

Estimation for Multivariate Stable Distributions

首都大学東京・都市教養学部経営学系 小方 浩明 (Hiroaki Ogata)
School of Business Administration,
Faculty of Urban Liberal Arts,
Tokyo Metropolitan University

1 Introduction

Apparently, Normal distribution has played a central role for the modeling of the financial data. However, it is well known that many actual financial data are skewed and heavy-tailed, which cannot be described by Normal distribution. For this reason, there has been increasing interest in the theory of skewed and heavy-tailed distributions over several decades.

One of the most important classes of skewed and heavy-tailed distributions is that of stable distributions. Since Mandelbrot (1963) and Fama (1965) suggested to use the stable distributions for analyzing financial data, many authors have paid attention to them. The main difficulties in dealing with stable distributions are that they do not have simple explicit forms of the densities except a few special cases, and that they do not necessarily have the first or/and second moments. These properties of stable distributions make us difficult to use ordinary maximum likelihood estimator (MLE) and method of moments. Despite of such difficulties, estimation methods of the parameters of the univariate stable distribution have been well investigated so far. For example, DuMouchel (1973) discusses the asymptotic normality of maximum likelihood estimation, McCulloch (1986) considered consistent estimators from five pre-determined sample quantiles and Kunitomo and Owada (2006) applies empirical likelihood method for the estimation.

Compared to the univariate case, not much is known about the estimation of the parameters of multivariate stable distributions. Yet, we can raise a few articles dealing with the estimation problems for multivariate stable distributions such as Rachev and Xin (1993), Cheng and Rachev (1995), Nolan et. al. (2001), etc. They investigated estimation theory especially of the spectral measure.

In this paper, motivated by Nolan et. al. (2001), we consider the joint estimation of the all parameters of stable distribution (characteristic exponent, spectral measure and location). For this purpose, we make use of the estimating function constructed by the empirical and theoretical characteristic functions. Using the estimating function, we obtain the estimators of the parameters by the generalized empirical likelihood (GEL) method. GEL method is a generalization of the empirical likelihood (EL) method and EL method can be considered as a generalization of generalized method of moments (GMM). The theory of GEL method is well investigated by Newey and Smith (2004) in which they mention that the asymptotic bias of the EL estimator does not grow with the number of moment restrictions, while the bias of GMM often does.

After describing the theory of the estimating method, we provide two Monte Carlo studies. They show that our estimating method works properly.

This article is organized as follows. Section 2 provides a review of the multivariate stable distribution and the estimating function for our estimation. In Section 3, we explain about GEL and give the asymptotic theory. Monte Carlo simulations are given in Section 4. Section 5 concludes.

2 Multivariate Stable Distributions

At the beginning of this section, we introduce the definition of the multivariate stable random vector. The following definition and corollary are due to Samorodnitsky and Taqqu (1994, pp. 57–58).

Definition 1 A random vector $\mathbf{X} = (X_1, \dots, X_d)^\top$ is said to be a stable random vector in \mathbb{R}^d if for any positive numbers A and B there is a positive number C and a vector $\mathbf{D} \in \mathbb{R}^d$ such that

$$A\mathbf{X}^{(1)} + B\mathbf{X}^{(2)} \stackrel{d}{=} C\mathbf{X} + \mathbf{D},$$

where $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ are independent copies of \mathbf{X} .

Corollary 1 A random vector \mathbf{X} is stable if and only if for any $n \geq 2$, there is an $\alpha \in (0, 2]$ and a vector \mathbf{D}_n such that

$$\mathbf{X}^{(1)} + \mathbf{X}^{(2)} + \dots + \mathbf{X}^{(n)} \stackrel{d}{=} n^{1/\alpha}\mathbf{X} + \mathbf{D}_n, \quad (1)$$

where $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \dots, \mathbf{X}^{(n)}$ are independent copies of \mathbf{X} .

