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RECURRENCY AND UNPREDICTABILITY OF QUASI-PERIODIC ORBITS ESTIMATED BY SIMULTANEOUS DIOPHANTINE APPROXIMATIONS

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1. Introduction

The Kolmogorov-Arnold-Moser (KAM) theorem shows the persistence of quasi-periodic dynamical systems under the Diophantine condition on their irrational frequencies, which are simultaneously very well approximable by rational numbers with the same denominator. In our previous papers ([4], [6]) we introduced Extending Common Multiples (ECM) conditions on pairs of irrational numbers and we have shown some inequality relations between the parameters of Diophantine conditions and the ECM conditions. In this paper we investigate these conditions for extremal irrational numbers, which were recently termed by D. Roy in studying some optimality of Diophantine conditions. We show the unpredictability of the quasi-periodic orbits, which have these extremal irrational frequencies, by estimating the positive gaps of their recurrent dimensions.

Our plan of this paper is as follows. In section 2 we introduce the notations and definitions on valuations for integers and show some inequality relations between these valuations. In section 3 we give the definitions of ECM sequences for a pair of irrational numbers and introduce the Diophantine conditions of the KAM theorem. We also give the inequality relations between the parameters of the ECM conditions and the Diophantine conditions. In section 4 we study the case where the pair of irrational numbers are give by an extremal number and its square and investigate the values of the Lévy constants of these irrational numbers. In section 5 we consider a discrete quasi-periodic orbit, the frequencies of which are given by the extremal number and its square, and we estimate the gap values of the recurrent dimensions of the orbits.

In this paper we cannot contain the proves of our theorems, which will be shown in the forthcoming complete paper.

2. Valuations of integers by continued fractions

For a irrational number τ, let $\{n_j/m_j\}$ be its convergents.

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For each positive integer l we can consider the expansion of l by using the denominators $\{m_j\}$;

$$l = p_k m_k + p_{k-1} m_{k-1} + \cdots + p_u m_u \tag{2.1}$$

where $p_j \in \mathbb{N}_0; p_j \leq a_{j+1}, j = u, u+1, ..., k$ and $p_k, p_u \geq 1$. By introducing the following lexicographical order we have the uniqueness of this expansion. Assume that some number l has two expansions such that

$$\begin{aligned} l &= p_{k_1} m_{k_1} + p_{k_1-1} m_{k_1-1} + \cdots + p_{u_1} m_{u_1} := [l1], \\ l &= p_{k_2} m_{k_2} + p_{k_2-1} m_{k_2-1} + \cdots + p_{u_2} m_{u_2} := [l2]. \end{aligned}$$

Define $[l1] \leq [l2]$ if $k_1 < k_2$, or otherwise if $k_1 = k_2$ and $p_{k_1} < p_{k_2}$, or otherwise if $k_1 = k_2$ and

$$p_{k_1} = p_{k_2},\ p_{k_1-1} = p_{k_2-1}, \cdots ,\ p_{k_1-j+1} = p_{k_2-j+1},\ p_{k_1-j} < p_{k_2-j}$$

for some $j \in \mathbb{N}$. Then we can take the largest expansion for this order.

For example, note that $p_j \leq [m_{j+1}/m_j] = a_{j+1}$ and let

$$l = p_k m_k + a_k m_{k-1} + p_{k-2} m_{k-2} + \cdots + p_u m_u,\ p_k < a_{k+1},\ p_{k-2} \geq 1,$$

then we choose the expansion

$$l = (p_k + 1) m_k + (p_{k-2} - 1) m_{k-2} + \cdots + p_u m_u.$$

For $l \in \mathbb{N}$, define the mapping $\zeta : \mathbb{N} \to \mathbb{N}$ by $\zeta(l) = u$, which specifies the final subscript of its expansion. Now we define the two valuations $\|l\|_\tau$ and $[l]_\tau$ of a positive integer l, which has the expansion (2.1) by

$$\|l\|_\tau = \frac{1}{m_{\zeta(l)+1}} = \frac{1}{m_{u+1}}$$

and

$$[l]_\tau = \frac{k-u}{k}$$

where $\|l\|_\tau$ is a kind of modified "p-adic" type valuation and $[l]_\tau$ shows a relative length of its expansion.

