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A Solution of the Jacobian Problem in Boolean Algebra

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Abstract.

We propose to answer the Jacobian conjecture in boolean algebra. The boolean analogue of the Jacobian problem in $\{0,1\}^{n}$ has been proved: if a map from $\{0,1\}^{n}$ to itself defines a boolean network has the property that all the boolean eigenvalues of the discrete Jacobian matrix of this map evaluated at each element of $\{0,1\}^{n}$are zero, then it has a unique fixed point. We propose extending this result to any map F from the product space X of n finite boolean algebras to itself.

Key words: Jacobian conjecture; Combinatorial fixed point theorem, Discrete Boolean eigenvalues; Finite boolean algebras.

1. Introduction

In 2004, a boolean analogue of the Jacobian problem has been proved[12].

Theorem 1.1. If a boolean function $F : \{0,1\}^n \longrightarrow \{0,1\}^n$ has the property that all the boolean eigenvalues of the discrete Jacobian matrix of each element of $\{0,1\}^n$ are zero, then it has a unique fixed point.

It raised by Shih and Ho [13] in 1999 in automata networks. In this note, we attempted to extend this result to any map F from the product X of n finite boolean algebras to itself. In the course of Robert's analysis of boolean contraction and applications, he introduced the boolean vector distance, the discrete incidence matrix for the maps from $\{0,1\}^n$ to itself and the notion of spectra of boolean matrices $[1-6]$. Also we have studied some conclusions about the global convergence $[7-11]$.

In order to extend this theorem from the $\{0,1\}$ case to the finite boolean algebra case, we will introduce here, the Jacobian matrix for maps from the product X of n finite boolean algebras to itself which generalizes Robert's discrete derivative for maps from $\{0,1\}^n$ to itself. We will present and prove the Jacobian problem in boolean algebra: if the boolean spectra radius of the Jacobian matrix of a map F from the product X of n finite boolean algebras to itself is zero, then it has a unique fixed point.

2. Jacobian matrix

Let $(A,+,\cdot,-,0,1)$ be a finite boolean algebra. Define $a \in A$ to be an *atom* of A if $0 < a$ but there is no x in A satisfying $0 < x < a$. We denoted by $At(A)$ the set of atoms of A. Write the cardinality of $At(A)$ by $\#At(A)$ and the power set algebra of $At(A)$ by $P(At(A))$. Remark that for every boolean algebra A, the map φ from A to the power set algebra $P(At(A))$ defined by

$$\varphi(x) = \{a \in At(A) : a \leq x\}$$

is an isomorphism.(see[11, Lemma 3.1])

Given a finite boolean algebra $(A,+,\cdot,-,0,1)$ with $At(A) = \{a_1,\ldots,a_m\}$, consider a positive integer n, such that $X_i = A$ $(i = 1,\ldots,n)$. Then

$X = X_1 \times \ldots \times X_n$ is a product of n finite boolean algebras. For $x = (x_1, \ldots, x_n) \in X$, we denoted by $\tilde{x}^{j(k)}$ the $j(k)$th *neighbour* of x ($j = 1, \ldots, n$; $k = 1, \ldots, m$).

$$\tilde{x}_i^{j(k)} = \begin{cases} x_i & \text{if } i \neq j, \\ \varphi^{-1}(\varphi(x_j) \cup \{a_k\} & \text{if } i = j \text{ and } a_k \notin \varphi(x_j), \\ \varphi^{-1}(\varphi(x_j) \setminus \{a_k\}) & \text{if } i = j \text{ and } a_k \in \varphi(x_j). \end{cases}$$

$(i = 1, \ldots, n)$.

Furthermore, we denoted by c_k the element in $\{0,1\}^m$ whose kth component is 1 and whose other components are 0. Therefore, if $m = n$ then it is the kth unit vector e_k of $\{0,1\}^n$. For any $D \in P(At(A))$, we denoted by $I(D)$ the set $\{k : a_k \in D\}$. Define the map η from the power set algebra $P(At(A))$ to$\{0,1\}^m$ by

$$\eta(D) = \begin{cases} \mathbf{0} \text{ (zero vector)} & \text{if } D = \phi, \\ c_k & \text{if } D = \{a_k\}, \\ \sum\limits_{j \in I(D)} c_j & \text{otherwise.} \end{cases}$$

