EXISTENCE THEOREMS FOR SADDLE POINTS OF SET-VALUED MAPS VIA NONLINEAR SCALARIZATION METHODS*

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Abstract

In the paper, we introduce five types of concepts for saddle points of set-valued maps and show existence theorems for these saddle points by using nonlinear scalarizing functions for sets introduced by Kuwano, Tanaka, and Yamada in 2009.

1 Introduction

Let $X$ and $Y$ be two real topological vector spaces, $F$ a map on $X \times Y$. In real-valued case, $(x_0, y_0) \in X \times Y$ is a saddle point of $F$ if

$$F(x_0, y) \leq F(x_0, y_0) \leq F(x, y_0)$$

for any $x \in X$ and $y \in Y$. In vector-valued case, a saddle point $(x_0, y_0) \in X \times Y$ with respect to partial ordering $\leq_C$ induced by a convex cone $C$ is defined by

$$F(x, y_0) \not\leq_C F(x_0, y_0) \not\leq_C F(x_0, y)$$

for any $x \in X$ and $y \in Y$, and it is called $C$-saddle point of $F$. Many researchers have been investigated existence theorems for saddle points and $C$-saddle points. In [7] and [8], we consider five types of generalizations for $C$-saddle points and investigate sufficient conditions for the existence of these saddle points by using nonlinear scalarization methods for sets proposed in [4].

The aim of the paper is to introduce three types of existence theorems for cone saddle points of set-valued maps.

The organization of the paper is as follows. In Section 2, we review mathematical methodology proposed in [3] on comparison between two sets in an ordered vector space and some basic concepts of set-valued optimization. In Section 3, we consider two types of nonlinear scalarizing functions for sets proposed by the unified approach in [4], and

* This work is based on research 21540121 supported by Grant-in-Aid for Scientific Research (C) from Japan Society for the Promotion of Science.
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2000 Mathematics Subject Classification. 49J53, 54C60, 90C46, 90C47.
Key words and phrases. Set-valued analysis, set-valued optimization, nonlinear scalarization, minimax theorems.
investigate their properties. In Section 4, we introduce five types of concepts for cone saddle points of set-valued maps, and three types of existence theorems for these saddle points proved in [7, 8].

2 Mathematical Preliminaries

Throughout the paper, $X$ and $Y$ are two real topological vector spaces and $C$ is a proper closed convex cone in $Y$ (that is, $C \neq Y$, $C + C = C$ and $\lambda C \subset C$ for all $\lambda \geq 0$) with nonempty topological interior. We define a partial ordering $\leq_c$ on $Y$ as follows:

$$x \leq_c y \quad \text{if} \quad y - x \in C \quad \text{for} \quad x, y \in Y.$$ 

Let $F$ be a set-valued map from $S \subset X$ into $2^Y$ where $S := \{x \in X | F(x) \neq \emptyset\}$ and assume that $S$ is a convex set. For $A \in 2^Y \setminus \{\emptyset\}$, we denote the topological interior of $A$ by $\text{int} A$. Also, we denote the algebraic sum, algebraic difference of $A$ and $C$ by $A + C := \bigcup_{a \in A} (a + C)$, $A - C := \bigcup_{a \in A} (a - C)$, respectively. In addition, we denote the composite function of two functions $f$ and $g$ by $g \circ f$. When $x \leq_c y$ for $x, y \in Y$, we define the order interval between $x$ and $y$ by $[x, y] := \{z \in Y | x \leq_c z \text{ and } z \leq_c y\}$.

At first, we review some basic concepts of set-relation.

**Definition 2.1.** (See Ref. [3].) For any $A, B \in 2^Y \setminus \{\emptyset\}$ and convex cone $C$ in $Y$, we write

$$A \leq^{(1)}_C B \text{ by } A \subset \bigcap_{b \in B} (b - C), \text{ equivalently } B \subset \bigcap_{a \in A} (a + C),$$

$$A \leq^{(2)}_C B \text{ by } A \cap \left( \bigcap_{b \in B} (b - C) \right) \neq \emptyset,$$

$$A \leq^{(3)}_C B \text{ by } B \subset (A + C),$$

$$A \leq^{(4)}_C B \text{ by } \left( \bigcap_{a \in A} (a + C) \right) \cap B \neq \emptyset,$$

$$A \leq^{(5)}_C B \text{ by } A \subset (B - C),$$

$$A \leq^{(6)}_C B \text{ by } A \cap (B - C) \neq \emptyset, \text{ equivalently } (A + C) \cap B \neq \emptyset.$$ 

