A topology in a vector lattice and fixed point theorems for nonexpansive mappings

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Abstract

In the previous paper [4] we show Takahashi's and Fan-Browder's fixed point theorems in a vector lattice and in the previous paper [5] we show Schauder-Tychonoff's fixed point theorem using Fan-Browder's fixed point theorem. The purpose of this paper is to introduce a topology in a vector lattice and to show a fixed point theorem for a nonexpansive mapping and also common fixed point theorems for commutative family of nonexpansive mappings in a vector lattice.

1 Introduction

There are many fixed point theorems in a topological vector space, for instance, Kirk's fixed point theorem in a Banach space, and so on; see for example [8].

In this paper we consider fixed point theorems in a vector lattice. As known well every topological vector space has a linear topology. On the other hand, although every vector lattice does not have a topology, it has two lattice operators, which are the supremum $\vee$ and the infimum $\wedge$, and also an order is introduced from these operators; see also [6,9] about vector lattices. There are some methods how to introduce a topology to a vector lattice. One method is to assume that the vector lattice has a linear topology [1]. On the other hand, there is another method to make up a topology in a vector lattice, for instance, in [2] one method is introduced in case of the vector lattice with unit.

In the previous paper [4] we show Takahashi's and Fan-Browder's fixed point theorems in a vector lattice and in the previous paper [5] we show Schauder-Tychonoff's fixed point theorem using Fan-Browder's fixed point theorem. The purpose of this paper is to introduce a topology in a vector lattice and to show a fixed point theorem for a nonexpansive mapping and also common fixed point theorems for commutative family of nonexpansive mappings in a vector lattice.

2 Topology in a vector lattice

First we introduce a topology in a vector lattice introduced by [2]; see also [4,5].
Let $X$ be a vector lattice. $e \in X$ is said to be an unit if $e \land x > 0$ for any $x \in X$ with $x > 0$. Let $\mathcal{K}_X$ be the class of units of $X$. In case where $X$ is the set of real numbers $\mathbb{R}$, $\mathcal{K}_R$ is the set of positive real numbers. Let $X$ be a vector lattice with unit and let $Y$ be a subset of $X$. $Y$ is said to be open if for any $x \in Y$ and for any $e \in \mathcal{K}_X$ there exists $\varepsilon \in \mathcal{K}_R$ such that $[x - \varepsilon e, x + \varepsilon e] \subset Y$. Let $O_X$ be the class of open subsets of $X$. $Y$ is said to be closed if $\overline{Y^c} \in O_X$. For $e \in \mathcal{K}_X$ and for an interval $[a, b]$ we consider the following subset

$$[a, b]^e = \{x \mid \text{there exists some } \varepsilon \in \mathcal{K}_R \text{ such that } x - a \geq \varepsilon e \text{ and } b - x \geq \varepsilon e\}.$$ 

By the definition of $[a, b]^e$ it is easy to see that $[a, b]^e \subset [a, b]$. Every mapping from $X \times \mathcal{K}_X$ into $(0, \infty)$ is said to be a gauge. Let $\Delta_X$ be the class of gauges in $X$. For $x \in X$ and $\delta \in \Delta_X$, $O(x, \delta)$ is defined by

$$O(x, \delta) = \bigcup_{\varepsilon \in \mathcal{K}_X} [x - \delta(x, e)e, x + \delta(x, e)e]^e.$$ 

$O(x, \delta)$ is said to be a $\delta$-neighborhood of $x$. Suppose that for any $x \in X$ and for any $\delta \in \Delta_X$ there exists $U \in O_X$ such that $x \in U \subset O(x, \delta)$.

For a subset $Y$ of $X$ we denote by $\text{cl}(Y)$ and $\text{int}(Y)$, the closure and the interior of $Y$, respectively. Let $X$ and $Y$ be vector lattices with unit, $x_0 \in Z \subset X$ and $f$ a mapping from $Z$ into $Y$. $f$ is said to be continuous in the sense of topology at $x_0$ if for any $V \in O_Y$ with $f(x_0) \in V$ there exists $U \in O_X$ with $x_0 \in U$ such that $f(U \cap Z) \subset V$.