By applying the estimates in our previous paper [4] we obtain the following theorem.

Theorem 2.1. *For an irrational number τ there exist positive constants c_1, c_2 such that*

$$c_1 \|l\|_\tau \leq \{l\tau\} \leq c_2 \|l\|_\tau \tag{2.2}$$

for every positive integer l where $\{r\}$ is a fractional part of a positive real number r.

3. Extended common multiples

We say that a sequence $\{l_j\}$ of positive integers is a sequence of common multiples for an irrational pair $\{\tau_1, \tau_2\}$ if

$$\lim_{j\to\infty} \max\{\|l_j\|_{\tau_1}, \|l_j\|_{\tau_2}\} = 0$$

holds. Then we denote the set of the sequences of common multiples by $cm(\tau_1, \tau_2)$.

In $cm(\tau_1, \tau_2)$ we can choose an extremal common multiples sequence (abr. ecm sequence) $\{t_j\}$, which satisfies the following properties: $t_{j+1} > t_j$ for every $j \in \mathbb{N}$ and, if $t_j > l_k$ for $(l) \in cm(\tau_1, \tau_2)$,

$$\max\{\|t_j\|_{\tau_1}, \|t_j\|_{\tau_2}\} < \max\{\|l_k\|_{\tau_1}, \|l_k\|_{\tau_2}\}.$$

There exists the maximal ecm sequence $\{T_j\}$, which satisfies that $\{t_j\} \subset \{T_j\}$ for every ecm sequence $\{t_j\}$. We denote the maximal ecm sequence by $ECM(\tau_1, \tau_2)$. In [4] we introduced the construction method of the ECM sequence.

For the $ECM(\tau_1, \tau_2)$ sequence $\{T_j\}$, we define the following constants

$$\delta_0 = \liminf_{j\to\infty} \max\{[T_j]_{\tau_1}, [T_j]_{\tau_2}\},$$
$$\delta_1 = \limsup_{j\to\infty} \max\{[T_j]_{\tau_1}, [T_j]_{\tau_2}\}.$$

Let $\{n_j/m_j\}$ and $\{r_j/l_j\}$ be the convergents of τ_1, τ_2, respectively, and we consider the case where the sequences $\{(m_j)^{\frac{1}{j}}\}, \{(l_j)^{\frac{1}{j}}\}$ are bounded. We denote the upper Lévy constants of τ_1, τ_2 by $\lambda^*(\tau_1), \lambda^*(\tau_2)$ and the lower Lévy constants of τ_1, τ_2 by $\lambda_*(\tau_1), \lambda_*(\tau_2)$, respectively, as follows.

$$\limsup_{j\to\infty}(m_j)^{\frac{1}{j}} = \lambda^*(\tau_1), \quad \liminf_{j\to\infty}(m_j)^{\frac{1}{j}} = \lambda_*(\tau_1), \tag{3.1}$$

$$\limsup_{j\to\infty}(l_j)^{\frac{1}{j}} = \lambda^*(\tau_2), \quad \liminf_{j\to\infty}(l_j)^{\frac{1}{j}} = \lambda_*(\tau_2). \tag{3.2}$$

We also say that an irrational number τ has a Lévy constant if $\lambda^*(\tau) = \lambda_*(\tau)$. In 1935 Khinchin proved that almost all irrational numbers have the same Lévy constant value and in 1936 Lévy found the explicit expression for this constant; $e^{\frac{\pi^2}{12\log 2}} \sim 3.27582...$

Hereafter we use the following notations.

$$E_1 = \min\{\lambda_*(\tau_1), \lambda_*(\tau_2)\}, \quad E_2 = \max\{\lambda^*(\tau_1), \lambda^*(\tau_2)\}.$$

Usual definitions of the Diophanitine condition in KAM theorem are given as follows.