**Proposition 2.1.** (See [3].) For any $A, B \in 2^Y \setminus \{\emptyset\}$, the following statements hold:

$$A \leq^{(1)}_C B \text{ implies } A \leq^{(2)}_C B,$$

$$A \leq^{(1)}_C B \text{ implies } A \leq^{(4)}_C B,$$

$$A \leq^{(2)}_C B \text{ implies } A \leq^{(3)}_C B,$$

$$A \leq^{(4)}_C B \text{ implies } A \leq^{(5)}_C B,$$

$$A \leq^{(3)}_C B \text{ implies } A \leq^{(6)}_C B,$$

$$A \leq^{(5)}_C B \text{ implies } A \leq^{(6)}_C B.$$ 

**Proposition 2.2.** (See [4].) For any $A, B \in 2^Y \setminus \{\emptyset\}$, the following statements hold:

(i) For each $j = 1, \ldots, 6$,

$$A \leq^{(j)}_C B \text{ implies } (A + y) \leq^{(j)}_C (B + y) \text{ for } y \in Y, \text{ and}$$

$$A \leq^{(j)}_C B \text{ implies } \alpha A \leq^{(j)}_C \alpha B \text{ for } \alpha \geq 0.$$ 

(ii) For each $j = 1, \ldots, 5$, $\leq^{(j)}_C$ is transitive.

(iii) For each $j = 3, 5, 6$, $\leq^{(j)}_C$ is reflexive.

From (b) and (c) of Proposition 2.2, $\leq^{(6)}_C$ is difficult to say as order. Hence, we consider mainly the cases of $j = 1, \ldots, 5$ in the paper.

By using the set-relations defined in Definition 2.1, we consider the following five kinds
of set-valued optimization problems with $j = 1, \ldots, 5$:

$$(j\text{-SVOP}) \quad \begin{cases} \text{j-Optimize } F(x) \\ \text{Subject to } x \in S. \end{cases}$$

Then, we introduce some concepts of solutions for $(j\text{-SVOP})$. Let $x_0 \in S$. For each $j = 1, \ldots, 5$, $x_0$ is a \textit{minimal solution} of $(j\text{-SVOP})$ if for any $x \in S \setminus \{x_0\}$,

$$F(x) \leq_C^{(j)} F(x_0) \quad \text{implies} \quad F(x_0) \leq_C^{(j)} F(x);$$

and $x_0$ is a \textit{maximal solution} of $(j\text{-SVOP})$ if for any $x \in S \setminus \{x_0\}$,

$$F(x_0) \leq_C^{(j)} F(x) \quad \text{implies} \quad F(x) \leq_C^{(j)} F(x_0).$$

If $C$ is replaced by $\text{int}C$, then $x_0$ is a \textit{weak minimal solution} (resp., \textit{weak maximal solution}) of $(j\text{-SVOP})$. We denote the family of sets satisfying (2.1) (resp., (2.2)) by $\text{Min}_{(j)} F(S)$ (resp., $\text{Max}_{(j)} F(S)$) and the case of weak minimal (resp., weak maximal) by $\text{WMin}_{(j)} F(S)$ (resp., $\text{WMax}_{(j)} F(S)$) where $F(S) = \{F(x) \mid x \in S\}$. It is clear that if $x_0$ is a minimal (resp., maximal) solution of $(j\text{-SVOP})$ then $x_0$ is a weak minimal (resp., weak maximal) solution of $(j\text{-SVOP})$.

Let us recall some definitions of $C$-notions (see [2].) A subset $A$ of $Y$ is said to be $C$-convex (resp., $C$-closed) if $A + C$ is convex (resp., closed). Moreover, we say that $F$ is $C$-\textit{notion} on $S$ if $F(x)$ has the property $C$-notion for every $x \in S$.

Next, we introduce several definitions of $C$-convexity and $C$-continuity for set-valued maps. These notions are used in Sections 3 and 4.