The index α in (1) is called the *index of stability* or the *characteristic exponent* of the vector \mathbf{X} . It represents a tail thickness or a kurtosis of the distribution.

The characteristic function of multivariate stable distribution is known as follows.

$$\phi(\boldsymbol{\nu}) = E[\exp\{i\langle \mathbf{X}, \boldsymbol{\nu} \rangle\}] = \exp\{-I(\boldsymbol{\nu})\}$$

where the exponent function is

$$I(\boldsymbol{\nu}) = \int_{S_d} \psi(\langle \mathbf{s}, \boldsymbol{\nu} \rangle) \Gamma(d\mathbf{s}) + i\langle \boldsymbol{\nu}, \boldsymbol{\mu} \rangle.$$

Here $S_d = \{\mathbf{s} : \|\mathbf{s}\| = 1\}$ is the unit sphere in \mathbb{R}^d , the symbol $\langle \cdot, \cdot \rangle$ shows the inner product, Γ is the finite spectral measure on S_d , $\boldsymbol{\mu} = (\mu_1, \dots, \mu_d) \in \mathbb{R}^d$ is the location vector and

$$\psi(u) = \begin{cases} |u|^\alpha \left(1 - i \operatorname{sign}(u) \tan \frac{\pi\alpha}{2} \right) & (\alpha \neq 1) \\ |u| \left(1 + i \frac{2}{\pi} \operatorname{sign}(u) \ln |u| \right) & (\alpha = 1), \end{cases}$$

where $\alpha \in (0, 2]$ is the characteristic exponent. We denote this d -dimensional stable distribution by $S^d(\alpha, \Gamma, \boldsymbol{\mu})$.

The purpose of this paper is to estimate the spectral measure Γ , the location vector $\boldsymbol{\mu}$ and the characteristic exponent α , given an i.i.d. sample $\mathbf{X}_1, \dots, \mathbf{X}_n$ of d -dimensional random vectors drawn from this distribution. However, we encounter several difficulties even for the case of univariate stable distribution. One is the complexity of their density functions. Except a few cases, we do not have a simple explicit form of the densities and this makes us difficult to use the usual MLE method. Moreover, the stable distributions do not necessarily have the second or even first moment. This means that we cannot use the ordinary method of moments.

One remedy for this difficulty is to use the empirical characteristic function

$$\hat{\phi}_n(\boldsymbol{\nu}) = \frac{1}{n} \sum_{j=1}^n \exp\{i\langle \mathbf{X}_j, \boldsymbol{\nu} \rangle\}$$

Nolan et. al. (2001) proposed the estimation method of spectral measure Γ with the empirical characteristic function as follows. First, they considered a discrete approximation

$$\Gamma^* = \sum_{\ell=1}^L \gamma_\ell \delta_{\mathbf{s}_\ell} \quad (2)$$

where $\gamma_\ell = \Gamma(A_\ell)$, $\ell = 1, \dots, L$, are the weights and $\delta_{\mathbf{s}_\ell}$ are point masses at the point $\mathbf{s}_\ell \in S_d$. Here A_ℓ are patches that partition the sphere S_d , with some "center" \mathbf{s}_ℓ . Next, choose arbitrary frequencies $\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_L \in \mathbb{R}^d$ and define $L \times L$ matrix Ψ whose (k, ℓ) -th element is $\psi(\langle \boldsymbol{\nu}_k, \mathbf{s}_\ell \rangle)$. After eliminating the location parameter $\boldsymbol{\mu}$ by usual sample mean, we can obtain the expression $\mathbf{I} = \Psi \boldsymbol{\gamma}$ where $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_L)^\top$ and $\mathbf{I} = (I(\boldsymbol{\nu}_1), \dots, I(\boldsymbol{\nu}_L))^\top$. By choosing $\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_L \in \mathbb{R}^d$ so that Ψ^{-1} exists, and substituting $\hat{\mathbf{I}} = (\hat{I}(\boldsymbol{\nu}_1), \dots, \hat{I}(\boldsymbol{\nu}_L))^\top$ into \mathbf{I} where $\hat{I}(\boldsymbol{\nu}) = -\ln \hat{\phi}_n(\boldsymbol{\nu})$, we obtain the discretized estimator $\hat{\boldsymbol{\gamma}} = \Psi^{-1} \hat{\mathbf{I}}$ of spectral measure Γ . This method is simple and easy to implement but we have to know α , or at least, to obtain the estimator $\hat{\alpha}$ in order to compute the Ψ . In their paper, they calculate the estimator $\hat{\alpha}_j$ ($j = 1, \dots, d$) for each of the coordinates of the d -dimensional data set, and define the ultimate estimator as $\hat{\alpha} = (\sum_{i=1}^d \hat{\alpha}_i)/d$. However, this method is of course a last resort.