Note that η is also an isomorphism.(see [11, Lemma 3.1])

For a map $F = (f_1, \ldots, f_n)$ from the product X of n finite boolean algebras to itself. Define a map $\bar{F} = (\bar{f}_{1(1)}, \ldots, \bar{f}_{1(m)}, \bar{f}_{2(1)}, \ldots, \bar{f}_{2(m)}, \ldots, \bar{f}_{n(1)}, \ldots, \bar{f}_{n(m)})$ from X to $\{0,1\}^{nm}$ by

$$\bar{f}_{i(k)}(x) = [\eta(\varphi(f_i(x)))]_k \quad (i = 1, \ldots, n;\ k = 1, \ldots, m).$$

Now, it is in position to introduce the notion of Discrete Jacobian matrix in boolean algebra. Given a map $F = (f_1, \ldots, f_n)$ from the product X of n finite boolean algebras with $\#At(X_i) = m$ to itself and $x \in X$. We call *discrete Jacobian matrix* of F evaluated at x, and we denote by

$$F'(x) = (f_{i(k_1)\ j(k_2)}(x))$$

$$= \left(\begin{array}{ccccccc} f_{1(1)1(1)}(x) \cdots f_{1(1)1(m)}(x) & & \cdots & & f_{1(1)n(1)}(x) \cdots f_{1(1)n(m)}(x) \\ \vdots \quad \ddots \quad \vdots & \vdots & \ddots & \vdots & \vdots \quad \ddots \quad \vdots \\ f_{1(m)1(1)}(x) \cdots f_{1(m)1(m)}(x) & & \cdots & & f_{1(m)n(1)}(x) \cdots f_{1(m)n(m)}(x) \\ \cdots & & \cdots & & \cdots \\ \vdots \quad \ddots \quad \vdots & \vdots & \ddots & \vdots & \vdots \quad \ddots \quad \vdots \\ \cdots & & \cdots & & \cdots \\ f_{n(1)1(1)}(x) \cdots f_{n(1)1(m)}(x) & & \cdots & & f_{n(1)n(1)}(x) \cdots f_{n(1)n(m)}(x) \\ \vdots \quad \ddots \quad \vdots & \vdots & \ddots & \vdots & \vdots \quad \ddots \quad \vdots \\ f_{n(m)1(1)}(x) \cdots f_{n(m)1(m)}(x) & & \cdots & & f_{n(m)n(1)}(x) \cdots f_{n(m)n(m)}(x) \end{array} \right),$$

the $nm \times nm$ matrix over $\{0, 1\}$ defined by

$$f_{i(k_1)j(k_2)}(x) = \begin{cases} 0 & \text{if } \bar{f}_{i(k_1)}(x) = \bar{f}_{i(k_1)}(\tilde{x}^{j(k_2)}), \\ 1 & \text{if } \bar{f}_{i(k_1)}(x) \neq \bar{f}_{i(k_1)}(\tilde{x}^{j(k_2)}). \end{cases}$$

$(i, j = 1, \ldots, n; k_1, k_2 = 1, \ldots, m)$

The order "$\leq$" in $\{0, 1\}$ is given by $0 \leq 0 \leq 1 \leq 1$. Obviously, this structure $(\{0, 1\}, +, \cdot, -, 0, 1)$ is the two-element boolean algebra. Let $F : \{0, 1\}^n \longrightarrow \{0, 1\}^n$. Then $\#At(\{0, 1\}) = 1$. By the definitions of the maps φ and η, we obtain

$$\bar{F} = (\bar{f}_{1(1)}, \bar{f}_{2(1)}, \ldots, \bar{f}_{n(1)}) = (f_1, \ldots, f_n) = F.$$

Note also that $\tilde{x}_j^{j(1)} = -x_j$, and then $\tilde{x}^{j(1)} = (x_1, \ldots, -x_j, \ldots, x_n) = \tilde{x}^j$, which is the jth neighbor of x in $\{0, 1\}^n$ [3], so that now the Jacobian matrix is the Robert's $n \times n$ discrete derivative [3]. Hence, if $\#At(X_i) = 1$ for all $i = 1, \ldots, n$, then our theorem is equivalent to Theorem 1.1.