\textbf{Definition 2.2.} (See [4].) For each $j = 1, \ldots, 5$,

(i) $F$ is called a \textit{type $(j)$ naturally quasi $C$-convex function} if for each $x, y \in S$ and $\lambda \in (0, 1)$, there exists $\mu \in [0, 1]$ such that

$$F(\lambda x + (1 - \lambda)y) \leq_C^{(j)} \mu F(x) + (1 - \mu) F(y).$$

(ii) $F$ is called a \textit{type $(j)$ naturally quasi $C$-concave function} if for each $x, y \in S$ and $\lambda \in (0, 1)$, there exists $\mu \in [0, 1]$ such that

$$\mu F(x) + (1 - \mu) F(y) \leq_C^{(j)} F(\lambda x + (1 - \lambda)y).$$

\textbf{Definition 2.3.} (See [8].) For each $j = 1, \ldots, 5$,

(i) $F$ is called a \textit{type $(j)$ $C$-convexlike function} if for every $x, y \in S$ and $\lambda \in (0, 1)$, there exists $z \in S$ such that

$$F(z) \leq_C^{(j)} \lambda F(x) + (1 - \lambda) F(y).$$

(ii) $F$ is called a \textit{type $(j)$ $C$-concavetype function} if for every $x, y \in S$ and $\lambda \in (0, 1)$, there exists $z \in S$ such that

$$\lambda F(x) + (1 - \lambda) F(y) \leq_C^{(j)} F(z).$$
Definition 2.4. (See [2].) Let \( x \in S \). Then,

(i) \( F \) is called \( C \)-lower continuous at \( x \) if for every open set \( V \) with \( F(x) \cap V \neq \emptyset \), there exists an open neighborhood \( U \) of \( x \) such that \( F(y) \cap (V + C) \neq \emptyset \) for all \( y \in U \). We shall say that \( F \) is \( C \)-lower continuous on \( S \) if it is \( C \)-lower continuous at every point \( x \in S \).

(ii) \( F \) is called \( C \)-upper continuous at \( x \) if for every open set \( V \) with \( F(x) \subset V \), there exists an open neighborhood \( U \) of \( x \) such that \( F(y) \subset V + C \) for all \( y \in U \). We shall say that \( F \) is \( C \)-upper continuous on \( S \) if it is \( C \)-upper continuous at every point \( x \in S \).

3 Unified Types of Scalarizing Functions for Sets

In [4], we propose the following nonlinear scalarizing functions for sets: Let \( V, V' \in 2^Y \setminus \{\emptyset\} \), and direction \( k \in \text{int} C \). For each \( j = 1, \ldots, 5 \), we define \( I_{k,V}^{(j)} : 2^Y \setminus \{\emptyset\} \rightarrow \mathbb{R} \cup \{-\infty, \infty\} \) by

\[
I_{k,V}^{(j)}(V) := \inf \{ t \in \mathbb{R} \mid V \leq c(j)(tk + V') \}.
\]

In this section, we introduce some properties of these functions and several sufficient conditions for the existence of solutions of \((j\text{-SVOP})\).

Proposition 3.1. (See [6].) Let \( A, B \in 2^Y \setminus \{\emptyset\} \). Then, the following statements hold:

(i) If \( A \leq_{C}^{(1)} B \), \( A \) is \( (-C) \)-closed and \( B \) is \( C \)-closed then

\[ I_{k,V}^{(1)}(A) < I_{k,V}^{(1)}(B). \]

(ii) For each \( j = 2, 3 \), if \( A \leq_{\text{int} C}^{(j)} B \) and \( B \) is \( C \)-closed then

\[ I_{k,V}^{(j)}(A) < I_{k,V}^{(j)}(B). \]

(iii) For each \( j = 4, 5 \), if \( A \leq_{\text{int} C}^{(j)} B \) and \( A \) is \( (-C) \)-closed then

\[ I_{k,V}^{(j)}(A) < I_{k,V}^{(j)}(B). \]

Next, we introduce certain inherited properties on cone-convexity and cone-continuity of set-valued maps proved in [4, 5, 8, 10].

Lemma 3.1. (See [4, 5].) Let \( k \in \text{int} C \) and \( V' \in 2^Y \setminus \{\emptyset\} \). Then, the following statements hold:

(i) For each \( j = 1, 2, 3 \), if \( F \) is type \( (j) \) naturally quasi \( C \)-convex, then \( I_{k,V'}^{(j)} \circ F \) is quasi convex. Moreover, if \( F \) is type \( (j) \) naturally quasi \( C \)-concave, then \( I_{k,V'}^{(j)} \circ F \) is quasi concave.