In this paper, we consider to estimate the parameters jointly based on the estimating function constructed by the empirical and theoretical characteristic functions. The method of empirical characteristic function estimation was investigated by many authors. For details, see Yu (2004), which gives the summary and many references. of this procedure. Suppose that $\{\mathbf{X}_j\}_{j=1}^n$ is an i.i.d. sequence from $S^d(\alpha, \Gamma^*, \boldsymbol{\mu})$ where Γ^* is a discrete spectral measure defined in (2). Denote the parameters by $\boldsymbol{\theta} = (\alpha, \boldsymbol{\gamma}^\top, \boldsymbol{\mu}^\top)^\top \in (0, 2] \times \mathbb{R}_+^L \times \mathbb{R}^d$ and define the estimating function as

$$h(\boldsymbol{\nu}, \mathbf{X}_j, \boldsymbol{\theta}) = \exp\{i\langle \boldsymbol{\nu}, \mathbf{X}_j \rangle\} - \phi_\theta(\boldsymbol{\nu})$$

where ϕ_θ is the theoretical characteristic function with a parameter $\boldsymbol{\theta}$. When we denote the true parameter value by $\boldsymbol{\theta}_0$, it is obviously $E[h(\boldsymbol{\nu}, \mathbf{X}_j, \boldsymbol{\theta}_0)] = 0$ for any frequency

$\boldsymbol{\nu} \in \mathbb{R}^d$. Choose arbitrary frequencies $\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_K \in \mathbb{R}^d$, and to avoid the calculation of complex numbers, we redefine the estimating function as

$$\mathbf{g}(\mathbf{X}_j, \boldsymbol{\theta}) = (\Re[h(\boldsymbol{\nu}_1, \mathbf{X}_j, \boldsymbol{\theta})], \dots, \Re[h(\boldsymbol{\nu}_K, \mathbf{X}_j, \boldsymbol{\theta})], \Im[h(\boldsymbol{\nu}_1, \mathbf{X}_j, \boldsymbol{\theta})], \dots, \Im[h(\boldsymbol{\nu}_K, \mathbf{X}_j, \boldsymbol{\theta})])^\top, \quad (3)$$

where $\Re[\cdot]$ and $\Im[\cdot]$ are the real and imaginary parts of a complex number. Obviously, we can see $E[\mathbf{g}(\mathbf{X}_j, \boldsymbol{\theta}_0)] = \mathbf{0}$. Using this estimating function, we consider to estimate the parameters with GEL method in the following section.

3 Generalized Empirical Likelihood

This section provides the theory of GEL method and its application to the estimation problem of multivariate stable distributions. GEL method can be considered as a generalization of GMM. One advantage of using GEL method is that GEL estimator has smaller bias than GMM estimator in overidentified cases. Before describing the GEL method, we start with the review of the EL method.

EL method was originally introduced by Owen (1988) for the nonparametric confidence interval. It was investigated by many authors because of its generality and effectiveness. Among many studies of empirical likelihood method, Qin and Lawless (1994) considered to link estimating functions and empirical likelihood. Empirical likelihood and estimating function methods are well suited each other. Using estimating function, they defined the maximum EL estimator and gave its asymptotic property.