Let $X$ be a vector lattice with unit. $X$ is said to be Hausdorff if for any $x_1, x_2 \in X$ with $x_1 \neq x_2$ there exists $O_1, O_2 \in O_X$ such that $x_1 \in O_1, x_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$. A subset $Y$ of $X$ is said to be compact if for any open covering of $Y$ there exists a finite sub-covering. A subset $Y$ of $X$ is said to be normal if for any closed subsets $F_1$ and $F_2$ with $F_1 \cap F_2 \cap Y = \emptyset$ there exists $O_1, O_2 \in O_X$ such that $F_1 \subset O_1, F_2 \subset O_2$ and $O_1 \cap O_2 \cap Y = \emptyset$.

A vector lattice is said to be Archimedean if it holds that $x = 0$ whenever there exists $y \in X$ with $y \geq 0$ such that $0 \leq rx \leq y$ for any $r \in \mathcal{K}_R$.

Let $X$ be a vector lattice with unit and $Y$ a vector lattice, $x_0 \in Z \subset X$ and $f$ a mapping from $Z$ into $Y$. $f$ is said to be continuous at $x_0$ if there exists $\{v_e \mid e \in \mathcal{K}_X\}$ satisfying the conditions (U1), (U2)$^d$, and (U3)$^*$ such that for any $e \in \mathcal{K}_X$ there exists $\delta \in \mathcal{K}_R$ such that for any $x \in Z$ if $|x - x_0| \leq \delta e$, then $|f(x) - f(x_0)| \leq v_e$; where

(U1) \hspace{0.5cm} $v_e \in Y$ with $v_e > 0$;

(U2)$^d$ \hspace{0.5cm} $v_{e_1} \geq v_{e_2}$ if $e_1 \geq e_2$;

(U3)$^*$ \hspace{0.5cm} For any $e \in \mathcal{K}_X$ there exists $\theta(e) \in \mathcal{K}_R$ such that $v_{\theta(e)e} \leq \frac{1}{2} v_e$.

Let $X$ be an Archimedean vector lattice. Then there exists a positive homomorphism $f$ from $X$ into $\mathbb{R}$, that is, $f$ satisfies the following conditions:

(H1) \hspace{0.5cm} $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for any $x, y \in X$ and for any $\alpha, \beta \in \mathbb{R}$;

(H2) \hspace{0.5cm} $f(x) \geq 0$ for any $x \in X$ with $x \geq 0$;
see [5, Example 3.1]. Suppose that there exists a homomorphism \( f \) from \( X \) into \( \mathbb{R} \) satisfying the following condition instead of (H2):

\[(H2)^* \quad f(x) > 0 \text{ for any } x \in X \text{ with } x > 0.\]

**Example 2.1.** We consider of a sufficient condition to satisfy \((H2)^*\). Let \( X \) be a Hilbert lattice with unit, that is, \( X \) has an inner product \( \langle \cdot, \cdot \rangle \) and for any \( x, y \in X \) if \( |x| \leq |y| \), then \( \langle x, x \rangle \leq \langle y, y \rangle \). For any \( e \in \mathcal{K}_X \) let \( f \) be a function from \( X \) into \( \mathbb{R} \) defined by \( f(x) = \langle x, e \rangle \). Then \( f \) satisfies (H1) and \((H2)^*\) clearly.

## 3 Fixed point theorem for a nonexpansive mapping

Let \( X \) be a vector lattice and \( Y \) a subset of \( X \). A mapping \( f \) from \( Y \) into \( Y \) is said to be nonexpansive if \( |f(x) - f(y)| \leq |x - y| \) for any \( x, y \in Y \). In this section we consider a fixed point theorem for a nonexpansive mapping.