(ii) For each \( j = 4, 5 \), if \( F \) is type \( (j) \) naturally quasi \( C \)-convex and \( V' \) is \( (-C) \)-convex, then \( I_{k,V'}^{(j)} \circ F \) is quasi convex. Moreover, if \( F \) is type \( (j) \) naturally quasi \( C \)-concave and \( V' \) is \( (-C) \)-convex, then \( I_{k,V'}^{(j)} \circ F \) is quasi concave.
Lemma 3.2. (See [8].) Let \( k \in \text{int} C \) and \( V' \in 2^Y \setminus \{\emptyset\} \). Then, the following statements hold:

(i) For each \( j = 1, 2, 3 \), if \( F \) is type \( (j) \) \( C \)-convexlike and \( V' \) is \( C \)-convex, then \( I_{k, V}^{(j)}, o F \) is convexlike.

(ii) For each \( j = 4, 5 \), if \( F \) is type \( (j) \) \( C \)-convexlike and \( V' \) is \( (-C) \)-convex, then \( I_{k, V}^{(j)}, o F \) is convexlike.

Lemma 3.3. (See [8].) Let \( k \in \text{int} C \) and \( V' \in 2^Y \setminus \{\emptyset\} \). Then, the following statements hold:

(i) For each \( j = 1, 2, 3 \), if \( F \) is type \( (j) \) \( C \)-concavelike and \( V' \) is \( C \)-convex, then \( I_{k, V}^{(j)}, o F \) is concavelike.

(ii) For each \( j = 4, 5 \), if \( F \) is type \( (j) \) \( C \)-concavelike and \( V' \) is \( (-C) \)-convex, then \( I_{k, V}^{(j)}, o F \) is concavelike.

Lemma 3.4. (See [10].) Let \( k \in \text{int} C \) and \( V' \in 2^Y \setminus \{\emptyset\} \). Then, the following statements hold:

(i) For each \( j = 1, 4, 5 \), if \( F \) is \( C \)-lower continuous on \( S \) then \( I_{k, V}^{(j)}, o F \) is lower semicontinuous on \( S \). Moreover, if \( F \) is \( (-C) \)-upper continuous on \( S \) then \( I_{k, V}^{(j)}, o F \) is upper semicontinuous on \( S \).

(ii) For each \( j = 2, 3 \), if \( F \) is \( (-C) \)-lower continuous on \( S \) then \( I_{k, V}^{(j)}, o F \) is upper semicontinuous on \( S \). Moreover, if \( F \) is \( C \)-upper continuous on \( S \) then \( I_{k, V}^{(j)}, o F \) is lower semicontinuous on \( S \).

Let \( V' \in 2^Y \setminus \{\emptyset\} \) and direction \( k \in \text{int} C \). To show sufficient conditions for the existence of solutions of \((j\text{-SVOP})\) by using properties of \( I_{k, V}^{(j)}, \), we consider the following two kinds of scalar optimization problems:

\[
\inf_{x \in S} (I_{k, V}^{(j)}, o F)(x) \quad \text{and} \quad \sup_{x \in S} (I_{k, V}^{(j)}, o F)(x).
\]

Lemma 3.5. (See [7].) Assume that \( F \) is \( C \)-closed on \( S \) and \( x_0 \in S \). Let \( k \in \text{int} C \). For each \( j = 1, 2, 3 \), the following statements hold:

(i) If \( x_0 \) is a solution of \( \inf_{x \in S} (I_{k, V}^{(j)}, o F)(x) \), then \( x_0 \) is a weak minimal solution of \((j\text{-SVOP})\).

(ii) If \( x_0 \) is a solution of \( \sup_{x \in S} (I_{k, V}^{(j)}, o F)(x) \), then \( x_0 \) is a weak maximal solution of \((j\text{-SVOP})\).

Lemma 3.6. (See [7].) Assume that \( F \) is \( (-C) \)-closed on \( S \) and \( x_0 \in S \). Let \( k \in \text{int} C \). For each \( j = 4, 5 \), the following statements hold:

(i) If \( x_0 \) is a solution of \( \inf_{x \in S} (I_{k, V}^{(j)}, o F)(x) \), then \( x_0 \) is a weak minimal solution of \((j\text{-SVOP})\).