The empirical likelihood ratio function for $\boldsymbol{\theta}$ is defined as

$$\mathcal{R}(\boldsymbol{\theta}) = \max_{(p_1, \dots, p_n)} \left\{ \prod_{j=1}^n np_j \mid \sum_{j=1}^n p_j \mathbf{g}(\mathbf{X}_j, \boldsymbol{\theta}) = \mathbf{0}, \sum_{j=1}^n p_j = 1, p_j \geq 0 \right\}, \quad (4)$$

where $\mathbf{g}(\mathbf{X}_j, \boldsymbol{\theta})$ is estimating function defined in (3) and p_j is a probability mass on $\mathbf{g}(\mathbf{X}_j, \boldsymbol{\theta})$. An explicit form of $\mathcal{R}(\boldsymbol{\theta})$ is derived by a Lagrange multiplier argument: the maximum of $\prod_{j=1}^n np_j$ subject to the constrains in (4) is attained when

$$p_j = p_j(\boldsymbol{\theta}) = \frac{1}{n} \frac{1}{1 + \boldsymbol{\lambda}^\top \mathbf{g}(\mathbf{X}_j, \boldsymbol{\theta})} \quad (5)$$

where $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\boldsymbol{\theta}) \in \mathbb{R}^{2K}$ is Lagrange multiplier given as the solution to

$$\sum_{j=1}^n \frac{\mathbf{g}(\mathbf{X}_j, \boldsymbol{\theta})}{1 + \boldsymbol{\lambda}^\top \mathbf{g}(\mathbf{X}_j, \boldsymbol{\theta})} = \mathbf{0}. \quad (6)$$

Then, from (5), the minus of empirical log-likelihood ratio is given by

$$l(\boldsymbol{\theta}) = -\log \left(\prod_{j=1}^n np_j \right) = \sum_{j=1}^n \log(1 + \boldsymbol{\lambda}^\top \mathbf{g}(\mathbf{X}_j, \boldsymbol{\theta}))$$

and maximum EL estimator is defined by

$$\hat{\boldsymbol{\theta}}_{\text{EL}} = \arg \min_{\boldsymbol{\theta}} \sum_{j=1}^n \log(1 + \boldsymbol{\lambda}^\top \mathbf{g}(\mathbf{X}_j, \boldsymbol{\theta}))$$

where $\boldsymbol{\lambda}$ satisfies (6), or, equivalently, it can be written as

$$\hat{\boldsymbol{\theta}}_{\text{EL}} = \arg \min_{\boldsymbol{\theta}} \sup_{\boldsymbol{\lambda}} \sum_{j=1}^n \log(1 + \boldsymbol{\lambda}^\top \mathbf{g}(\mathbf{X}_j, \boldsymbol{\theta})). \quad (7)$$

Smith (1997, 2001) generalized above EL estimator. Let $\rho(v)$ be a concave function on its domain, an open interval $\mathcal{V} \subset \mathbb{R}$ containing zero. Then, GEL estimator is defined by

$$\hat{\boldsymbol{\theta}}_{\text{GEL}} = \arg \min_{\boldsymbol{\theta}} \sup_{\boldsymbol{\lambda} \in \hat{\Lambda}_n(\boldsymbol{\theta})} \sum_{j=1}^n \rho(\boldsymbol{\lambda}^\top \mathbf{g}(\mathbf{X}_j, \boldsymbol{\theta}))$$

where $\hat{\Lambda}_n(\boldsymbol{\theta}) = \{\boldsymbol{\lambda} : \boldsymbol{\lambda}^\top \mathbf{g}(\mathbf{X}_j, \boldsymbol{\theta}) \in \mathcal{V}, j = 1, \dots, n\}$. EL estimator (7) is a special case with $\rho(v) = \log(1 + v)$ and $\mathcal{V} = (-1, \infty)$. Besides, we can obtain the exponential tilting (ET) estimator in Kitamura and Stutzer (1997) and Smith (1997) by setting $\rho(v) = -e^v$, and continuous updating estimator (CUE) of Hansen et al. (1996) by setting $\rho(v)$ as quadratic.