**Lemma 3.1.** Let \( X \) be a Hausdorff Archimedean vector lattice with unit and \( K \) a non-empty compact convex subset of \( X \). Then

\[
c(K) = \left\{ x \in K \mid \bigvee_{y \in K} |x - y| = \bigwedge_{x \in K} \bigvee_{y \in K} |x - y| \right\}
\]

is non-empty compact convex.

**Proof.** For any \( x \in K \) and for any \( e \in \mathcal{K}_X \) let

\[
F(x, e) = \left\{ y \in K \mid |x - y| \leq \bigwedge_{x \in K} \bigvee_{y \in K} |x - y| + e \right\}.
\]

Then \( F(x, e) \) is non-empty compact convex. Let \( C(e) = \bigcap_{x \in K} F(x, e) \). Since \( \bigcap_{i=1}^{n} F(x_i, e) \neq \emptyset \) for any \( x_1, \ldots, x_n \in K \), \( C(e) \) is non-empty compact convex. Since \( C(e_1) \supset C(e_2) \) for any \( e_1, e_2 \in \mathcal{K}_X \) with \( e_1 \geq e_2 \), \( \bigcap_{e \in \mathcal{K}_X} C(e) \) is non-empty compact convex. Moreover \( c(K) = \bigcap_{e \in \mathcal{K}_X} C(e) \). Indeed \( c(K) \subset \bigcap_{e \in \mathcal{K}_X} C(e) \) is clear. Let \( x \in C(e) \) for any \( e \in \mathcal{K}_X \). Then

\[
|x - y| \leq \bigwedge_{x \in K} \bigvee_{y \in K} |x - y| + e
\]

for any \( y \in K \). Therefore

\[
\bigvee_{y \in K} |x - y| \leq \bigwedge_{x \in K} \bigvee_{y \in K} |x - y| + \bigwedge_{e \in \mathcal{K}_X} e = \bigwedge_{x \in K} \bigvee_{y \in K} |x - y|.
\]

By definition

\[
\bigvee_{y \in K} |x - y| \geq \bigwedge_{x \in K} \bigvee_{y \in K} |x - y|.
\]

Therefore

\[
\bigvee_{y \in K} |x - y| = \bigwedge_{x \in K} \bigvee_{y \in K} |x - y|.
\]
Therefore
\[ \bigvee_{y \in K} |x - y| = \bigwedge_{z \in K} \bigvee_{y \in K} |x - y|, \]
that is, \( x \in c(K) \).

Let \( X \) be a Hausdorff Archimedean vector lattice with unit and \( Y \) a subset of \( X \). We say that \( Y \) has the normal structure if for any compact convex subset \( K \), which contains two points at least, of \( Y \) there exists \( x \in K \) such that
\[ \bigvee_{y \in K} |x - y| < \bigvee_{x,y \in K} |x - y|. \]

**Lemma 3.2.** Let \( X \) be a Hausdorff Archimedean vector lattice with unit and \( K \) a non-empty compact convex subset, which contains two points at least, of \( X \). Suppose that \( K \) has the normal structure. Then
\[ \bigvee_{x,y \in c(K)} |x - y| < \bigvee_{x,y \in K} |x - y|. \]

**Proof.** Since \( K \) has the normal structure, there exists \( z \in K \) such that
\[ |x - y| \leq \bigvee_{y \in K} |x - y| = \bigwedge_{z \in K} \bigvee_{y \in K} |x - y| \leq \bigvee_{y \in K} |z - y| < \bigvee_{x,y \in K} |x - y| \]
for any \( x, y \in c(K) \). Therefore
\[ \bigvee_{x,y \in c(K)} |x - y| < \bigvee_{x,y \in K} |x - y|. \]

**Theorem 3.3.** Let \( X \) be a Hausdorff Archimedean vector lattice with unit and \( K \) a non-empty compact convex subset of \( X \). Suppose that \( K \) has the normal structure. Then every nonexpansive mapping from \( K \) into \( K \) has a fixed point.

