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SOMMERFELD RADIATION CONDITION AT THRESHOLD

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Dedicated to Hiroshi Isozaki on the occasion of his Sixtieth Birthday.

1. Introduction

We give an account of various recent results [Sk1]. These include Besov space bounds of the resolvent at low energies in any dimension for a class of potentials that are negative and obey a virial condition with these conditions imposed at infinity only. We do not require spherical symmetry. There are two boundary values of the resolvent at zero energy which we characterize by radiation conditions. These radiation conditions are zero energy versions of the well-known Sommerfeld radiation condition.

We study low-energy spectral theory for the Schrödinger operator $H=-\Delta+V$ on $\mathcal{H}=L^2(\mathbb{R}^d)$, $d\geq 1$, where the potential V obeys the following condition. We use the notation $\langle x\rangle=\sqrt{x^2+1}$, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, and for $\mu\in(0,2)$ the notation $s_0=1/2+\mu/4$.

Condition 1.1. Let $V=V_1+V_2$ be a real-valued function defined on $\mathbb{R}^d$; $d\geq 1$. There exists $\mu\in(0,2)$ such that the following conditions (1)–(5) hold.

(1) There exists $\epsilon_1>0$ such that $V_1(x)\leq-\epsilon_1\langle x\rangle^{-\mu}$.
(2) $V_1\in C^\infty(\mathbb{R}^d)$. For all $\alpha\in\mathbb{N}_0^d$ there exists $C_\alpha>0$ such that
$$\langle x\rangle^{\mu+|\alpha|}|\partial^\alpha V_1(x)|\leq C_\alpha.$$
(3) There exists $\tilde{\epsilon}_1>0$ such that $-|x|^{-2}\left(x\cdot\nabla(|x|^2V_1)\right)\geq-\tilde{\epsilon}_1V_1$.
(4) There exists $\delta,C,R>0$ such that
$$|V_2(x)|\leq C|x|^{-2s_0-\delta},$$
for $|x|>R$.
(5) $V_2\in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ for $d=1,2,3$, $V_2\in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ for some $p>2$ if $d=4$ while $V_2\in L^{d/2}_{\mathrm{loc}}(\mathbb{R}^d)$ for $d\geq 5$.

Due to (4) and (5) the operator $V_2(-\Delta+i)^{-1}$ is a compact operator on $L^2(\mathbb{R}^d)$, see for example [RS, Theorems X.20 and X.21] for the case $d\geq 4$. Whence H is self-adjoint. The Schrödinger operator with an attractive Coulomb potential in dimension $d\geq 3$ is a particular example.

While low-energy spectral asymptotics for Schrödinger operators is a well studied subject for classes of potentials of fast decay the literature is more sparse for classes of potentials with decay $O(r^{-2})$ or slower. We refer to [Ya, Na, FS, SW] and references therein. We remark that Condition 1.1 is closely related to the conditions used in [FS], in fact our goal is to present more precise resolvent bounds than appearing in [FS] (Besov space bounds), and characterize boundary values $R(0\pm \mathrm{i}0)$ of the resolvent at zero energy. The latter is achived in terms of certain microlocal estimates traditionally referred to as Sommerfeld radiation conditions. For positive energies the limiting absorption principle (LAP) and Sommerfeld radiation conditions are fundamental for stationary scattering theory and they are used at many places in the literature, see for example [Sa, AH, GY] and [Hö1, Section 30.2] (two-body problems), [Is, Va] (many-body problems) and [Mø] (abstract framework). Moreover radiation conditions are an integral part of for one of the oldest method of proving LAP, cf. for example [Sa]. We consider them as interesting of their own right, in particular including the case of zero energy cf. [DS1].

Neither [FS] nor the present work deal with scattering theory. On the other hand there are indeed applications to scattering theory at low energies as demonstrated in the recent works [DS1, DS2]. However this is for a smaller class of potentials (essentially radially symmetric potentials). We plan in a future publication [Sk2] to study scattering theory at low energies for a larger class than the ones of [DS1, DS2] however within the one defined by Condition 1.1. For this study Besov space bounds and uniqueness induced by versions of the Sommerfeld radiation condition will be useful. We remark that these results are also present in some form in [DS1] although, as indicated above, this is under stronger conditions on the potentials. Besides the Besov space bounds are not obtained in the strongest form as done here and they are shown somewhat indirectly (in fact only the imaginary part of the boundary value of the resolvent is estimated). We present our results in Section 2. They are all under Condition 1.1.

