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VISCOSITY APPROXIMATION METHODS FOR FIXED POINTS PROBLEMS

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1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H. Then, a mapping $T:C\to C$ is called nonexpansive if $\|Tx-Ty\|\leq\|x-y\|$ for all $x,y\in C$. We denote by $F(T)$ the set of fixed points of T. Let $\{T_n\}$ be a family of nonexpansive mappings of C into itself and let F be the set of common fixed points of $\{T_n\}$, i.e., $F=\cap_{n=1}^{\infty}F(T_n)$. Browder [3] introduced the following iterations and proved strong convergence theorem:

$$u_n=\alpha_n u+(1-\alpha_n)Tu_n \quad \text{for every} \quad n=1,2,\ldots. \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ converging to 0, and $u\in C$. Reich [14] and Takahashi and Ueda [20] extended Browder's result to those of a Banach space. Wittmann [23] obtained a strong convergence theorem in Hilbert spaces by using the iteration procedure which was initially introduced by Halpern [6]:

$$\begin{aligned} &x_1\in C \quad \text{and} \\ &x_{n+1}=\alpha_n x_1+(1-\alpha_n)Tx_n, \quad n=1,2,\ldots, \end{aligned} \tag{1.2}$$

where $\alpha_n\in[0,1]$ (see [23, 18] for the proof). Moudafi[9] generalize Browder's and Halpern's theorems [3, 6]. Moudafi's generalizations are called viscosity approximations. Xu extend Moudafi's theorems toe uniformly smooth Banach spaces (see also [19]). Petrusel and Yao [12] studied viscosity approximations with generalized contraction mappings and nonexpansive mappings, and they proved strong convergence theorems for the mappings. Wangkeeree [22] studied viscosity approximations with nonself nonexpansive mappings and proved strong convergence theorems for the mappings. On the other hand, Cho and Kang [4] studied implicit viscosity approximations for pseudocontractive semigroups and proved strong convergence theorems for the semigroups (see also [15]).

In this paper, we study implicit and explicit viscosity approximations with generalized contraction mappings and nonself nonexpansive mappings. We prove strong convergence theorems for the nonself nonexpansive mappings. Further, we study implicit and explicit viscosity approximations with generalized contraction mappings and pseudocontractive semigroups, and prove strong convergence theorems for the pseudocontractive semigroups.

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2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the set of all positive integers, the set of all real numbers, respectively. We also denote by $\mathbb{R}G^+$ the set of all nonnegative real numbers. Let E be a real Banach space with norm $\|\cdot\|$. We denote by B_r the set $\{x \in E : \|x\| \leq r\}$. Let E^* be the dual space of a Banach space E. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$. Let E be a real Banach space and let C be a nonempty closed convex subset of E. We denote by I the identity operator on E. The multi-valued mapping J from E into E^* defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \quad \text{for every} \quad x \in E$$

is called the duality mapping of E. From the Hahn-Banach theorem, we see that $J(x) \neq \emptyset$ for all $x \in E$.

A Banach space E is said to be strictly convex if

$$\frac{\|x+y\|}{2} < 1$$

for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space, we have that if $\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\|$ for $x, y \in E$ and $\lambda \in (0,1)$, then $x = y$. For every ε with $0 \leq \varepsilon \leq 2$, we define the modulus $\delta(\varepsilon)$ of convexity of E by

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} \ : \ \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\right\}.$$

A Banach space E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If E is uniformly convex, then for r, ε with $r \geq \varepsilon > 0$, we have $\delta\left(\frac{\varepsilon}{r}\right) > 0$ and

$$\left\|\frac{x+y}{2}\right\| \leq r\left(1 - \delta\left(\frac{\varepsilon}{r}\right)\right)$$

for every $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x-y\| \geq \varepsilon$. It is well-known that a uniformly convex Banach space is reflexive and strictly convex. Banach space E is said to be smooth if

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each x and y in S_1, where $S_1 = \{u \in E : \|u\| = 1\}$. The norm of E is said to be uniformly Gâteaux differentiable if for each y in S_1, the limit is attained uniformly for x in S_1. We know that if E is smooth, then the duality mapping is single-valued and norm to weak star continuous and that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping is single-valued and norm to weak star, uniformly continuous on each bounded subset of E.