**Proof.** Let \( f \) be a nonexpansive mapping from \( K \) into \( K \) and \( \{ K_{\lambda} \mid \lambda \in \Lambda \} \) the family of non-empty compact convex subsets of \( K \) satisfying that \( f(K_{\lambda}) \subset K_{\lambda} \). By Zorn's lemma there exists a minimal element \( K_0 \) of \( \{ K_{\lambda} \mid \lambda \in \Lambda \} \). Assume that \( K_0 \) contains two points at least. By Lemma 3.1 \( c(K_0) \) is non-empty compact convex. Let \( x \in c(K_0) \). For any \( y \in K_0 \)
\[ |f(x) - f(y)| \leq |x - y| \leq \bigvee_{y \in K_0} |x - y| = \bigwedge_{x \in K_0} \bigvee_{y \in K_0} |x - y|. \]

Let
\[ M = \left\{ y \in K \mid f(x) - y \leq \bigwedge_{x \in K_0} \bigvee_{y \in K_0} |x - y| \right\}. \]
Then \( f(K_0) \subset M \) and hence \( f(K_0 \cap M) \subset K_0 \cap M \). Since \( K_0 \) is a minimal element, it holds that \( K_0 \subset M \). Therefore

\[
\bigvee_{y \in K_0} |f(x) - y| \leq \bigwedge_{x \in K_0 \bigvee_{y \in K_0} |x - y|.
\]

By definition

\[
\bigvee_{y \in K_0} |f(x) - y| \geq \bigwedge_{x \in K_0 \bigvee_{y \in K_0} |x - y|.
\]

Therefore

\[
\bigvee_{y \in K_0} |f(x) - y| = \bigwedge_{x \in K_0 \bigvee_{y \in K_0} |x - y|,
\]

that is, \( f(x) \in c(K_0) \). Since \( K_0 \) is a minimal element, it holds that \( c(K_0) = K_0 \) and hence

\[
\bigvee_{x,y \in c(K_0)} |x - y| = \bigvee_{x,y \in K_0} |x - y|.
\]

However by Lemma 3.2

\[
\bigvee_{x,y \in c(K_0)} |x - y| < \bigvee_{x,y \in K_0} |x - y|.
\]

It is a contradiction. Therefore \( K_0 \) only contains a unique point. The point is a fixed point. \( \square \)

4 Fixed point theorem for the commutative family of nonexpansive mappings

For any nonexpansive mapping \( f \) from \( K \) into \( K \) let \( F_K(f) \) be the set of fixed points of \( f \).

**Lemma 4.1.** Let \( X \) be a Hausdorff Archimedean vector lattice with unit, \( Y \) a subset of \( X \) and \( f \) a nonexpansive mapping from \( Y \) into \( Y \). Suppose that there exists a homomorphism from \( X \) into \( \mathbb{R} \) satisfying the condition (H2)*. Then \( F_Y(f) \) is closed.

**Proof.** Assume that \( F_Y(f) \) is not closed. Then for any \( \delta \in \Delta_X \) there exists \( x \in F_Y(f)^C \) such that \( O(x, \delta) \not\subset F_Y(f)^C \). Take \( y_\delta \in O(x, \delta) \cap F_Y(f) \). Then \( f(y_\delta) = y_\delta \). Note that every nonexpansive mapping is continuous and hence by [5, Lemma 3.2] it is also continuous in the sense of topology. Since \( \{y_\delta \mid \delta \in \Delta_X\} \) is convergent to \( x \) in the sense of topology, \( \{f(y_\delta) \mid \delta \in \Delta_X\} \) is convergent to \( f(x) \) in the sense of topology. Since \( X \) is Hausdorff, \( f(x) = x \). It is a contradiction. Therefore \( F_Y(f) \) is closed. \( \square \)
Lemma 4.2. Let $X$ be a vector lattice. If $|x - z| = |x - w|$, $|y - z| = |y - w|$ and $|x - z| + |y - z| = |x - y|$, then $z = w$.