There exist constants $\gamma, d : \gamma > 0,\ d > 2$, which satisfy

$$|(\tau_1 m_1 + \tau_2 m_2) - n| \geq \frac{\gamma}{|m|^d}$$

for every integers $m = (m_1, m_2) \in \mathbb{Z}^2,\ n \in \mathbb{Z}$ where $|\cdot|$ denotes a usual Euclidean norm.

Here we say that $\{\tau_1, \tau_2\}$ satisfies d_0-(D) condition or we call the pair a d_0-(D) class pair if there exists a constant $d_0 : d_0 \geq 2$, such that, for each $d > d_0$, there exists $\gamma_d > 0$, which satisfies

$$(3.3) \qquad |(\tau_1 m_1 + \tau_2 m_2) - n| \geq \frac{\gamma_d}{|m|^d}$$

for every integers $m = (m_1, m_2) \in \mathbb{Z}^2$, $n \in \mathbb{Z}$ and furthermore, for each d : $0 < d < d_0$ and each $\gamma > 0$, there exist integers $m_\gamma = (m_{\gamma,1}, m_{\gamma,2}) \in \mathbb{Z}^2$ and $n_\gamma \in \mathbb{Z}$, which satisfy

$$(3.4) \qquad |(\tau_1 m_{\gamma,1} + \tau_2 m_{\gamma,2}) - n_\gamma| < \frac{\gamma}{|m_\gamma|^d}.$$

By (3.4) the constant d_0 specifies the infimum value of d, which satisfies (3.3).

We call a pair $\{\tau_1, \tau_2\}$ a Liouville type pair if, for every $d_0 > 0$, there exists $d : d > d_0$ such that for each $\gamma > 0$, there exists $m_\gamma = (m_{\gamma,1}, m_{\gamma,2})$, which satisfies

$$|(\tau_1 m_{\gamma,1} + \tau_2 m_{\gamma,2}) - n_\gamma| < \frac{\gamma}{|m_\gamma|^d}.$$

Theorem 3.1. *Let τ_1, τ_2 have the upper and lower Lévy constants and belong to a d_0-(D) class. Then for the constants d_0, δ_0 we have*

$$(3.5) \qquad 1 - \frac{d_0 - 1}{2} \cdot \frac{\log E_2}{\log E_1} \leq \delta_0 \leq 1 - \frac{d_0}{d_0 + 2} \cdot \frac{\log E_1}{\log E_2}.$$

4. Extremal numbers

For the d_0-(D) condition, if $\{1, \tau_1, \tau_2\}$ are linearly independent over $\mathbb{Q}$, it is known that the following inequalities

$$2 \leq d_0 \leq \gamma^2 = 2.618\cdots$$

hold where $\gamma = (1 + \sqrt{5})/2$. Furthermore, almost all pairs of irrational numbers with respect to Lebesgue's measure satisfy $d_0 = 2$. For the case $\tau_1 = \xi, \tau_2 = \xi^2$ where ξ is not quadratic irrational Davenport and Schmidt estimated the upper bound γ^2 in 1969 and In [10] D.Roy introduced the irrational numbers, called extremal numbers, which satisfy $d_0 = \gamma^2$ and proved that the set of extremal numbers is countable. He gave some explicit examples of extremal numbers by using continued fractions of Fibonacci sequences as follows.

Let $\{a, b\}$ be a pair of distinct positive integers and define the sequence $\{w_i\}$ recursively by

$$w_0 = b, \; w_1 = a, \; w_i = w_{i-1} w_{i-2} \; (i \geq 2),$$

$$\begin{aligned} w_2 &= ab \\ w_3 &= w_2 w_1 = aba \\ w_4 &= w_3 w_2 = abaab \\ w_5 &= w_4 w_3 = abaababa \\ &\vdots \end{aligned}$$

and put the infinite word $w = abaababaabaab\cdots$, which is also given by a fixed point of the substitution $a \to ab,\ b \to a$.

