THE EASTON COLLAPSE AND A SATURATED FILTER

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Abstract. Suppose that there is a huge cardinal. We prove that a two-stage iteration of Easton collapses produces a saturated filter on the successor of a regular cardinal.

1. Introduction

In the pioneering work [10] Kunen established:

Theorem (Kunen). Suppose that $\kappa$ is huge with target $\lambda$. Then in some forcing extension $\kappa = \omega_1$, $\lambda = \omega_2$ and $\omega_1$ carries an $\omega_2$-saturated filter.

Kunen's forcing has the form $P*\dot{S}(\kappa, \lambda)$, where $P$ forces that $\kappa = \omega_1$ and $\dot{S}(\kappa, \lambda)$ is the Silver collapse introduced in [15]. The poset $P$ is constructed by recursion so that $P*\dot{S}(\kappa, \lambda)$ can be completely embedded into $j(P)$, where $j : V \rightarrow M$ is the original huge embedding. Kunen's construction has since been modified to get models containing filters that are strongly saturated in various senses. We refer the reader to [5] for a comprehensive survey of the development.

In [7] Foreman, Magidor and Shelah proved the following striking result: If $\lambda$ is supercompact, then the Levy collapse $C(\omega_1, \lambda)$ forces that $(\lambda = \omega_2$ and) $\omega_1$ carries a saturated filter. The hypothesis was later reduced by Todorčević (see [2]) to $\lambda$ being Woodin, which follows from Kunen's hypothesis as well. In contrast Foreman and Magidor [6] showed that $C(\omega_2, \lambda)$ forces the nonexistence of a saturated filter on $\omega_2$ under PFA.

Let us assume again $\kappa$ is huge with target $\lambda$. Todorčević's result implies that a saturated filter on $\omega_1$ can be forced to exist by the iteration $C(\omega_1, \kappa) * \dot{C}(\kappa, \lambda)$ as well. What about $\omega_2$? Namely we ask:

Question. Does $C(\omega_1, \kappa) * \dot{C}(\kappa, \lambda)$ force that $\omega_2$ carries an $\omega_3$-saturated filter?

One motivation for the question comes from the following unpublished result of Woodin: $C(\omega_1, \kappa) * \dot{C}(\kappa, \lambda)$ forces that an $\omega_2$-dense filter on $\omega_2$ exists in some inner model. (See [5] for an exposition in the case of $\omega_1$.) Moreover if the answer is positive, then we would get saturated filters on many cardinals by simply iterating Levy collapses. This would in turn help to simplify Foreman's construction [3, 4] of a model in which every regular uncountable cardinal carries a saturated filter.

In this paper we define a poset $E(\mu, \kappa)$ for a pair of regular cardinals $\mu < \kappa$, and call it the Easton collapse. It is the product of standard collapsing posets with Easton support, and forces $\kappa = \mu^+$ if $\kappa$ is Mahlo. In place of the original question, we answer the corresponding question for the iteration of Easton collapses:

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Theorem. Suppose that $\kappa$ is huge with target $\lambda$. Let $\mu < \kappa$ be regular. Then $E(\mu, \kappa) \ast E(\kappa, \lambda)$ forces that $\kappa$ carries a $\lambda$-saturated filter.

In §4 we prove our theorem in somewhat refined form.

2. PRELIMINARIES

We refer the reader to [9] for background material.
Throughout the paper we use $\mu, \kappa$ and $\lambda$ to denote a regular cardinal. Unless otherwise stated it is understood that $\mu < \kappa < \lambda$.

Let $P$ and $Q$ be posets. We say that a map $\pi : P \to Q$ is a projection if the following hold:

(1) $\pi$ is order-preserving, i.e. $p' \leq_P p \to \pi(p') \leq_Q \pi(p)$,
(2) $\pi(1_P) = 1_Q$ and
(3) $q \leq_Q \pi(p) \to \exists p^* \leq_P p(\pi(p^*) \leq_Q q)$.