(ii) If \( x_0 \) is a solution of \( \sup_{x \in S} (I_{k, V}^{(j)}, o F)(x) \), then \( x_0 \) is a weak maximal solution of \((j\text{-SVOP})\).
4 Existence Theorems for Saddle Points of Set-Valued Maps

At first, we introduce definitions of saddle points for set-valued maps proposed in [8]. For each $j = 1, \ldots, 5$, if $(x_0, y_0) \in X \times Y$ satisfies the following properties:

(i) $F(x, y_0) \leq_C^{(j)} F(x_0, y_0)$ implies $F(x_0, y_0) \leq_C^{(j)} F(x, y_0),$

(ii) $F(x_0, y_0) \leq_C^{(j)} F(x_0, y)$ implies $F(x_0, y) \leq_C^{(j)} F(x_0, y_0),$

for any $x \in X$ and $y \in Y$, then we call it type $(j)$ $C$-saddle point of $F$. It is equivalent to

$F(x_0, y_0) \in \{\text{Min}_{(j)} F(X, y_0)\} \cap \{\text{Max}_{(j)} F(x_0, Y)\}.$

If $C$ is replaced by $\text{int}C$ then we call it type $(j)$ weak $C$-saddle point of $F$.

In this section, we give three types of existence theorems for type $(j)$ cone saddle points of set-valued maps. At first, we introduce the first existence theorems which are natural extensions of Sion's minimax theorem (see [9]).

Theorem 4.1. (See [7].) Let $X$ and $Y$ be nonempty compact convex subsets of two real topological vector spaces, respectively, $Z$ a real topological vector space with the partial ordering $\leq_C$, $k \in \text{int}C$, $V'$ a nonempty subset of $Z$ and $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. Assume that $F$ is $C$-closed and $(-C)$-closed on $X \times Y$. If $F$ satisfies the following conditions:

(i) $x \to F(x, y)$ is $C$-lower continuous and type (1) naturally quasi $C$-convex on $X$ for every $y \in Y$,

(ii) $x \to F(x, y)$ is $(-C)$-upper continuous and type (1) naturally quasi $C$-concave on $Y$ for every $x \in X$,

then $F$ has at least one type (1)-weak saddle point.

Theorem 4.2. (See [7].) Let $X$ and $Y$ be nonempty compact convex subsets of two real topological vector spaces, respectively, $Z$ a real topological vector space with the partial ordering $\leq_C$, $k \in \text{int}C$, $V'$ a nonempty subset of $Z$ and $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. Assume that $F$ is $C$-closed on $X \times Y$. For each $j = 2, 3$, if $F$ satisfies that

(i) $x \to F(x, y)$ is $C$-upper continuous and type $(j)$ naturally quasi $C$-convex on $X$ for every $y \in Y$,

(ii) $x \to F(x, y)$ is $(-C)$-lower continuous and type $(j)$ naturally quasi $C$-concave on $Y$ for every $x \in X$,

then $F$ has at least one type $(j)$-weak saddle point.

Theorem 4.3. (See [7].) Let $X$ and $Y$ be nonempty compact convex subsets of two real topological vector spaces, respectively, $Z$ a real topological vector space with the partial ordering $\leq_C$, $k \in \text{int}C$, $V'$ a nonempty subset of $Z$ and $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. Assume that $F$ is $(-C)$-closed on $X \times Y$ and $V'$ is $(-C)$-convex. For each $j = 4, 5$, if $F$ satisfies that

(i) $x \to F(x, y)$ is $C$-lower continuous and type $(j)$ naturally quasi $C$-convex on $X$ for every $y \in Y$,

(ii) $x \to F(x, y)$ is $(-C)$-upper continuous and type $(j)$ naturally quasi $C$-concave on $Y$ for every $x \in X$, 
then $F$ has at least one type $(j)$-weak saddle point.

Next, we introduce the second existence theorems which are natural extensions of Fan type minimax theorem (see [1]).

**Theorem 4.4.** (See [8].) Let $X$ be a nonempty compact subset of real topological space, $Y$ any space, $Z$ a real topological vector space with the partial ordering $\leq_C$, $k \in \text{int}C$, $V'$ a nonempty subset of $Z$ and $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. Assume that $F$ is $C$-closed and $(-C)$-closed on $X \times Y$. If $F$ satisfies that

(i) $x \to F(x, y)$ is type (1) $C$-convexlike on $X$ for every $y \in Y$,
(ii) $x \to F(x, y)$ is $(-C)$-upper continuous and type (1) $C$-concavelike on $Y$ for every $x \in X$,

then $F$ has at least one type (1)-weak saddle point.

