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一般化された sine-Gordon 方程式の厳密解法

Exact method of solution for the generalized sine-Gordon equation

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Abstract

We develop a direct method for solving the generalized sine-Gordon equation $u_{tx}=(1+\partial_x^2)\sin u$. Using the bilinear transformation method, we construct exact multisoliton solutions and investigate their properties. In particular, we show that the equation exhibits kink and breather solutions and does not admit multi-valued solutions like loop solitons. We also demonstrate that the equation reduces to the short pulse and sine-Gordon equations in appropriate scaling limits. The limiting form of the multisoliton solutions are also presented. Finally, we derive an infinite number of conservation laws by using a novel Bäcklund transformation connecting solutions of the sine-Gordon and generalized sine-Gordon equations.

1. Introduction

The generalized sine-Gordon (sG) equation

$$u_{tx}=(1+\nu\partial_x^2)\sin u, \tag{1.1}$$

where $u=u(x,t)$ is a scalar-valued function, ν is a real parameter, $\partial_x^2=\partial^2/\partial x^2$ and the subscripts t and x appended to u denote partial differentiation, has been derived by Fokas [1]. In the case of $\nu=-1$, its integrability was established by constructing a Lax pair associated with it and the initial value problem was formulated for decaying initial data by means of the inverse scattering method [2]. Quite recently, we developed a systematic method for solving equation (1.1) with $\nu=-1$ and obtained soliton solutions in the form of parametric representation [3].

Here, we consider equation (1.1) with $\nu=1$

$$u_{tx}=(1+\partial_x^2)\sin u. \tag{1.2}$$

One of the remarkable features of equation (1.2) is that it does not admit multi-valued solutions like loop solitons as obtained in the case of $\nu=-1$. The detail of this report has been published in [4].

2. Exact method of solution

2.1. Hodograph transformation

First, we introduce the new dependent variable r in accordance with the relation

$$r^2 = 1 - u_x^2, \qquad (0 < r < 1), \tag{2.1}$$

to transform equation (1.2) into the conservation law of the form

$$r_t - (r \cos u)_x = 0. \tag{2.2}$$

This expression makes it possible to define the hodograph transformation $(x, t) \to (y, \tau)$ by

$$dy = r dx + r \cos u \, dt, \qquad d\tau = dt. \tag{2.3}$$

The x and t derivatives are then rewritten in terms of the y and τ derivatives as

$$\frac{\partial}{\partial x} = r\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + r \cos u \frac{\partial}{\partial y}. \tag{2.4}$$

With the new variables y and τ, (2.1) and (2.2) are recast into the form

$$r^2 = 1 - r^2 u_y^2, \tag{2.5}$$

$$\left(\frac{1}{r}\right)_\tau + (\cos u)_y = 0, \tag{2.6}$$

respectively. Further reduction is possible if one defines the variable ϕ by

$$u_y = \sinh \phi, \qquad \phi = \phi(y, \tau). \tag{2.7}$$

It follows from (2.5) and (2.7) that

$$\frac{1}{r} = \cosh \phi. \tag{2.8}$$

Substituting (2.7) and (2.8) into equation (2.6), we find

$$\phi_\tau = \sin u. \tag{2.9}$$

If we eliminate the variable ϕ from (2.7) and (2.9), we obtain a single PDE for u

$$\frac{u_{\tau y}}{\sqrt{1 + u_y^2}} = \sin u. \tag{2.10}$$

Similarly, elimination of the variable u gives a single PDE for ϕ

$$\frac{\phi_{\tau y}}{\sqrt{1 - \phi_\tau^2}} = \sinh \phi. \tag{2.11}$$

By inverting the hodograph transformation (2.3) and using (2.8), the equation that determines the inverse mapping $(y,\tau) \to (x,t)$ is found to be governed by the system of linear PDEs for $x = x(y,\tau)$

$$x_y = \cosh\phi, \tag{2.12a}$$

$$x_\tau = -\cos u. \tag{2.12b}$$

2.2. Bilinear formalism

Let σ and σ' be solutions of the sG equation

$$\sigma_{\tau y} = \sin\sigma, \qquad \sigma = \sigma(y,\tau), \tag{2.13a}$$

$$\sigma'_{\tau y} = \sin\sigma', \qquad \sigma' = \sigma'(y,\tau). \tag{2.13b}$$

The solutions of the above equations can be put into the form

$$\sigma = 2\mathrm{i}\,\ln\frac{f'}{f}, \qquad \sigma' = 2\mathrm{i}\,\ln\frac{g'}{g}. \tag{2.14a, b}$$

For soliton solutions, the tau functions f, f', g and g' satisfy the following system of bilinear equations:

$$D_\tau D_y f\cdot f = \frac{1}{2}(f^2 - f'^2), \qquad D_\tau D_y f'\cdot f' = \frac{1}{2}(f'^2 - f^2), \tag{2.15a, b}$$

$$D_\tau D_y g\cdot g = \frac{1}{2}(g^2 - g'^2), \qquad D_\tau D_y g'\cdot g' = \frac{1}{2}(g'^2 - g^2), \tag{2.16a, b}$$

where the bilinear operators D_τ and D_y are defined by

$$D_\tau^m D_y^n f\cdot g = (\partial_\tau - \partial_{\tau'})^m (\partial_y - \partial_{y'})^n f(\tau,y)g(\tau',y')|_{\tau'=\tau,\, y'=y}, \qquad (m,n = 0,1,2,\ldots). \tag{2.17}$$