Suppose that $\pi : P \to Q$ is a projection. Then ran $\pi$ is dense in $Q$. It is straightforward to check that the map $q \mapsto \sum \{p \in P : \pi(p) \leq q\}$ is a complete embedding of $Q$ into $B(P)$, the completion of $P$. It is also easy to see that if $D$ is dense open in $Q$, $\pi^{-1}(D)$ is dense in $P$. So if $G \subseteq P$ is generic, $\pi^*G$ generates a generic filter over $Q$. Let $H \subseteq Q$ be $V$-generic. In $V[H]$ let $P/H$ be the set $\pi^{-1}(H)$ ordered by $\leq_P$. It is straightforward to check that the map $p \mapsto (\pi(p), \hat{p})$, where $\hat{p}$ is a $Q$-name with $\pi(p) \Vdash Q \hat{p} = p$, is a dense embedding of $P$ into $Q \ast (P/H)$. Finally note that the composition of two projections is a projection.

We say that a cardinal $\gamma$ is strongly regular if $\gamma^{<\gamma} = \gamma$. A set $d$ of strongly regular cardinals is called Easton if $\text{sup}(d \cap \gamma) < \gamma$ for all regular $\gamma$.

Suppose that $X$ be a set of ordinals and $P_\gamma$ is a poset for $\gamma \in X$. Define

$$E \prod_{\gamma \in X} P_\gamma = \{p : \text{dom } p \subseteq X \text{ is Easton } \land \forall \gamma \in \text{dom } p(p(\gamma) \in P_\gamma)\}.$$ 

$E \prod_{\gamma \in X} P_\gamma$ is ordered coordinatewise: $p' \leq p$ iff $\text{dom } p' \supset \text{dom } p$ and $p'(\gamma) \leq_\gamma p(\gamma)$ for all $\gamma \in \text{dom } p$.

Let $Y \subseteq X$. Then $E \prod_{\gamma \in X} P_\gamma$ is canonically isomorphic to $E \prod_{\gamma \in Y} P_\gamma \times E \prod_{\gamma \notin X-Y} P_\gamma$.

Suppose in addition $\pi_\gamma : P_\gamma \to Q_\gamma$ is a projection for $\gamma \in Y$. Then it is easy to see that the map $p \mapsto (\pi_\gamma(p(\gamma)) : \gamma \in \text{dom } p \cap Y)$ is a projection from $E \prod_{\gamma \in X} P_\gamma$ to $E \prod_{\gamma \in Y} Q_\gamma$.

We say that $P$ has $(\kappa, \kappa, \mu)$-cc if for every $X \in [P]^\kappa$ there is $Y \in [X]^\kappa$ such that every $Z \in [Y]^\mu$ has a common extension. Needless to say, $(\kappa, \kappa, \mu)$-cc implies $\kappa$-cc. If $Q$ is separative and can be completely embedded into $P$, then the $(\kappa, \kappa, \mu)$-cc of $P$ implies that of $Q$.

Lemma 1. Suppose that $\kappa$ is Mahlo and $P_\gamma$ is a poset of size $< \kappa$ for $\gamma < \kappa$. Then $E \prod_{\mu \leq \gamma < \kappa} P_\gamma$ has $(\kappa, \kappa, \mu)$-cc.

Proof. Let $\{p_\xi : \xi < \kappa\} \subseteq E \prod_{\mu \leq \gamma < \kappa} P_\gamma$. It suffices to find $X \in [\kappa]^\kappa$ and $\delta < \kappa$ such that $\text{dom } p_\xi - \delta$ is mutually disjoint and $p_\xi|\delta$ is constant for $\xi \in X$.

Since $\text{dom } p_\xi$ is Easton, $\text{sup}(\text{dom } p_\xi \cap \xi) < \xi$ for all regular $\xi < \kappa$. Since $\kappa$ is Mahlo, we get a stationary $S \subseteq \kappa$ and $\delta < \kappa$ such that $\text{dom } p_\xi \cap \xi \subseteq \delta$ for all $\xi \in S$. 


Since dom $p_\xi$ is bounded in $\kappa$, $C = \{ \zeta < \kappa : \forall \zeta < \zeta (\text{dom } p_\xi \subset C) \}$ is club. Note that if $\xi < \zeta$ are both from $S \cap C$, we have $\text{dom } p_\xi \cap \text{dom } p_\zeta = \text{dom } p_\xi \cap \zeta \cap \text{dom } p_\zeta \subset \delta$. Since $|\prod_{\mu \leq \gamma \leq \delta} F_\gamma| < \kappa$, there is $X \in [S \cap C]^\kappa$ such that $p_\xi|\delta$ is constant for $\xi \in X$, as desired. \qed

For $\gamma \geq \mu$ we equip the set $<^\mu \gamma$ with reverse inclusion. Needless to say, $<^\mu \gamma$ is $\mu$-closed and forces $|\gamma| = \mu$. Let us sketch a proof of

**Lemma 2.** If $\gamma <^\kappa \gamma = \gamma$, then $<^\mu \gamma$ is isomorphic to a dense subset of $<^\mu \kappa \times <^\kappa \gamma$.