**Theorem 4.5.** (See [8].) Let $X$ be a nonempty compact subset of real topological space, $Y$ any space, $Z$ a real topological vector space with the partial ordering $\leq_C$, $k \in \text{int}C$, $V'$ a nonempty subset of $Z$ and $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. Assume that $F$ is $C$-closed on $X \times Y$. For each $j = 2, 3$, if $F$ satisfies that

(i) $x \to F(x, y)$ is type $(j)$ $C$-convexlike on $X$ for every $y \in Y$,
(ii) $x \to F(x, y)$ is $(-C)$-lower continuous and type $(j)$ $C$-concavelike on $Y$ for every $x \in X$,

then $F$ has at least one type $(j)$-weak saddle point.

**Theorem 4.6.** (See [8].) Let $X$ be a nonempty compact subset of real topological space, $Y$ any space, $Z$ a real topological vector space with the partial ordering $\leq_C$, $k \in \text{int}C$, $V'$ a nonempty subset of $Z$ and $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. Assume that $F$ is $(-C)$-closed on $X \times Y$. For each $j = 4, 5$, if $F$ satisfies that

(i) $x \to F(x, y)$ is type $(j)$ $C$-convexlike on $X$ for every $y \in Y$,
(ii) $x \to F(x, y)$ is $(-C)$-upper continuous and type $(j)$ $C$-concavelike on $Y$ for every $x \in X$,

then $F$ has at least one type $(j)$-weak saddle point.

Finally, we give the third existence theorems for type $(j)$ cone saddle points of set-valued maps with separated form.

**Theorem 4.7.** (See [7].) Let $X$ and $Y$ be nonempty compact subsets of two real topological spaces, respectively, $Z$ a real ordered topological vector space with the partial ordering $\leq_C$, $k \in \text{int}C$, $V'$ a nonempty subset of $Z$ and $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. If $F$ satisfies that

(i) $F(x, y) := G_1(x) \cup G_2(y)$,
(ii) $G_1$ is $C$-closed and $C$-lower continuous on $X$,
(iii) $G_2$ is $(-C)$-closed and $(-C)$-upper continuous on $Y$,

where $G_1 : X \to 2^Z \setminus \{\emptyset\}$ and $G_2 : Y \to 2^Z \setminus \{\emptyset\}$, then $F$ has at least one type (1) $C$-saddle point.

**Theorem 4.8.** (See [7].) Let $X$ and $Y$ be nonempty compact subsets of two real topological spaces, respectively, $Z$ a real ordered topological vector space with the partial ordering $\leq_C$, $k \in \text{int}C$, $V'$ a nonempty subset of $Z$ and $F : X \times Y \to 2^Z \setminus \{\emptyset\}$. For each $j = 2, 3$, if $F$ satisfies that
(i) $F(x, y) := G_1(x) \cup G_2(y)$,
(ii) $G_1$ is $C$-closed and $C$-upper continuous on $X$,
(iii) $G_2$ is $C$-closed and $(-C)$-lower continuous on $Y$,

where $G_1 : X \rightarrow 2^Z \setminus \{\emptyset\}$ and $G_2 : Y \rightarrow 2^Z \setminus \{\emptyset\}$, then $F$ has at least one type (j) weak $C$-saddle point.

**Theorem 4.9.** (See [7].) Let $X$ and $Y$ be nonempty compact subsets of two real topological spaces, respectively, $Z$ a real topological vector space with the partial ordering $\leq_C$, $k \in \text{int}C$, $V'$ a nonempty subset of $Z$ and $F : X \times Y \rightarrow 2^Z \setminus \{\emptyset\}$. For each $j = 4, 5$, if $F$ satisfies that

(i) $F(x, y) := G_1(x) \cup G_2(y)$,
(ii) $G_1$ is $(-C)$-closed and $C$-lower continuous on $X$,
(iii) $G_2$ is $(-C)$-closed and $(-C)$-upper continuous on $Y$,

where $G_1 : X \rightarrow 2^Z \setminus \{\emptyset\}$ and $G_2 : Y \rightarrow 2^Z \setminus \{\emptyset\}$, then $F$ has at least one type (j) weak $C$-saddle point.

**References**