Some geometric constants related with the modulus of convexity of a Banach space

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We shall consider the constant $C_f(X)$ for a Banach space $X$, where $f(u, v)$ is a real valued continuous function which is non-decreasing in $u$ and $v$ in $[0,2]$. Some geometric constants of $X$ are unifyingly described by this constant $C_f(X)$ with a suitable $f$ and some previous results are derived.

Let $X$ be a real Banach space with $\dim X \geq 2$. The *modulus of convexity* of $X$ is defined by

$$\delta_X(\epsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : x, y \in S_X, \left\| x - y \right\| = \epsilon \right\} \quad (0 \leq \epsilon \leq 2),$$

where $S_X$ is the unit sphere of $X$. $S_X$ may be replaced by the unit ball $B_X$. The function $\delta_X$ is continuous on $[0,2)$, increasing on $[0,2]$ and strictly increasing on $[\epsilon_0,2]$, where $\epsilon_0 = \epsilon_0(X) = \sup \{ \epsilon \in [0,2] : \delta_X(\epsilon) = 0 \}$ is the *coefficient of convexity* of $X$. The function $\delta_X(\epsilon)/\epsilon$ is also increasing on $(0,2]$ (Figiel, 1976).

The *James constant* of $X$ is defined by

$$J(X) = \sup \{ \min(\|x + y\|, \|x - y\|) : x, y \in S_X \}.$$  

$X$ is called *uniformly non-square* if $J(X) < 2$. It is well-known that $X$ is uniformly non-square if and only if $\epsilon_0(X) < 2$. If $J(X) < 2$, we have

$$J(X) = 2(1 - \delta_X(J(X)))$$

(Casini [4]).
In this note we shall consider the following constant: Let \( f(u, v) \) be a real valued continuous function satisfying \( f(u_1, v_1) \leq f(u_2, v_2) \) for all \( 0 \leq u_1 \leq u_2 \leq 2 \) and \( 0 \leq v_1 \leq v_2 \leq 2 \). We define the constant \( C_f(X) \) to be

\[
C_f(X) = \sup \left\{ f(\|x - y\|, \|x + y\|) : x, y \in S_X \right\}.
\] (1)

One should note that

\[
\begin{align*}
J(X) &= C_f(X) \quad \text{if } f(u, v) = \min(u, v), \\
A_2(X) &= C_f(X) \quad \text{if } f(u, v) = (u + v)/2, \text{
} \\
T(X) &= C_f(X) \quad \text{if } f(u, v) = \sqrt{uv}, \text{
} \\
C_{NJ}'(X) &= C_f(X) \quad \text{if } f(u, v) = (u^2 + v^2)/4.
\end{align*}
\]

We recall the definitions of these constants. The constant \( A_2(X) \) ([3]) is given by

\[
A_2(X) := \rho_X(1) + 1,
\]

where \( \rho_X(\tau) \) is the modulus of smoothness of \( X \),

\[
\rho_X(\tau) = \sup \left\{ \frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : x, y \in S_X \right\} \quad (\tau > 0).
\]

The constant \( T(X) \) is defined in [1] by

\[
T(X) := \sup \{ \sqrt{\|x - y\| \|x + y\|} : x, y \in S_X \}.
\]

The von Neumann-Jordan constant of \( X \) is

\[
C_{NJ}(X) := \sup \left\{ \frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \text{ are not both } 0 \right\},
\] (2)

where the supremum can be taken over all \( x \in S_X \) and \( y \in B_X \). The constant defined by taking supremum over all \( x, y \in S_X \) in (2) is denoted by \( C_{NJ}'(X) \) ([2]). We have \( C_{NJ}'(X) \leq C_{NJ}(X) \) and they do not coincide in general.