2. Results

2.1. Resolvent bounds. Let us recall a main result from [FS]. Let $\theta \in (0, \pi)$, $\lambda_0 > 0$ and define

$$\Gamma_\theta = \{z \in \mathbb{C} \setminus \{0\} \mid \arg z \in (0, \theta),\ |z| \le \lambda_0\}. \tag{2.1}$$

In [FS] λ_0 is exclusively taken equal to one although this is only for convenience of presentation. We fix any $\lambda_0 > 0$ at this point and suppress henceforth any dependence of this constant (as done in the notation (2.1)). At this point we also fix $\theta \in (0, \pi)$, but keep (somewhat inconsistently) the dependence of θ of the set (2.1).

For $\mu \in (0, 2)$, $K > 0$ and $\lambda \ge 0$ let

$$f = f_\lambda(x) = (\lambda + K\langle x\rangle^{-\mu})^{1/2};\ x \in \mathbb{R}^d. \tag{2.2}$$

Here λ will be taken as $|z|$ for z in the closure of Γ_θ and K can for parts of our presentation be taken arbitrary. More precisely the latter is true for Theorems 2.1 and 2.2 (presented below). Consequently we take, for convenience, $K = 1$ in these theorems. As for Theorems 2.3 and 2.5 we choose a different value of K, see (2.5).

For a Hilbert space $\mathcal{H}$ (which in our case will be $L^2(\mathbb{R}^d)$) we denote by $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators on $\mathcal{H}$ (a similar notation will be used for Banach spaces). A $\mathcal{B}(\mathcal{H})$-valued function $T(\cdot)$ on Γ_θ is said to be uniformly Hölder continuous in Γ_θ if there exist $C, \gamma > 0$ such that

$$\|T(z_1) - T(z_2)\| \le C|z_1 - z_2|^\gamma \text{ for all } z_1, z_2 \in \Gamma_\theta.$$

We consider the Schrödinger operator $H = -\Delta + V$ on $L^2(\mathbb{R}^d)$ under Condition 1.1. The resolvent is denoted by $R(z) = (H - z)^{-1}$. In the statement below of (a version of) [FS, Theorem 1.1] some conditions on the potential V are slightly changed. Comments at this point are given after the statement.

Theorem 2.1 (LAP). *Suppose Condition 1.1. For all $s > s_0$ the family of operators $T(z) = \langle x\rangle^{-s}R(z)\langle x\rangle^{-s}$ is uniformly Hölder continuous in Γ_θ. In particular the limits*

$$T(0+\mathrm{i}0) = \langle x\rangle^{-s}R(0+\mathrm{i}0)\langle x\rangle^{-s} = \lim_{z\to 0, z\in\Gamma_\theta} T(z),$$
$$T(0-\mathrm{i}0) = \langle x\rangle^{-s}R(0-\mathrm{i}0)\ \langle x\rangle^{-s} = \lim_{z\to 0, z\in\Gamma_\theta} T(\bar z)$$

exist in $\mathcal{B}(L^2(\mathbb{R}^d))$.

For all $s > 1/2$ there exists $C > 0$ such that for all $z \in \Gamma_\theta$

$$\|\langle x\rangle^{-s} f_{|z|}^{1/2} R(z) f_{|z|}^{1/2}\langle x\rangle^{-s}\| \le C. \tag{2.3}$$

We have already noted that due to (4) and (5) the operator $V_2(-\Delta + i)^{-1}$ is a compact, which occurs as a separate condition in [FS, Theorem 1.1] (the condition (5) is a new condition compared to [FS, Theorem. 1.1]). Another condition from [FS, Theorem 1.1] that we omitted above is a version of unique continuation at infinity. This version, [FS, Assumption 2.1], is automatically satisfied, given (5), due to results of [FH] (for $d \le 3$) and [JK] (for $d \ge 3$). Applying it with $V \to V - \lambda$ for any $\lambda \ge 0$ in conjunction with [FS, Theorem 2.4] we have $\sigma_{\mathrm{pp}}(H) \cap [0,\infty) = \emptyset$ (for $d = 1, 2, 3$ absence of strictly positive eigenvalues follows alternatively from [FH, Corollary 1.4]). The absence of non-negative eigenvalues is of course a consequence of (2.3), however this property is part of the proof of the latter bound.

We note that imposing the conditions (1) and (3) only near infinity may seem, with the other conditions of Condition 1.1, to weaken the assumptions. However this is not the case cf. a discussion in [FS, Section 3]. On the other hand it suffices to have the bounds (2) for $|\alpha| \le 2$. More precisely this is the case for Theorems 2.1 and 2.2. For Theorems 2.3 and 2.5 we need V_1 to be a "symbol" and all bounds of (2) are then needed.

Notice also that (2.3) is stronger than boundedness of the family $T(\cdot)$ (which is a consequence of the uniform Hölder continuity).