Let μ be a mean on positive integers $\mathbb{N}$, i.e., a continuous linear functional on l^∞ satisfying $\|\mu\| = 1 = \mu(1)$. We know that μ is a mean on $\mathbb{N}$ if and only if

$$\inf\{a_n : n \in \mathbb{N}\} \leq \mu(f) \leq \sup\{a_n : n \in \mathbb{N}\}$$

for each $f = (a_1, a_2, \ldots) \in l^\infty$. Occasionally, we use $\mu_n(a_n)$ instead of $\mu(f)$. So, a Banach limit μ is a mean on $\mathbb{N}$ satisfying $\mu_n(a_n) = \mu_n(a_{n+1})$. Let $f = (a_1, a_2, \ldots) \in l^\infty$

and let μ be a Banach limit on $\mathbb{N}$. Then,

$$\varliminf_{n\to\infty} a_n \le \mu(f) = \mu_n(a_n) \le \varlimsup_{n\to\infty} a_n.$$

Specially, if $a_n \to a$, then $\mu(f) = \mu_n(a_n) = a$ (see [17, 18]).

Let E be a real Banach space and let C be a nonempty closed convex subset of E. Then, a mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. A function ψ: $\mathbb{R}G^+ \to \mathbb{R}G^+$ is said to be L-function if $\psi(0) = 0$, $\psi(t) > 0$ for $t > 0$ and for any $s > 0$, there exists $u > s$ such that $\psi(t) \le s$ for $t \in [s, u]$. A mapping f from E into E is said to be (ψ, L)-contraction if $\psi : \mathbb{R}G^+ \to \mathbb{R}G^+$ is L-function and $\|f(x) - f(x)\| < \psi(\|x - y\|)$ for all $x, y \in E$ with $x \ne y$. A mapping f: $C \to C$ is said to be Meir-Keeler type mapping if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for any $x, y \in E$ with $\|x - y\| < \varepsilon + \delta \quad \|f(x) - f(y)\| < \varepsilon$ (see [10]). If f is k-contractive, then f is a Meir-Keeler type mapping and (ϕ, L)-contraction. By a generalized contraction mapping we mean a Meir-Keeler type mapping or (ϕ, L)-contraction (see [2, 8, 10, 12, 13, 16]).

3. Strong convergence theorems for nonself mappings

In this section, we study implicit and explicit viscosity approximations with generalized contraction mappings and nonself nonexpansive mappings (see [1]). Now we can prove a strong convergence theorem by an implicit viscosity approximation method (see [1]).

Theorem 3.1. Let E be a uniformly convex Banach space which admits a weakly sequentially continuous duality mapping J from E to E^*. Let C be a nonempty closed convex subset of E. Suppose that C is a sunny nonexpansive retract of E. Let P be a sunny nonexpansive retraction of E onto C, let T be a nonself nonexpansive mapping of C into E such that $F(T) \ne \emptyset$ and let f be a generalized contraction mapping. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < \alpha_n < 1$ and $\lim_{n\to\infty} \alpha_n = 0$. If $\{x_n\}$ is given by

$$x_n = \frac{1}{n}\sum_{j=1}^{n} P(\alpha_n f(x_n) + (1 - \alpha_n)(TP)^j x_n)$$

for every $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to $p \in F(T)$. Further, p is the unique solution of the variational inequality :

$$\langle (f - I)p, j(u - p)\rangle \le 0$$

for all $u \in F(T)$.

We can prove a strong convergence theorem by an explicit viscosity approximation method (see [1]).

Theorem 3.2. Let E be a uniformly convex Banach space which admits a weakly sequentially continuous duality mapping J from E to E^*. Let C be a nonempty closed convex subset of E. Suppose that C is a sunny nonexpansive retract of E. Let P be a sunny nonexpansive retraction of E onto C, let T be a nonself nonexpansive mapping of C into E such that $F(T) \neq \emptyset$ and let f be a generalized contraction mapping. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < \alpha_n < 1$, $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $\{x_n\}$ is given by $x_1 = x \in C$ and

$$x_{n+1} = \frac{1}{n}\sum_{j=1}^{n} P(\alpha_n f(x_n) + (1-\alpha_n)(TP)^j x_n)$$

for every $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to $p \in F(T)$. Further, p is the unique solution of the variational inequality :

$$\langle (f - I)p, j(u - p)\rangle \leq 0$$

for all $u \in F(T)$.

We also have a strong convergence theorem by an explicit viscosity approximation method (see [1]).

Theorem 3.3. Let E be a uniformly convex Banach space which admits a weakly sequentially continuous duality mapping J from E to E^*. Let C be a nonempty closed convex subset of E. Suppose that C is a sunny nonexpansive retract of E. Let P be a sunny nonexpansive retraction of E onto C, let T be a nonself nonexpansive mapping of C into E such that $F(T) \neq \emptyset$ and let f be a generalized contraction mapping. Let $\{\alpha_n\}$ a sequence of real numbers such that $0 < \alpha_n < 1$, $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $\{x_n\}$ is given by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n$$

for every $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to $p \in F(T)$. Further, p is the unique solution of the variational inequality :

$$\langle (f - I)p, j(u - p)\rangle \leq 0$$

for all $u \in F(T)$.