The example of extremal numbers is given by the continued fraction

$$\xi_{a,b} = [0; w] = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \ddots}}}.$$

In 1998 M.Queffélec proved that any real number whose continued fraction is given by a fixed point of substitutions has a Lévy constant. Thus we put $\lambda_1 := \lambda^*(\xi) = \lambda_*(\xi)$.

However, up to now we have not yet known any results about Lévy constants of ξ^2 and so, we put

$$E_1 = \min\{\lambda_*(\xi^2), \lambda_1\}, \quad E_2 = \max\{\lambda^*(\xi^2), \lambda_1\}.$$

For the pair $\tau_1 = \xi,\ \tau_2 = \xi^2$ given by an extremal number it follows from Theorem 3.1 that we have

$$1 - \frac{\gamma^2 - 1}{2} \cdot \frac{\log E_2}{\log E_1} \le \delta_0 \le 1 - \frac{\gamma^2}{\gamma^2 + 2} \cdot \frac{\log E_1}{\log E_2}. \tag{4.1}$$

In [10] D.Roy obtained an ecm sequence $\{t_j\}$ of $\{\xi, \xi^2\}$, which is constructed by denominators of the convergents of ξ by using palindrome words (see the next section 5). Since $[t_j]_\xi = 0$, we can show that $\delta_0 = 0$. It follows from (4.1) that we can estimate

$$\frac{\log E_2}{\log E_1} \le \frac{2}{\gamma^2 - 1}.$$

Since almost all irrational numbers have the Lévy constant $\lambda_0 = e^{\pi^2/12\log 2}$, here we assume that

$$\lambda_*(\xi^2) = \lambda^*(\xi^2) = \lambda_0 = 3.27582.... \tag{4.2}$$

Then, if $\lambda_0 = \lambda_1$, that is, $E_1 = E_2$, we have the contradiction: $\gamma^2 \ge 3$. Thus a Lévy constant λ_1 is not equal to the Lévy constant λ_0 or the equalities (4.2) do not hold: $\lambda^*(\xi^2) > \lambda_*(\xi^2)$ or $\lambda_*(\xi^2) = \lambda^*(\xi^2) \ne \lambda_0$. Our numerical calculations show that (4.2) holds and the value of λ_1 depends on the values of the partial quotients a, b. For example, in case $a = 3,\ b = 2$ we can see that $\lambda_*(\xi^2) = \lambda^*(\xi^2) \sim 3.27582...$ and $\lambda_1 \sim 2.916$.

5. Quasi-periodic orbits

In this section we estimate the gap values of recurrent dimesnions of a simple quasi-periodic orbit, using the extremal numbers. In our previous papers ([6], [7], [9]) we have investigated these gap values in the other examples of quasi-periodic orbits.

For an irrational pair $\{\tau_1, \tau_2\}$ as frequency we consider the following discrete quasi-periodic orbit in the unit interval $[0, 1)$

$$\Sigma = \{\varphi(n) :\ n = 0, 1, 2, ...\}, \quad \varphi(n) = \max\{\{n\tau_1\}, \{n\tau_2\}\}$$

where $\{a\}$ denotes the fractional part of a.

The first ε-recurrent time M_ε to 0 is defined by

$$M_\varepsilon = \min\{n \in \mathbb{N} : \varphi(n) < \varepsilon\}$$

and the upper and the lower recurrent dimensions are defined by

$$\overline{D}(\Sigma) = \limsup_{\varepsilon \to 0} \frac{\log M_\varepsilon}{-\log \varepsilon},$$
$$\underline{D}(\Sigma) = \liminf_{\varepsilon \to 0} \frac{\log M_\varepsilon}{-\log \varepsilon}.$$