It is readily seen that

\[
C_f(X) = \sup \left\{ f(\varepsilon, 2(1 - \delta_X(\varepsilon)) : 0 < \varepsilon < 2 \right\}.
\] (3)
With regard to a lower bound of $C_f(X)$ we easily have
\[ C_f(X) \geq \max \left\{ f(J(X), J(X)), f(\epsilon_0(X), 2) \right\}. \quad (4) \]

In particular we have $C_f(X) = f(2, 2)$ if $J(X) = 2$. It follows from (4) that
\[ T(X) \geq \sqrt{2\epsilon_0(X)} \] ([1]) and $C_{NJ}'(X) \geq 1 + \epsilon_0(X)^2 / 4$ ([2]), where we have equality
in both inequalities if $X$ is not uniformly non-square.

**Theorem 1.** Let $J(X) < 2$ and assume that $f(u, v) = f(v, u)$ for all $u, v \in [0, 2]$. Then
\[ C_f(X) = \sup \left\{ f(\epsilon, 2(1 - \delta_X(\epsilon)) : J(X) \leq \epsilon < 2 \right\}. \quad (5) \]

We shall present some applications of (5): Let $J(X) < 2$. Then
\[ \rho_X(1) = \sup \left\{ \frac{\epsilon}{2} - \delta_X(\epsilon) : J(X) \leq \epsilon < 2 \right\} \leq 2 \left( 1 - \frac{1}{J(X)} \right) \quad (6) \]
and
\[ C_{NJ}'(X) = \sup \left\{ \frac{\epsilon^2}{4} + (1 - \delta_X(\epsilon))^2 : J(X) \leq \epsilon < 2 \right\} \leq 1 + 4 \left( 1 - \frac{1}{J(X)} \right)^2. \quad (7) \]

We shall give simple proofs of (6) and (7). We write $J$ and $\delta(\epsilon)$ for $J(X)$ and $\delta_X(\epsilon)$ respectively. Since $\delta(\epsilon)/\epsilon$ is increasing, $\delta(\epsilon) \geq \delta(J)\epsilon/J$ for all $J \leq \epsilon < 2$. Noting $2\delta(J) = 2 - J$ we have
\[ \frac{\epsilon}{2} - \delta(\epsilon) \leq \frac{\epsilon}{2} - \delta(J)\epsilon/J \leq 1 - 2\delta(J)/J = 1 - (2 - J)/J = 2(1 - 1/J), \]
which proves (6). Similarly we have
\[ \frac{\epsilon^2}{4} + (1 - \delta_X(\epsilon))^2 \leq \frac{\epsilon^2}{4} + (1 - \delta(J)\epsilon/J)^2 \leq 1 + (1 - 2\delta(J)/J)^2 = 1 + 4(1 - 1/J)^2, \]
which proves (7).

In 2008 Alonso et al. [2] showed that
\[ C_{NJ}'(X) \leq J(X), \]
which is useful to estimate the von Neumann-Jordan constant $C_{NJ}(X)$ by $J(X)$.

It was shown in [2] that
\[ C_{NJ}(X) \leq 1 + (\sqrt{2C_{NJ}'(X)} - 1)^2 \leq 1 + (\sqrt{2J(X)} - 1)^2, \]
while by using (7) we easily have
\[ C'_{NJ}(X) \leq 1 + 4(1 - 1/J(X))^2 \leq (1 + \sqrt{J(X) - 1})^2/2, \]
which yields that
\[ C_{NJ}(X) \leq 1 + (\sqrt{2C'_{NJ}(X)} - 1)^2 \leq J(X) \]
(Kato-Takahashi [6]; see also [8], [9]). The simple inequality
\[ C_{NJ}(X) \leq J(X) \] (8)
concerning the von Neumann-Jordan and James constants was first proved by Takahashi and Kato [7] in 2009, which answered affirmatively a question posed in Alonso et al. [2]. In [7] they proved (8) as
\[ C_{NJ}(X) \leq \frac{2}{2 - \rho_X(1)} \leq J(X), \]
where the second inequality is equivalent to (6).

References