4. STRONG CONVERGENCE THEOREMS FOR PSEUDOCONTRACTIVE SEMIGROUPS

In this section, we study implicit and explicit viscosity approximations with L-Lipschitz semigroup pseudocontraction on C. We prove strong convergence theorems for the L-Lipschitz semigroup pseudocontraction.

A mapping $T : C \to C$ is called pseudocontractive if there exists some $j(x-y) \in J(x-y)$ such that $\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2$ for all $x, y \in C$. A mapping $T : C \to C$ is called strongly pseudocontractive if there exists a constant $\alpha \in (0,1)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \alpha\|x-y\|^2 \quad (x, y \in C)$$

for some $j(x-y) \in J(x-y)$. A mapping $T : C \to C$ is said to be Lipschitz if there exists a constant $L > 0$ such that $\|Tx - Ty\| \leq L\|x-y\|$ for all $x, y \in C$. If $L = 1$, then T is said to be nonexpansive. Deimling [5] proved the following theorem.

Theorem 4.1 ([5]). Let E be a Banach space, let C be a nonempty closed convex subset of E. Let T be a continuous and strong pseudocontractive mapping. Then, T has a unique fixed point of T.

A family $\mathcal{S} = \{T(t) : t \geq 0\}$ of mappings of C into itself is said to be a pseudocontraction semigroup on C.

(i) $T(0)x = x$ for all $x \in C$;
(ii) $T(t+s) = T(t)T(s)$ for each $t, s \in S$;
(iii) $\lim_{t\to 0} T(t)x = x$ for all $x \in C$;
(iv) for each $t \in S$, $T(t)$ is a pseudocontractive mapping of C into itself, that is,

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2$$

for all $x, y \in C$.

We denote by $F(\mathcal{S})$ the set of common fixed points of $\mathcal{S}$, i.e., $F(\mathcal{S}) = \bigcap_{t\geq 0} F(T(t))$. Note that the class of pseudocontraction semigroups includes the class of nonexpansive semigroups. Now, we can prove a strong convergence theorem by an implicit viscosity approximation method (see [1]).

Theorem 4.2. Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let C be a nonempty closed convex subset of E. Let $\mathcal{S} = \{T(t) : t \geq 0\}$ be a strongly continuous, and L-Lipschitz semigroup of pseudocontractions of C into itself such that $F(\mathcal{S}) \neq \emptyset$. Let f be a generalized contraction mapping. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \alpha_n < 1, t_n > 0$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0$. Let μ be a Banach limit. Let $\{x_n\}$ be a sequence defined by

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n$$

for every $n \in \mathbb{N}$. Assume that $\mu_n\|T(t)x_n - T(t)z\| \leq \mu_n\|x_n - z\|$ for each $z \in K$ and $t \geq 0$, where $K = \{z \in C : \mu_n\|x_n - z\|^2 = \min_{x\in C}\mu_n\|x_n - x\|^2\}$. Then, $\{x_n\}$ converges strongly to $p \in F(\mathcal{S})$. Further, p is the unique solution of the variational inequality :

$$\langle (f-I)p, j(u-p)\rangle \leq 0$$

for all $u \in F(\mathcal{S})$.

Now we can prove a strong convergence theorem by an explicit viscosity approximation method (see [1]).

Theorem 4.3. Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let C be a nonempty closed convex subset of E. Let $\mathcal{S} = \{T(t) : t \geq 0\}$ be a strongly continuous, and nonself L-Lipschitz semigroup of pseudocontractions of C into itself such that $F(\mathcal{S}) \neq \emptyset$. Let f be a generalized contraction mapping. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \alpha_n < 1, t_n > 0$, $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let μ be a Banach limit. Let $\{x_n\}$ be a sequence defined by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n$$

for every $n \in \mathbb{N}$. Assume that $\mu_n\|T(t)x_n - T(t)z\| \leq \mu_n\|x_n - z\|$ for each $z \in K$ and $t \geq 0$, where $K = \{z \in C : \mu_n\|x_n - z\|^2 = \min_{x \in C} \mu_n\|x_n - x\|^2\}$. Then, $\{x_n\}$ converges strongly to $p \in F(\mathcal{S})$. Further, p is the unique solution of the variational inequality :

$$\langle (f - I)p, j(u - p) \rangle \leq 0$$

for all $u \in F(\mathcal{S})$.

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