Let $\boldsymbol{\Omega} = E[\mathbf{g}(\mathbf{X}_j, \boldsymbol{\theta})\mathbf{g}(\mathbf{X}_j, \boldsymbol{\theta})^\top]$ and $\mathbf{G} = E[\partial \mathbf{g}(\mathbf{X}_j, \boldsymbol{\theta})/\partial \boldsymbol{\theta}]$. For the consistency and the asymptotic normality of GEL estimator, we impose the following assumptions.

Assumption 1

1. $\boldsymbol{\theta}_0 \in \text{int}(\Theta)$ where $\Theta = [\epsilon, 1 - \epsilon] \cup [1 + \epsilon, 2 - \epsilon] \times [\epsilon, M]^L \times [-M, M]^d$ for sufficiently small $\epsilon > 0$ and sufficiently large $M > 0$.
2. $\boldsymbol{\theta}_0 \in \Theta$ is the unique solution to $E[\mathbf{g}(\mathbf{X}_j, \boldsymbol{\theta})] = \mathbf{0}$.
3. $\boldsymbol{\Omega}$ is nonsingular.
4. $\rho(v)$ is twice continuously differentiable in a neighborhood of zero.
5. $\text{rank}(\mathbf{G}) = 1 + L + d$.

Then, we obtain the following theorem.

Theorem 1 *If Assumption 1 holds, then $\hat{\boldsymbol{\theta}}_{\text{GEL}} \xrightarrow{p} \boldsymbol{\theta}_0$ and*

$$\sqrt{n}(\hat{\boldsymbol{\theta}}_{\text{GEL}} - \boldsymbol{\theta}_0) \xrightarrow{d} \mathcal{N}(\mathbf{0}, (\mathbf{G}^\top \boldsymbol{\Omega}^{-1} \mathbf{G})^{-1}).$$

Proof. See Theorems 3.1 and 3.2 of Newey and Smith (2004).

GEL estimator is affected by the choice of the frequencies ν_1, \dots, ν_K . In Section 2.1 of Yu (2004), we can find the statement about this issue. Saying from the viewpoint of the minimum asymptotic variance, many and fine frequencies will be fine. Feuerverger and McDunnough (1981) shows that the asymptotic variance can be made arbitrarily close to the Cramér-Rao lower bound if the number of frequencies K is sufficiently large and the grid is sufficiently fine and extended. On the other hand, Carrasco and Florens (2002) insists that too fine frequencies lead to a singular asymptotic variance matrix and we can not calculate its inverse.

4 Monte Carlo Studies

This section provides two Monte Carlo studies of stable parameters estimation based on Theorem 1. In both cases, we consider bivariate stable distributions. For generating the multivariate stable samples, we rely on the STABLE programs developed by John Nolan (see Nolan (1997)). For calculating the GEL estimator, we use the R package `gmm` produced by Pierre Chaussé (see Chaussé (2010)). The program calculates several GEL estimators such as EL, ET and CUE. We simulate 100 draws of 2,000 observations in each case.

Example 1 Data was simulated from the characteristic exponent $\alpha = 1.5$, the discrete spectral measure (2) with $L = 3$, $\gamma_\ell = 1/3$, $\mathbf{s}_\ell = (\cos \theta_\ell, \sin \theta_\ell)^\top$, $\ell = 1, \dots, 3$ for $\theta_1 = \pi/3$, $\theta_2 = \pi$, $\theta_3 = 3\pi/2$, and the location vector $\boldsymbol{\mu} = (0, 0)^\top$. The estimation result with GEL is in Table 1. We use three estimators, EL, ET and CUE. For constructing estimating function (3), we use four frequencies, $\nu_1 = (0.1, 0.2)^\top$, $\nu_2 = (0.3, 0.4)^\top$, $\nu_3 = (0.5, 0.6)^\top$ and $\nu_4 = (0.7, 0.8)^\top$.