A new result presented here is the following improvement of (2.3) in terms of Besov spaces defined as follows (abstractly):

Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Let $R_0 = 0$ and $R_j = 2^{j-1}$ for $j \in \mathbb{N}$. We define correspondingly characteristic functions $F_j = F(R_{j-1} \le |\cdot| < R_j)$ and the space

$$B = B(A) = \big\{u \in \mathcal{H} \big| \sum_{j\in\mathbb{N}} R_j^{1/2}\|F_j(A)u\| =: \|u\|_B < \infty\big\}.$$

We can identify (using the embeddings $\langle A\rangle^{-1}\mathcal{H} \subsetneq B \subsetneq \mathcal{H} \subsetneq B^*$, $\langle A\rangle := \sqrt{A^2+1}$) the dual space B^* as

$$B^* = B(A)^* = \big\{u \in \langle A\rangle\mathcal{H} \big| \sup_{j\ge 1} R_j^{-1/2}\|F_j(A)u\| =: \|u\|_{B^*} < \infty\big\}.$$

Alternatively, the elements u of B^* are those sequences $u = (u_j) \subseteq \mathcal{H}$ with $u_j \in \mathrm{Ran}(F_j(A))$ and $\sup_{j\in\mathbb{N}} R_j^{-1/2}\|u_j\| < \infty$.

We use these constructions with the operator A being given as multiplication by $|x|$ on the complex Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d_x)$.

Theorem 2.2 (Besov space bound). *There exists $C > 0$ such that for all $z \in \Gamma_\theta$*

$$\|f_{|z|}^{1/2} R(z) f_{|z|}^{1/2}\|_{\mathcal{B}(B(|x|),B(|x|)^*)} \le C. \tag{2.4}$$

The proof of Theorem 2.2 is partly based on Theorem 2.1 and various ideas of [AH, Mo1, Mo2, JP, FS].

2.2. **Sommerfeld radiation condition.** We shall give an outline of some microlocal estimates and characterization of solutions to the equation $Hu = v$. In particular we estimate and characterize the particular solution provided by Theorems 2.1 and 2.2. This particular solution is constructed as follows in terms of Besov spaces. First note that the relevant Besov space at zero energy is $B^\mu := B(\langle x\rangle^{2s_0}) = \langle x\rangle^{-\mu/4}B(|x|)$, cf. Theorem 2.2 (recall $s_0 := 1/2 + \mu/4$). We have the following characterization of the corresponding dual space

$$u \in (B^\mu)^* \Leftrightarrow u \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \text{ and } \sup_{R>1} R^{-s_0}\|F(|x| < R)u\| < \infty.$$

A slightly smaller space is given by

$$u \in (B^\mu)^*_0 \Leftrightarrow u \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \text{ and } \lim_{R\to\infty} R^{-s_0}\|F(|x| < R)u\| = 0.$$

Now suppose $v \in B^\mu$. Then due to Theorems 2.1 and 2.2 there exists the weak-star limit

$$u = R(0 + \mathrm{i}0)v = \operatorname*{w^\star\text{-}lim}_{z\to 0, z\in\Gamma_\theta} R(z)v \in (B^\mu)^*.$$

Note that indeed this u is a (distributional) solution to the equation $Hu = v$.

Let us state a microlocal property of this solution. We shall use (2.2) with

$$K = \epsilon_1\tilde{\epsilon}_1/(2-\mu), \tag{2.5}$$

where the ϵ's come from Condition 1.1. In terms of f_0 we then introduce symbols

$$a_0 = \frac{\xi^2}{f_0(x)^2}, \quad b_0 = \frac{\xi}{f_0(x)} \cdot \frac{x}{\langle x\rangle},$$

and we prove that

$$\mathrm{Op^w}(\chi_-(a_0)\tilde{\chi}_-(b_0))u \in (B^\mu)^*_0 \text{ for all } \chi_- \in C_c^\infty(\mathbb{R}) \text{ and } \tilde{\chi}_- \in C_c^\infty((-\infty, 1)). \tag{2.6a}$$

Here we use Weyl quantization (although this is not the only choice). These estimates are accompanied by "high energy estimates", stated as follows: Let us note that

$$f_{|z|}^{-2}(x)\big|V_1(x) - z\big| \le C_0' := \max(C_0/K, 1),$$

where C_0 is given in Condition 1.1 (2) (i.e. the constant with $\alpha = 0$). Consider real-valued $\chi_- \in C_c^\infty(\mathbb{R})$ such that $\chi_-(t) = 1$ in a neighbourhood of $[0, C_0']$, and let $\chi_+ := 1 - \chi_-$. We prove that for all such functions χ_+

$$\mathrm{Op^w}(\chi_+(a_0))u \in (B^\mu)^*_0. \tag{2.6b}$$

The support property of $\tilde{\chi}_-$ in (2.6a) mirrors that the particular solution studied is "outgoing", and we refer to (2.6a) as a *Sommerfeld radiation condition.* This condition (in fact a weaker version) suffices for a characterization as expresssed in the following result. Here and henceforth $L^2_s := \langle x\rangle^{-s}L^2(\mathbb{R}^d)$.