Throughout this paper, a *boolean matrix* is meant to be a matrix over $\{0, 1\}$. Here the discrete Jacobian matrices are the boolean matrices. Boolean matrix multiplication and addition are the same as in the case of complex matrices but the concerned products of entries are boolean. A non-zero element $u \in \{0, 1\}^n$ is called the (*boolean*) *eigenvector* of a boolean matrix M if there exists an λ in $\{0, 1\}$ such that $Mu = \lambda u$; λ is called the (*boolean*) *eigenvalue* associated with eigenvector. For any boolean matrix M, the symbol $\sigma(M)$ denotes the (*boolean*) *spectrum* of M, it is the set of all eigenvalues of M, so that $\sigma(M) \subset \{0, 1\}$. The (*boolean*) *spectral radius* of M, which is denoted by $\rho(M)$, is defined to be the largest eigenvalue of M.

The main result is the following theorem.

Theorem 2.1. Given a finite boolean algebra $(A, +, \cdot, -, 0, 1)$ with $At(A) = \{a_1, \ldots, a_m\}$. Let X be the product of n finite boolean algebras with $X_i = A$ $(i = 1, \ldots, n)$. If a map F from X to itself is such that $\rho(F'(x))) = 0$ for all $x \in X$, then it has a unique fixed point.

Theorem 2.1 can be viewed as a discrete version of the Jacobian conjecture in boolean algebra. The aim of this note is to prove it.

3. Iteration graph

Define maps $h : X \longrightarrow [P\,(At(A))]^n$ and $g : [P\,(At\,(A))]^n \longrightarrow \{0,1\}^{nm}$ by

$$\begin{aligned} h\,(x) &= h\,(x_1, ..., x_n) \\ &= (\varphi\,(x_1)\,, ..., \varphi\,(x_n)), \end{aligned}$$

and

$$\begin{aligned} g(D) &= g\,(D_1, ..., D_n) \\ &= (\eta\,(D_1)\,, ..., \eta\,(D_n)) \end{aligned}$$

In this section, we state one result we have studied[7, 11].

Lemma 3.1. Given a finite boolean algebra $(A, +, \cdot, -, 0, 1)$ with $At(A) = \{a_1, \ldots, a_m\}$. Let X be the product of n finite boolean algebras with $X_i = A$ $(i = 1, \ldots, n)$. For a map F from X to itself, there is a map $\hat{F}$ from $\{0,1\}^{nm}$ to itself and two isomorphisms $h : X \longrightarrow [P\,(At(A))]^n$ and $g : [P\,(At\,(A))]^n \longrightarrow \{0,1\}^{nm}$ such that

$$F(x) = (gh)^{-1}\,\hat{F}gh(x) \qquad \text{for all } x \in X.$$

Recall that the *iteration graph* for a map F is the digraph consisting of vertices which are elements of X and the following directed arcs: for all x in X, a directed arc connects x to $F(x)$. Since $g \circ h$ is an isomorphism, the iteration graphs for F and $\hat{F}$ have the same pattern. Particularly, if there is a unique fixed point in the iteration graphs for $\hat{F}$ then there is also a unique fixed point in the iteration graphs for F. Let $c \in \{0,1\}^{nm}$ be the unique fixed point in the iteration graphs for $\hat{F}$. Put $\xi = (gh)^{-1}(c)$. Then $\xi \in X$, the product of n finite boolean algebras and we have

$$\begin{aligned} & \hat{F}(c) = c \\ \Rightarrow\ & ghF(gh)^{-1}(c) = c \\ \Rightarrow\ & F(gh)^{-1}(c) = (gh)^{-1}(c) \\ \Rightarrow\ & F(\xi) = \xi. \end{aligned}$$

Since $(gh)^{-1}$ is an isomorphism, ξ is the unique fixed point in the iteration graphs for F.

Lemma 3.2. The iteration graphs for F and $\hat{F}$ have the same pattern.

4. Proof of Theorem 2.1

Given a finite boolean algebra $(A, +, \cdot, -, 0, 1)$ with $At(A) = \{a_1, \ldots, a_m\}$. Let X be the product of n finite boolean algebras with $X_i = A$ $(i = 1, \ldots, n)$. For a map F from X to itself is such that $\rho(F'(x)) = 0$ for all $x \in X$. By Lemma 3.1, there is a map $\hat{F} : \{0,1\}^{nm} \longrightarrow \{0,1\}^{nm}$ and two isomorphisms $h : X \longrightarrow [P(At(A))]^n$ and $g : [P(At(A))]^n \longrightarrow \{0,1\}^{nm}$ such that $F(x) = (gh)^{-1}\hat{F}gh(x)$ for all $x \in X$.