**Proof.** Define

$$D = \{ (q, r) \in <^\mu \kappa \times <^\kappa \gamma : \sup \{ \beta + 1 : \beta \in \text{ran } q \} = \text{dom } r \}.$$  

It is easy to see that $D$ is dense in $<^\mu \kappa \times <^\kappa \gamma$. The following three facts should suffice to construct an isomorphism between $<^\mu \gamma$ and $D$ by recursion.

First if $(\theta, 0) \in D$. Second each $(q, r) \in D$ has $\gamma$ immediate extensions in $D$. Third if $(\langle q_\alpha, r_\alpha \rangle : \alpha < \delta)$ is a descending sequence in $D$ with $\delta < \mu$, then we have $(\bigcup_{\alpha < \delta} q_\alpha, \bigcup_{\alpha < \delta} r_\alpha) \in D$. \qed

**Corollary 3.** If $\gamma \geq \kappa$ is strongly regular, there is a projection from $<^\mu \gamma$ to $<^\kappa \gamma$.

Let $F$ be a filter on a set. We denote by $F^+$ the set of $F$-positive sets ordered by: $X' \leq X$ iff $\exists C \in F(X' \cap C \subset X)$. Then $F^+$ is a separative poset. We say that $F$ is $(\kappa, \kappa, \mu)$-saturated if $F^+$ has $(\kappa, \kappa, \mu)$-cc.

### 3. THE EASTON COLLAPSE

In this section we define the Easton collapse $E(\mu, \kappa)$ and prove its basic properties.

For a set $X$ of ordinals define

$$E(\mu, X) = \prod_{\mu \leq \gamma \leq X} <^\mu \gamma.$$  

It is easy to see that $E(\mu, X)$ is $\mu$-directed closed and forces $|\gamma| \leq \mu$ for all strongly regular $\gamma \in X$. $E(\mu, \kappa)$ is a subset of $V_\kappa$, hence has size $\kappa$ if $\kappa$ is inaccessible.

If $\kappa$ is Mahlo, then $E(\mu, \kappa)$ has $\kappa$-$cc$ by Lemma 1, and hence forces $\kappa = \mu^+$. If $\mu < \kappa \leq \nu < \lambda$ are all regular, Corollary 3 provides a projection from $E(\mu, \lambda - \kappa) = \prod_{\kappa \leq \gamma \leq \lambda} <^\mu \gamma$ to $\prod_{\mu \leq \gamma \leq \lambda} <^\mu \gamma = E(\nu, \lambda)$.

Here is the main result of this section:

**Lemma 4.** Suppose that $P$ has $\kappa$-$cc$ and size $\leq \kappa$. Then there is a projection $\pi : P \times E(\kappa, \lambda) \to P \ast E(\kappa, \lambda)$ such that $\pi(\mu, \rho)$ has the form $(\mu, \rho)$, where

- $\forces \mu \| \rho \| \hat{\gamma} = \text{dom } q$ and
- each $\hat{\rho}(\gamma)$ depends only on $q(\gamma)$, i.e. if in addition $\pi(p', q') = (p', \hat{q}')$ and $q(\gamma) = q'(\gamma)$, then $\forces \rho \hat{\gamma} = \hat{q}'(\gamma)$.

**Proof.** Since $P$ has $\kappa$-$cc$ and size $\leq \kappa$, forcing with $P$ does not change the class of (strongly) regular cardinals $\geq \kappa$. If $\gamma \geq \kappa$ is regular and $\forces \dot{\alpha} < \gamma$, then there is $\beta < \gamma$ with $\forces \dot{\alpha} < \beta$. If $\gamma \geq \kappa$ is strongly regular, there exist exactly $\gamma$ representatives from the $P$-names $\dot{\alpha}$ such that $\forces \dot{\alpha} < \gamma$. Thus we can take $P$-names $\dot{\tau}(\xi)$ so that for every strongly regular $\gamma \geq \kappa$