Theorem 2.3 (Uniqueness of outgoing solution, data in B^μ). *Suppose $v \in B^\mu$. Suppose u is a distributional solution to the equation $Hu = v$ belonging to the space L^2_s for some $s \in \mathbb{R}$, and suppose that there there exists $\sigma \in (0, 1]$ such that*

$$\mathrm{Op^w}(\chi_-(a_0)\tilde{\chi}_-(b_0))u \in (B^\mu)^*_0 \text{ for all } \chi_- \in C_c^\infty(\mathbb{R}) \text{ and } \tilde{\chi}_- \in C_c^\infty((-\infty, \sigma)). \tag{2.7}$$

Then $u = R(0 + \mathrm{i}0)v$. In particular (2.6a) *and* (2.6b) *hold.*

Corollary 2.4. *Suppose $u \in (B^\mu)^*_0$ solves the equation $Hu = 0$. Then $u = 0$.*

We have another version of uniqueness of the outgoing solution (with $\delta > 0$ as in Condition 1.1).

Theorem 2.5 (Uniqueness of outgoing solution, data in L^2_s). *Suppose $v \in L^2_s$ for some $s > s_0$.*

i) Suppose u is a distributional solution to the equation $Hu = v$ belonging to the space $L^2_{-s_0-\delta}$, and suppose that there there exists $\sigma > 0$ such that

$$\mathrm{Op^w}(\chi_-(a_0)\tilde{\chi}_-(b_0))u \in L^2_{-s_0} \text{ for all } \chi_- \in C_c^\infty(\mathbb{R}) \text{ and } \tilde{\chi}_- \in C_c^\infty((-\infty, \sigma-1)). \tag{2.8}$$

Then $u = R(0 + \mathrm{i}0)v$.

ii) Moreover the state $u = R(0 + \mathrm{i}0)v$ obeys: For all $t < \min(s - s_0, \delta) - s_0$

$$\mathrm{Op^w}(\chi_-(a_0)\tilde{\chi}_-(b_0))u \in L^2_t \text{ for all } \chi_- \in C_c^\infty(\mathbb{R}) \text{ and } \tilde{\chi}_- \in C_c^\infty((-\infty, \sigma)), \tag{2.9a}$$

$$\mathrm{Op^w}(\chi_+(a_0))u \in L^2_t \text{ for all functions } \chi_+ \text{ as in (2.6b)}. \tag{2.9b}$$

In particular we can take $t = -s_0$ in (2.9a) and (2.9b).

The proof of the uniqueness statements of Theorems 2.3 and 2.5 relies partly on a "propagation of singularities" result in the spirit of [Hö2, Me, Va, DS1]. For example it follows that (2.8) is valid for all $\sigma < 2$ if it is valid for some $\sigma > 0$, and consequently Theorem 2.5 i) follows from Theorem 2.3. The bounds (2.9a) and (2.9b) essentially follow from [FS]. We note that the "incoming" solution $u = R(0 - \mathrm{i}0)v$ can be characterized similarly. Our results generalize [DS1, Proposition 4.10] at zero energy. For similar results for positive energies and for larger classes of potentials see [Hö1, Theorem 30.2.10] and [GY].

Remark 2.6. Define under Condition 1.1 the operator

$$\delta(0) = (2\pi \mathrm{i})^{-1}\big(R(0+\mathrm{i}0) - R(0-\mathrm{i}0)\big) = \pi^{-1}\mathrm{Im}\,\big(R(0+\mathrm{i}0)\big) \in \mathcal{B}(B^\mu, (B^\mu)^*),$$

and note that its range

$$\mathrm{Ran}(\delta(0)) \subseteq \mathcal{E} := \{u \in (B^\mu)^* \,|\, Hu = 0\}.$$

Under some stronger conditions it follows from [DS1, Theorem 8.2] that $\mathrm{Ran}(\delta(0)) = \mathcal{E}$ (proved in terms of wave matrices at zero energy). Equality is an open problem under Condition 1.1, in fact it is only known [FS] that $\delta(0) \neq 0$. More generally "scattering theory at zero energy" in the spirit of [DS1, Theorem 8.2] is an open problem under Condition 1.1.

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