The gap of the recurrent dimensions, which gives the unpredictability level of orbits, is defined by

$$G(\Sigma) = \overline{D}(\Sigma) - \underline{D}(\Sigma).$$

Since we can show the following estimates by applying the argument in our previous paper [4]

$$\underline{D}(\Sigma) \leq \frac{\log E_2}{(1-\delta_0)\log E_1},$$
$$\overline{D}(\Sigma) \geq \frac{\log E_1}{(1-\delta_1)\log E_2},$$

we have

$$G(\Sigma) \geq \frac{\log E_1}{(1-\delta_1)\log E_2} - \frac{\log E_2}{(1-\delta_0)\log E_1}. \tag{5.1}$$

Now we consider the orbit given by

$$\varphi(n) = \max\{\{n\xi\}, \{n\xi^2\}\}$$

To show the optimality of the Diophantine condition, that is, $d_0 = \gamma^2$, D.Roy used the palindrome words $\{m_i\}$ in the Fibonacci sequence:

$$m_1 = a,\ m_2 = aba,\ m_i = m_{i-1}s_{i-1}m_{i-2}\quad (i \geq 3)$$

where $s_i = ab$ for even i and $s_i = ba$ for odd i.

m_i is a word of w_{i+2} without its last two terms.

$$\begin{aligned}
&w_2 = ab \\
&w_3 = w_2w_1 = a[ba] && \to m_1 = w_3 - [ba] = a \\
&w_4 = w_3w_2 = aba[ab] && \to m_2 = w_4 - [ab] = aba \\
&w_5 = w_4w_3 = abaaba[ba] && \to m_3 = w_5 - [ba] = abaaba \\
&\vdots && \qquad\qquad\vdots
\end{aligned}$$

Let $\{p_i/q_i\}$ be the convergents of ξ, then

$$[0; m_i] = \frac{p_{f_i-2}}{q_{f_i-2}}$$

holds where f_i is the usual Fibonacci sequence;

$$f_1 = 1,\ f_2 = 1,\ f_3 = 2, f_4 = 2+1 = 3, \ldots, f_i = f_{i-1} + f_{i-2},$$

$$f_i = \frac{5+\sqrt{5}}{10}(\frac{\sqrt{5}+1}{2})^{i-1} + \frac{5-\sqrt{5}}{10}(\frac{-\sqrt{5}+1}{2})^{i-1}.$$

The essential estimates, which were used in the proof by Roy, are

$$|\xi - \frac{p_{f_i-2}}{q_{f_i-2}}| < \frac{1}{q^2_{f_i-2}}, \quad |\xi^2 - \frac{p_{f_i-3}}{q_{f_i-2}}| < \frac{c}{q^2_{f_i-2}}$$

for some constant $c > 0$.

Since the sequence $\{q_{f_i-2}\}$ of common denominators is a subsequence of $\mathrm{ECM}(\xi, \xi^2)$, which satisfies $[q_{f_i-2}]_\xi = 0$, we can show that $\delta_0 = 0$. On the other hand, investigating the sequence $\{q_{f_{i+1}-2} - q_{f_i-2}\}$, which is also a subsequence of $\mathrm{ECM}(\xi, \xi^2)$, we can estimate $\delta_1 \geq 1/2$. It follows from (5.1) that we have

$$G(\Sigma) \geq \frac{2\log E_1}{\log E_2} - \frac{\log E_2}{\log E_1}.$$

For instance, according to our numerical calculations, considering the case $a = 3, b = 2$ where

$$\lambda_*(\xi^2) = \lambda^*(\xi^2) \sim \lambda_0 = 3.27582...., \quad \lambda_*(\xi) = \lambda^*(\xi) = \lambda_1 \neq \lambda_0,$$

we have

$$\log E_1 = \log \lambda_1 \sim 1.0705..., \quad \log E_2 = \log \lambda_0 \sim 1.18657...$$

Then we obtain a strictly positive gap value:

$$G(\Sigma) \geq 0.6959... > 0.$$

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