By the result of Woodin, it is also known that the converse is true, thus the above two statements are equiconsistent.

The reason why Conjecture 6.3 or the equiconsistency of $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$ is plausible to achieve, is that one can produce similar generic embeddings to the one in Theorem 6.7:

Proposition 6.8. Assume Bl-$\mathrm{AD}_{\mathbb{R}}$. Then for any $\alpha < \Theta$ and $A \subseteq \mathbb{R}$, there is a generic embedding $j\colon \mathrm{L}(A, \mathbb{R}) \to M$ such that

1. M is transitive, $\mathbb{R}^{V[G]} \subseteq M$, and α is countable in M,
2. G is a generic filter of a homogeneous forcing, and
3. $j \restriction \mathrm{Ord}$ is definable in V.

Core Model Induction seems very useful to establish consistency of large cardinals from several principles in set theory such as the existence of ideals on ω_2 or $\mathcal{P}_\kappa(\lambda)$ with strong properties and one could expect to have new interesting results in this direction.

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