Proof. Note that $|a + b| = |a - b|$ if and only if $|a| \wedge |b| = 0$. Since

$$|x - z| = \left| x - \frac{1}{2}(z + w) - \frac{1}{2}(z - w) \right|$$

and

$$|x - w| = \left| x - \frac{1}{2}(z + w) + \frac{1}{2}(z - w) \right|,$$

it holds that $|x - \frac{1}{2}(z + w)| \wedge \frac{1}{2}|z - w| = 0$. In the same way it holds that $|y - \frac{1}{2}(z + w)| \wedge \frac{1}{2}|z - w| = 0$. Note that $(a + b) \wedge c \leq a \wedge b \wedge c$ for any $a, b, c \geq 0$. Therefore

$$|x - y| \wedge \frac{1}{2}|z - w| \leq \left( \left| x - \frac{1}{2}(z - w) \right| + \frac{1}{2}|z - w| \right) \wedge \frac{1}{2}|z - w|$$

$$\leq \left| x - \frac{1}{2}(z - w) \right| \wedge \frac{1}{2}|z - w| + \left| y - \frac{1}{2}(z + w) \right| \wedge \frac{1}{2}|z - w|$$

$$= 0.$$ 

Assume that $z \neq w$. Note that, if $|b| \wedge |c| = 0$, then $||a| - |b| \wedge |c| = |a| \wedge |c|$. Therefore

$$\left( |x - z| + |y - z| \right) \wedge \frac{1}{2}|z - w| \geq |x - z| \wedge \frac{1}{2}|z - w|$$

$$\geq \left| x - \frac{1}{2}|z - w| \right| - \frac{1}{2}|z - w| \wedge \frac{1}{2}|z - w|$$

$$= \frac{1}{2}|z - w| > 0.$$ 

It is a contradiction. Therefore $z = w$. \(\square\)

Lemma 4.3. Let $X$ be a Hausdorff Archimedean vector lattice with unit, $Y$ a subset of $X$ and $f$ a nonexpansive mapping from $Y$ into $Y$. Then $F_Y(f)$ is convex.

Proof. Let $x, y \in F_Y(f)$ and $0 \leq \alpha \leq 1$. Then

$$|x - f((1 - \alpha)x + \alpha y)| = |f(x) - f((1 - \alpha)x + \alpha y)|$$

$$\leq |x - ((1 - \alpha)x + \alpha y)| = \alpha|x - y|,$$

$$|y - f((1 - \alpha)x + \alpha y)| = |f(y) - f((1 - \alpha)x + \alpha y)|$$

$$\leq |y - ((1 - \alpha)x + \alpha y)| = (1 - \alpha)|x - y|.$$ 

Since

$$|x - y| \leq |x - f((1 - \alpha)x + \alpha y)| + |y - f((1 - \alpha)x + \alpha y)|$$

$$\leq |x - ((1 - \alpha)x + \alpha y)| + |y - ((1 - \alpha)x + \alpha y)| = |x - y|,$$
it holds that

\[|x - f((1 - \alpha)x + \alpha y)| = |x - ((1 - \alpha)x + \alpha y)|,\]
\[|y - f((1 - \alpha)x + \alpha y)| = |y - ((1 - \alpha)x + \alpha y)|,\]

and hence

\[|x - f((1 - \alpha)x + \alpha y)| + |y - f((1 - \alpha)x + \alpha y)| = |x - y|\]

By Lemma 4.2 \(f((1 - \alpha)x + \alpha y) = (1 - \alpha)x + \alpha y,\) that is, \(F_Y(f)\) is convex.

\[\square\]

**Theorem 4.4.** Let \(X\) be a Hausdorff Archimedean vector lattice with unit, \(K\) a compact convex subset of \(X\) and \(\{f_i \mid i = 1, \cdots, n\}\) the finite commutative family of nonexpansive mappings from \(K\) into \(K\). Suppose that there exists a homomorphism from \(X\) into \(R\) satisfying the condition (H2)* and \(K\) has the normal structure. Then \(\bigcap_{i=1}^{n} F_K(f_i)\) is non-empty.