- if $\xi < \gamma$, then $\forces \dot{\tau}(\xi) < \gamma$ and
\begin{itemize}
  \item if \( \models \alpha < \gamma \), then there is \( \xi < \gamma \) with \( \models \alpha = \tau(\xi) \).
\end{itemize}

For \( (p, q) \in P \times E(\kappa, \lambda) \) define
\[ \pi(p, q) = (p, q), \]
where \( q \) is a \( P \)-name such that
\begin{itemize}
  \item \( \models dom \dot{q} = dom q \) and
  \item \( \models \dot{q}(\gamma) = \langle \dot{\tau}(q(\gamma)(\eta)) : \eta \in dom q(\gamma) \rangle \) for every \( \gamma \in dom q \).
\end{itemize}

Since \( P \) has \( \kappa \)-cc, \( dom q \) remains an Easton subset of \( \lambda - \kappa \) after forcing with \( P \).
Moreover \( \models \dot{q}(\gamma)(\eta) < \gamma \) by \( q(\gamma)(\eta) < \gamma \) and the choice of \( \tau(\xi) \). Thus \( \pi(p, q) \in P \ast \dot{E}(\kappa, \lambda) \).

\textbf{Claim.} \( \pi \) is a projection.

\textbf{Proof.} It is easy to see that \( \pi \) is order-preserving and \( \pi(1_{P}, \emptyset) = (1_{P}, \emptyset) \).

Now assume \( (p, q) \in P \times E(\kappa, \lambda) \) and \( (p', q') \leq (p, q) \) in \( P \ast \dot{E}(\kappa, \lambda) \). Let \( (p, \dot{q}) = \pi(p, q) \).
Define
\[ p^{*} = p'. \]

Then \( p^{*} \leq p \) by \( (p', q') \leq (p, \dot{q}) \). It remains to find \( q^{*} \leq q \) in \( E(\kappa, \lambda) \) such that \( \pi(p^{*}, q^{*}) \leq (p', q') \) in \( P \ast \dot{E}(\kappa, \lambda) \). Define
\[ d^{*} = \{ \gamma : \exists r \in P(r \models \gamma \in dom \dot{q}') \}. \]

Since \( P \) has \( \kappa \)-cc and \( \models \dot{q}' \in \dot{E}(\kappa, \lambda) \), \( d^{*} \) is an Easton subset of \( \lambda - \kappa \). Moreover \( dom q \subset d^{*} \) because
\[ p' \models dom \dot{q} = dom q \subset dom \dot{q}' \subset d^{*}. \]

The left equality follows from the definition of \( \dot{q} \), the middle inclusion from \( (p', q') \leq (p, \dot{q}) \), and the right inclusion from the definition of \( d^{*} \).

Fix \( \gamma \in d^{*} \). Since \( P \) has \( \kappa \)-cc and \( \models \dot{q}' \in \dot{E}(\kappa, \lambda) \), there is \( \delta_{\gamma}^{*} < \kappa \) such that
\[ \models \gamma \in dom \dot{q}' \rightarrow dom \dot{q}'(\gamma) \subset \delta_{\gamma}^{*}. \]
If \( \gamma \) is in \( dom q \), then \( dom q(\gamma) \subset \delta_{\gamma}^{*} \) because
\[ p' \models dom q(\gamma) = dom \dot{q}(\gamma) \subset dom \dot{q}'(\gamma) \subset \delta_{\gamma}^{*}. \]

The left equality follows from the definition of \( \dot{q} \), the middle inclusion from \( (p', q') \leq (p, \dot{q}) \), and the right inclusion from \( p' \models \gamma \in dom \dot{q}' \) and the choice of \( \delta_{\gamma}^{*} \).

Now define \( q^{*} \) with \( dom q^{*} = d^{*} \) and \( dom q^{*}(\gamma) = \delta_{\gamma}^{*} \) for every \( \gamma \in d^{*} \) so that
\begin{itemize}
  \item \( q^{*}(\gamma)(\eta) = q(\gamma)(\eta) \) if \( \gamma \in dom q \) and \( \eta \in dom q(\gamma) \), or else
  \item \( q^{*}(\gamma)(\eta) \) is the minimal \( \xi \) such that \( \models \gamma \in dom \dot{q}' \land \eta \in dom \dot{q}'(\gamma) \rightarrow \dot{q}'(\gamma)(\eta) = \dot{\tau}(\xi) \).
\end{itemize}

Note that \( q^{*}(\gamma)(\eta) < \gamma \) by \( q \in E(\kappa, \lambda) \) in the first case, and by \( \models \dot{q} \in \dot{E}(\kappa, \lambda) \) and the choice of \( \tau(\xi) \) in the second case. Thus \( q^{*} \in E(\kappa, \lambda) \) and \( q^{*} \leq q \).