[TABLE 1 ABOUT HERE]

EL gives the best estimate for the characteristic exponent among the three methods. However, for the spectral measure and the location, estimation results of ET and CUE are better than those of EL. Although almost all estimations have slightly upward biases, they are well estimated by each method. Regarding the RMSE, ET and CUE give smaller values than EL for almost all parameters. Roughly speaking, results of ET and CUE are similar.

Example 2 Data was simulated from the characteristic exponent $\alpha = 1.75$, the discrete spectral measure (2) with $L = 4$, $\gamma_\ell = 1/4$, $\mathbf{s}_\ell = (\cos \theta_\ell, \sin \theta_\ell)^\top$, $\ell = 1, \dots, 4$ for $\theta_1 = 0$, $\theta_2 = \pi/2$, $\theta_3 = \pi$, $\theta_4 = 3\pi/2$ and the location vector $\boldsymbol{\mu} = (0, 0)^\top$. The estimation result with GEL is in Table 2. We use three estimators, EL, ET and CUE. For constructing estimating function (3), we use five frequencies $\nu_1 = (0.1, 0.2)^\top$, $\nu_2 = (0.3, 0.4)^\top$, $\nu_3 = (0.5, 0.6)^\top$, $\nu_4 = (0.7, 0.8)^\top$ and $\nu_5 = (0.9, 1.0)^\top$.

[TABLE 2 ABOUT HERE]

Regarding the characteristic exponent, estimation results are almost same among three methods, but RMSE is decreasing in order of EL-ET-CUE. Although there are differences in the estimation results of the spectral measure, we cannot find out the best method by means of both bias and RMSE. About the estimation of the location, both bias and RMSE of ET and CUE are smaller than those of EL. Although almost all estimations have slightly upward biases, they are well estimated by each method.

5 Conclusions

In this article we deal with the estimation problem of the multivariate stable distributions. In principle, estimation is based on the estimating function. Since stable distributions do not have explicit forms of density functions except a few cases, and do not necessarily have the second or even first moment, estimating function is constructed by matching the empirical and theoretical characteristic functions. The main advantage of using the estimating functions is that we can simultaneously estimate all parameters of multivariate stable distributions. Given the estimating functions, we obtain the estimator by GEL method. GEL estimator is a very wide class of estimators and is known as a generalization of GMM. The Monte Carlo studies show that GEL estimators for multivariate stable distributions are reasonable. Extension to the time series model is possible.

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Tables

Table 1: Monte Carlo Study: Example 1

	EL		ET		CUE	
	mean	RMSE	mean	RMSE	mean	RMSE
α	1.515	0.037	1.523	0.036	1.524	0.039
γ_1	0.344	0.028	0.347	0.021	0.346	0.025
γ_2	0.356	0.049	0.344	0.040	0.339	0.038
γ_3	0.345	0.043	0.320	0.035	0.321	0.030
μ_1	0.037	0.062	0.035	0.049	0.039	0.053
μ_2	0.032	0.054	0.004	0.037	0.001	0.038

Table 2: Monte Carlo Study: Example 2

	EL		ET		CUE	
	mean	RMSE	mean	RMSE	mean	RMSE
α	1.790	0.063	1.790	0.051	1.789	0.048
γ_1	0.268	0.065	0.275	0.045	0.279	0.047
γ_2	0.248	0.061	0.249	0.039	0.252	0.036
γ_3	0.284	0.059	0.273	0.053	0.275	0.044
γ_4	0.258	0.052	0.242	0.040	0.223	0.050
μ_1	0.023	0.064	0.013	0.039	0.011	0.033
μ_2	0.026	0.060	0.012	0.033	0.010	0.033