**Proof.** Let \(\{K_\lambda \mid \lambda \in \Lambda\}\) be the family of non-empty compact convex subsets of \(K\) satisfying that \(f_i(K_\lambda) \subset K_\lambda\) for any \(i\). By Zorn's lemma there exists a minimal element \(K_0\) of \(\{K_\lambda \mid \lambda \in \Lambda\}\). Assume that \(K_0\) contains two points at least. By Theorem 3.3 \(F_{K_0}(f_1 \circ \cdots \circ f_n)\) is non-empty. Moreover by Lemma 4.1 and Lemma 4.3 \(F_{K_0}(f_1 \circ \cdots \circ f_n)\) is compact convex. It holds that \(f(F_{K_0}(f_1 \circ \cdots \circ f_n)) = F_{K_0}(f_1 \circ \cdots \circ f_n)\) for any \(i\). It is shown as follows. Let \(x \in F_{K_0}(f_1 \circ \cdots \circ f_n)\). Since

\[f_i(x) = f_i((f_1 \circ \cdots \circ f_n)(x)) = (f_1 \circ \cdots \circ f_n)(f_i(x))\]

for any \(i, f_i(x) \in F_{K_0}(f_1 \circ \cdots \circ f_n)\), that is, \(f_i(F_{K_0}(f_1 \circ \cdots \circ f_n)) \subset F_{K_0}(f_1 \circ \cdots \circ f_n)\). Next let \(x_i = (f_1 \circ \cdots \circ f_{i-1} \circ f_{i+1} \circ \cdots \circ f_n)(x)\). Since

\[(f_1 \circ \cdots \circ f_n)(x_i) = (f_1 \circ \cdots \circ f_{i-1} \circ f_{i+1} \circ \cdots \circ f_n)(x) = x_i,\]

it holds that \(x_i \in F_{K_0}(f_1 \circ \cdots \circ f_n)\). Moreover \(f_i(x_i) = x\). Therefore \(F_{K_0}(f_1 \circ \cdots \circ f_n) \subset F_i(F_{K_0}(f_1 \circ \cdots \circ f_n))\). Since \(K\) has the normal structure, there exists \(x_0 \in K_0\) such that

\[\bigvee_{y \in K_0} |x_0 - y| < \bigvee_{x, y \in K_0} |x - y|\]

Let

\[A = \left\{ x \mid x \in K_0, \bigvee_{y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x - y| \leq \bigvee_{y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x_0 - y| \right\} .\]

\(A\) is non-empty and convex clearly. Moreover since \(X\) is Archimedean, \(A\) is closed and hence compact. Let \(x \in A\). Then for any \(i\) and for any \(y \in F_{K_0}(f_1 \circ \cdots \circ f_n)\)

\[|f_i(x) - y| = |f_i(x) - f_i(y_i)| \leq |x - y_i| \leq \bigvee_{y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x - y| \leq \bigvee_{y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x_0 - y| \]
and hence $f_i(a) \in A$, that is, $f_i(A) \subset A$. Since $K_0$ is minimal, $A = K_0$. Therefore

$$\bigvee_{x,y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x - y| \leq \bigvee_{y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x_0 - y| < \bigvee_{x,y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x - y|.$$ 

It is a contradiction. Therefore $K_0$ only contains a unique point. The point is a common fixed point of $\{f_i \mid i = 1, \cdots, n\}$. \qed

**Theorem 4.5.** Let $X$ be a Hausdorff Archimedean vector lattice with unit, $K$ a compact convex subset of $X$ and $\{f_i \mid i \in I\}$ the commutative family of nonexpansive mappings from $K$ into $K$. Suppose that there exists a homomorphism from $X$ into $\mathbb{R}$ satisfying the condition $(H2)^\ast$ and $K$ has the normal structure. Then $\bigcap_{i \in I} F_K(f_i)$ is non-empty.

**Proof.** By Theorem 4.4 $\bigcap_{k=1}^{n} F_K(f_{i_k})$ is non-empty for any finite set $i_1, \cdots, i_n \in I$. Since $K$ is compact, $\bigcap_{i \in I} F_K(f_i)$ is non-empty. \qed

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**References**