Let \( (p^{*}, q^{*}) = \pi(p^{*}, q^{*}) \). Since \( p^{*} = p' \), it remains to prove that \( p' \models q^{*} \leq q' \).
First recall that
\[ p' \models dom \dot{q}' = dom q' \supset dom \dot{q}'. \]

It remains to prove that for every \( \gamma \in d^{*} \) and \( \eta \in \delta_{\gamma}^{*} \)
\[ p' \models \gamma \in dom \dot{q}' \land \eta \in dom \dot{q}'(\gamma) \rightarrow \dot{q}^{*}(\gamma)(\eta) = \dot{q}'(\gamma)(\eta). \]

If \( \gamma \) is in \( dom q \) and \( \eta \in dom q(\gamma) \), the claim follows from
\[ p' \models \dot{q}^{*}(\gamma)(\eta) = \dot{\tau}(q^{*}(\gamma)(\eta)) = \dot{\tau}(q(\gamma)(\eta)) = \dot{q}'(\gamma)(\eta). \]
The left equality follows from the definition of $\hat{q}^*$, the middle from that of $q^*$, and the right from $(p', \hat{q}') \leq (p, \hat{q})$.

In the remaining case the claim follows from

$$\Vdash \gamma \in \text{dom} \hat{q}' \land \eta \in \text{dom} \hat{q}'(\gamma) \rightarrow q^*(\gamma)(\eta) = \hat{q}^*(\gamma)(\eta) = \hat{q}'(\gamma)(\eta).$$

The left equality follows from the definition of $\hat{q}^*$, and the right from that of $q^*$. \(\Box\)

This completes the proof. \(\Box\)

**Remark.** Lemma 4 should hold for suitable modifications of the collapses of Levy and Silver. See [13] or [14] for the corresponding lemma for the modified Silver collapse and the resulting model in which a saturated filter exists and Chang's conjecture holds.

In [11] Laver introduced a poset $L(\kappa, \lambda)$, here called the Laver collapse. It is the product of collapsing posets with Easton support and bounded height. Using Kunen's method Laver constructed a forcing of the form $P*L(\kappa, \lambda)$, which produces an $(\omega_2, \omega_2, \omega)$-saturated filter on $\omega_1$. Although Lemma 4 should hold for a suitable modification of the Laver collapse as well, we need to work with the Easton collapse because a projection, say from $L(\mu, \lambda - \kappa)$ to $L(\kappa, \lambda)$ is not available to us. For the same reason we cannot substitute the collapses of Levy or of Silver for the Easton collapse.

For a $P$-name $\dot{Q}$ for a poset let $T(P, \dot{Q})$ denote the term forcing. It is known that the identity map from $P \times T(P, \dot{Q})$ to $P*\dot{Q}$ is a projection. See [5] for details. In [1] Cummings observed that $T(P, <\kappa \gamma)$ is equivalent to $<\kappa \gamma$ if $P$ has $\kappa$-cc and size $\leq \kappa$, and $\gamma^{<\kappa} = \gamma$. The proof of Lemma 4 shows in effect that $T(P, E(\kappa, \lambda))$ is equivalent to $E(\kappa, \lambda)$. To see that the filter in our model is $\lambda$-saturated only, it suffices to prove this fact or even Lemma 4 without additional clauses.

4. THE MAIN THEOREM

This section is devoted to the proof of

**Theorem.** Suppose that $\kappa$ is almost huge with target $\lambda$ and $\lambda$ is Mahlo. Let $\mu < \nu$ be both regular with $\mu < \kappa \leq \nu < \lambda$. Then $E(\mu, \kappa) * E(\nu, \lambda)$ forces a $(\lambda, \lambda, \mu)$-saturated normal filter.