Let $F'(x) = (f_{i(k_1)\,j(k_2)}(x))$, $(i, j = 1, \ldots, n; k_1, k_2 = 1, \ldots, m)$, be the discrete Jacobian matrix of F evaluated at x in X. For the map

$$\hat{F} = (\hat{f}_{1(1)}, \ldots, \hat{f}_{1(m)}, \hat{f}_{2(1)}, \ldots, \hat{f}_{2(m)}, \ldots, \hat{f}_{n(1)}, \ldots, \hat{f}_{n(m)}),$$

the discrete Jacobian matrix of $\hat{F}$ evaluated at y in $\{0,1\}^{nm}$ is the $nm \times nm$ matrix $\hat{F}'(x) = (\hat{f}_{i(k_1)j(k_2)}(x))$ over $\{0,1\}$ defined by

$$\hat{f}_{i(k_1)j(k_2)}(y) = \begin{cases} 0 & \text{if } \ \hat{f}_{i(k_1)}(y) = \hat{f}_{i(k_1)}(\tilde{y}^{j(k_2)}), \\ 1 & \text{otherwise.} \end{cases}$$

$(i, j = 1, \ldots, n; k_1, k_2 = 1, \ldots, m)$, where

$$\tilde{y}^{j(k_2)} = (y_{1(1)}, \ldots, y_{1(m)}, \ldots, -y_{j(k_2)}, \ldots, y_{n(1)}, \ldots, y_{n(m)})$$

is the $j(k_2)$th neighbor of y. Given $x \in X$, for any $i, j = 1, \ldots, n; k_1, k_2 = 1, \ldots, m$, we have

$$
\begin{aligned}
& f_{i(k_1)j(k_2)}(x) = 0 \\
\Leftrightarrow\ & \bar{f}_{i(k_1)}(x) = \bar{f}_{i(k_1)}(\tilde{x}^{j(k_2)}) \\
\Leftrightarrow\ & [\eta(\varphi(f_i(x)))]_{k_1} = [\eta(\varphi(f_i(\tilde{x}^{j(k_2)})))]_{k_1} \\
\Leftrightarrow\ & \hat{f}_{i(k_1)}(g(h(x))) = \hat{f}_{i(k_1)}(g(h(\tilde{x}^{j(k_2)}))) \\
\Leftrightarrow\ & \hat{f}_{i(k_1)}(y) = \hat{f}_{i(k_1)}(\tilde{y}^{j(k_2)}) \quad \text{where } y = g(h(x)) \in \{0,1\}^{nm} \\
\Leftrightarrow\ & \hat{f}_{i(k_1)j(k_2)}(y) = 0.
\end{aligned}
$$

So that, for $x \in X$ and $y = g(h(x))$, $F'(x) = \hat{F}'(y)$. Since $g \circ h$ is an isomorphism and $\rho(F'(x)) = 0$ for all $x \in X$, we obtain $\rho(\hat{F}'(y)) = 0$ for all $y \in \{0,1\}^{nm}$. Combining Theorems 1.1 and 3.2, we obtain that F has a unique fixed point. This completes the proof of Theorem 2.1.

5. Examples

If F is a map from X the product of n finite boolean algebras to itself.We denoted by $B(F) = (b_{i(k_1)j(k_2)})$ the *incidence matrix* of F (see[11,p.1137]). It is the $nm \times nm$ matrix over $\{0,1\}$ defined by

$$
b_{i(k_1)j(k_2)} = \begin{cases} 0 & \text{if } \bar{f}_{i(k_1)}(x) = \bar{f}_{i(k_1)}(\tilde{x}^{j(k_2)}) \text{ for all } x \in X, \\ 1 & \text{otherwise.} \end{cases}
$$

$(i, j = 1, \ldots, n; k_1, k_2 = 1, \ldots, m)$

Then $B(F) = \sup_{x \in X}\{F'(x)\}$, hence if all the boolean eigenvalues of the incidence matrix of this map are zero then all the boolean eigenvalues of the discrete Jacobian matrix of this map evaluated at each element of X are zero. But not vice versa (see[11,Example 1]).

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