**Proof.** Let $j : V \rightarrow M$ witness that $\kappa$ is almost huge with target $\lambda$, i.e. $\kappa = \text{crit}(j)$, $\lambda = j(\kappa)$ and $<^\lambda M \subset M$. Then we have $j(E(\mu, \kappa)) = E(\mu, \lambda)$, which is canonically isomorphic to $E(\mu, \kappa) * E(\mu, \lambda - \kappa)$. As stated in §3, there is a projection from $E(\mu, \lambda - \kappa)$ to $E(\mu, \lambda)$. Since $E(\mu, \kappa)$ has $\kappa$-cc and size $\kappa$, there is a projection from $E(\mu, \kappa) * E(\mu, \kappa)$ to $E(\mu, \kappa) * E(\nu, \lambda)$ as in Lemma 4. Thus we get a projection $\pi : E(\mu, \lambda) \rightarrow E(\mu, \kappa) * E(\nu, \lambda)$ such that $\pi(p)$ has the form $(p|\kappa, \dot{q})$, where $E(\mu, \kappa) \Vdash \text{dom} \hat{q} = \text{dom} p - \nu$ and each $\hat{q}(\gamma)$ depends only on $p(\gamma)$.

Now let $\dot{G} \subset E(\mu, \lambda)$ be $V$-generic. Then $\pi'^*\dot{G}$ generates a $V$-generic filter over $E(\mu, \kappa) * E(\nu, \lambda)$, say $\dot{G} \ast H$. We claim that $V[\dot{G}]\lceil[H]$ is the desired model. Since $j'G = G \subset \dot{G}$, we can lift $j : V \rightarrow M$ to $j : V[\dot{G}] \rightarrow M[\dot{G}]$ in $V[\dot{G}]$. Since $\lambda$ is Mahlo in $V$, $E(\mu, \lambda)$ has $\lambda$-cc in $V$. Hence we have $<^\lambda M[\dot{G}] \subset M[\dot{G}]$ in $V[\dot{G}]$ by $<^\lambda M \subset M$ in $V$. 

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Work in $V[G]$. Since $E(\mu, \kappa)$ has size $\kappa$ in $V$, $\lambda$ remains Mahlo and hence $E(\nu, \lambda)$ has $\lambda$-cc. Thus a nice $E(\nu, \lambda)$-name for a subset of $\mathcal{P}_\kappa \nu$ can be viewed as an $E(\nu, \xi)$-name for some $\xi < \lambda$. So we can list the set of all such names with cofinal repetition as $\{\check{X}_\xi : \xi < \lambda\}$.

Now work in $V[\check{G}]$. Since $^{<\lambda} M[\check{G}] \subset M[\check{G}]$, $E(j(\nu), j(\xi))^{M[\check{G}]}$ is $\lambda$-directed closed for $\xi < \lambda$. So we can define for $\xi < \lambda$

$$r_\xi = \text{the greatest lower bound of } j^\mu (H \cap E(\nu, \xi)^{V[\check{G}]}) \text{ in } E(j(\nu), j(\xi))^{M[\check{G}]}.$$

Note that $\xi < \zeta < \lambda$ implies $r_\zeta j(\xi) = r_\xi$. Thus we can define a descending sequence $(r_\xi : \xi < \lambda)$ in $E(j(\nu), j(\lambda))^{M[\check{G}]}$ by recursion so that

- $r_\xi \leq r_\zeta$ in $E(j(\nu), j(\xi))^{M[\check{G}]}$ and
- $\text{if } \check{X}_\xi \text{ is an } E(\nu, \xi)^{V[\check{G}]} \text{-name, then } r_\xi \text{ decides } j^\nu \in j(\check{X}_\xi) \text{ in } M[\check{G}]$.

Define

$$U = \{(\check{X}_\xi)_H : \xi < \lambda \land M[\check{G}] \models r_\xi \vdash j^\nu \in j(\check{X}_\xi)\}.$$ 

Standard arguments show that $U$ is a $V[G][H]$-normal ultrafilter on $\mathcal{P}_\kappa \nu^{V[G][H]}$.

Finally we work in $V[G][H]$. Since $E(\mu, \lambda)$ projects down to $E(\mu, \kappa) \ast E(\nu, \lambda)$ in $V$, there is an $E(\mu, \lambda)^V/(G \ast H)$-name $\dot{U}$ such that

$$E(\mu, \lambda)^V/(G \ast H) \models \dot{U} \text{ is a } V[G][H]\text{-normal ultrafilter on } \mathcal{P}_\kappa \nu^{V[G][H]}.$$

Define

$$F = \{X \subset \mathcal{P}_\kappa \nu : E(\mu, \lambda)^V/(G \ast H) \models X \in \dot{U}\}.$$ 

Standard arguments show that $F$ is a normal filter on $\mathcal{P}_\kappa \nu$. We claim that $F$ is $(\lambda, \lambda, \mu)$-saturated. Standard arguments show that

$$X \mapsto \sum\{p \in E(\mu, \lambda)^V/(G \ast H) : p \models X \in \dot{U}\}$$

defines a complete embedding of $F^+$ into $B(E(\mu, \lambda)^V/(G \ast H))$. So it suffices to prove that $E(\mu, \lambda)^V/(G \ast H)$ has $(\lambda, \lambda, \mu)$-cc. Let $\{p_\xi : \xi < \lambda\} \subset E(\mu, \lambda)^V/(G \ast H)$.

Since $E(\mu, \kappa)$ has $\kappa$-cc and forces $E(\nu, \lambda)$ to be $\kappa$-closed in $V$, it suffices to find $S \in [\lambda]^\lambda$ such that $\check{x} \in [S]^{\mu}$ and $\{p_\xi : \xi \in x\} \in V$, $\{p_\xi : \xi \in x\}$ has a common extension in $E(\mu, \lambda)^V/(G \ast H)$.

Let $R$ be the set of regular cardinals $< \lambda$ in $V$. Since $\lambda$ is Mahlo and $E(\mu, \kappa) \ast E(\nu, \lambda)$ has $\lambda$-cc in $V$, $R$ is stationary. As in the proof of Lemma 1 we get a stationary $S \subset R$ such that $\text{dom } p_\xi : \xi \in S\}$ forms a $\Delta$-system, say with root $d$. Moreover we may assume that $p_\xi|d$ is constant and $\text{dom } p_\xi \kappa \subset d$ for $\xi \in S$.

Suppose $x \in [S]^{\mu}$ and $\{p_\xi : \xi \in x\} \in V$. Define $p = \bigcup_{\xi \in x} p_\xi$. We claim that $p$ is a lower bound of $\{p_\xi : \xi \in x\}$ in $E(\mu, \lambda)^V/(G \ast H)$. Since $p_\xi|d$ is constant on $S$, $p$ is a lower bound of $\{p_\xi : \xi \in x\}$ in $E(\mu, \lambda)^V$.

It remains to prove that $\pi(p) \in G \ast H$. Let $\langle p^\nu, \check{q}_\xi \rangle = \pi(p)$ and $\langle p_\xi \kappa, \check{q}_\xi \rangle = \pi(p_\xi)$ for $\xi \in S$. Since $p_\xi|\kappa$ is constant on $S$, we have $p|\kappa = p_\xi|\kappa$ for every $\xi \in S$. Hence $p|\kappa \in G$ by $\langle p_\xi|\kappa, \check{q}_\xi \rangle = \pi(p_\xi) \in G \ast H$. To see that $\check{q}_\xi \in H$, note first that $(\check{q}_\xi)_G \in H$ by $\langle p_\xi|\kappa, \check{q}_\xi \rangle \in G \ast H$. Since $\text{dom } (\check{q}_\xi)_G = \text{dom } p_\xi - \nu$, $\{\text{dom } (\check{q}_\xi)_G : \xi \in S\}$ forms a $\Delta$-system with root $d - \nu$. Moreover $(\check{q}_\xi)_G|\overline{G} \ast (d - \nu)$ is constant on $S$. Thus $\check{q}_G = \bigcup_{\xi \in S} \check{q}_\xi)$ is the greatest lower bound of $\{(\check{q}_\xi)_G : \xi \in x\}$ in $E(\nu, \lambda)^V[G]$. Therefore $\check{q}_G \in H$, as desired.

$\square$
Remark. For the moment let us assume that $\kappa$ is huge with target $\lambda$. As remarked in §3, our strategy requires forcing with Easton collapses rather than with Laver collapses. This requires in turn invoking an argument of Magidor [8] that involves local master conditions, even under the stronger hypothesis as above. In fact we can dispense with the argument in the case $\nu > \kappa$. Moreover the proof in this case, if modified as in [12], shows that $[\lambda]^\kappa$ carries a $(\lambda, \lambda, \mu)$-saturated $\kappa$-complete filter in the extension.

In [11] Laver observed that a strong form of Chang's conjecture holds in his model. We do not know whether our model in the case $\nu = \kappa$ satisfies the conjecture.

References


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