Now, we seek solutions of equations (2.7) and (2.9) of the form

$$u = \mathrm{i}\,\ln\frac{F'}{F}, \qquad \phi = \ln\frac{G'}{G}, \tag{2.18a, b}$$

where F, F', G and G' are new tau functions. If we impose the condition

$$F'F = G'G, \tag{2.19}$$

among these tau functions, then equations (2.7) and (2.9) can be transformed to the following bilinear equations

$$\mathrm{i}\,D_y F'\cdot F = \frac{1}{2}(G'^2 - G^2), \tag{2.20}$$

$$\mathrm{i}\,D_\tau G' \cdot G = \frac{1}{2}(F^2 - F'^2), \tag{2.21}$$

respectively. The proposition below provides the tau functions F, F', G and G' in terms of f, f', g and g'.

Proposition 2.1. *If we impose the conditions for the tau functions f, f', g and g'*

$$\mathrm{i}\,D_y f \cdot g' = \frac{1}{2}(fg' - f'g), \qquad \mathrm{i}\,D_y f' \cdot g = \frac{1}{2}(f'g - fg'), \tag{2.22a, b}$$

$$\mathrm{i}\,D_\tau f \cdot g = -\frac{1}{2}(fg - f'g'), \qquad \mathrm{i}\,D_\tau f' \cdot g' = -\frac{1}{2}(f'g' - fg), \tag{2.23a, b}$$

then the solutions of bilinear equations (2.20) and (2.21) subjected to the condition (2.19) are given by

$$F = fg, \quad F' = f'g', \tag{2.24a}$$

$$G = fg', \quad G' = f'g. \tag{2.24b}$$

2.3. Parametric representation

Proposition 2.2. $\cosh\phi$ *is given in terms of the tau functions f, f', g and g' as*

$$\cosh\phi = 1 + \mathrm{i}\left(\ln\frac{g'g}{f'f}\right)_y. \tag{2.25}$$

Integrating (2.12a) with (2.25) by y yields the expression of x

$$x = y + \mathrm{i}\,\ln\frac{g'g}{f'f} + d(\tau), \tag{2.26}$$

where d is an integration constant which depends generally on τ. The expression (2.26) now leads to our main result:

Theorem 2.1. *The solution of equation (1.2) can be expressed by the parametric representation*

$$u(y,\tau) = \mathrm{i}\,\ln\frac{f'g'}{fg}, \tag{2.27a}$$

$$x(y,\tau) = y - \tau + \mathrm{i}\,\ln\frac{g'g}{f'f} + y_0, \tag{2.27b}$$

where the tau functions f, f', g and g' satisfy equations (2.15), (2.16), (2.22) and (2.23) and y_0 is an arbitrary constant independent of y and τ.

An interesting feature of the parametric solution (2.27) is that it never exhibits singularities as encountered in the case of equation (1.1) with $\nu = -1$. Indeed

$$u_x = ru_y = \tanh\phi, \tag{2.28}$$

showing that u_x always takes a finite value.

2.4. Multisoliton solutions

Theorem 2.2. *The tau-functions f, f', g and g' given below satisfy both the bilinear forms (2.15) and (2.16) of the gG equation and the bilinear equations (2.22) and (2.23),*

$$f = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j \left(\xi_j + d_j + \frac{\pi}{2}\mathrm{i}\right) + \sum_{1\le j<k\le N} \mu_j\mu_k\gamma_{jk}\right], \tag{2.29a}$$

$$f' = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j \left(\xi_j + d_j - \frac{\pi}{2}\mathrm{i}\right) + \sum_{1\le j<k\le N} \mu_j\mu_k\gamma_{jk}\right], \tag{2.29b}$$

$$g = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j \left(\xi_j - d_j + \frac{\pi}{2}\mathrm{i}\right) + \sum_{1\le j<k\le N} \mu_j\mu_k\gamma_{jk}\right], \tag{2.30a}$$

$$g' = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j \left(\xi_j - d_j - \frac{\pi}{2}\mathrm{i}\right) + \sum_{1\le j<k\le N} \mu_j\mu_k\gamma_{jk}\right], \tag{2.30b}$$

where

$$\xi_j = p_j y + \frac{1}{p_j}\tau + \xi_{j0}, \qquad (j = 1, 2, ..., N), \tag{2.31a}$$

$$e^{\gamma_{jk}} = \left(\frac{p_j - p_k}{p_j + p_k}\right)^2, \qquad (j, k = 1, 2, ..., N; j \neq k), \tag{2.31b}$$

$$e^{d_j} = \sqrt{\frac{1 + \mathrm{i}p_j}{1 - \mathrm{i}p_j}}, \qquad (j = 1, 2, ..., N). \tag{2.31c}$$

Here, p_j and ξ_{j0} are arbitrary complex parameters satisfying the conditions $p_j \neq \pm p_k$ for $j \neq k$, $\mathrm{i} = \sqrt{-1}$ and N is an arbitrary positive integer. The notation $\sum_{\mu=0,1}$ implies the summation over all possible combination of $\mu_1 = 0, 1, \mu_2 = 0, 1, ..., \mu_N = 0, 1$.

The parametric solution (2.27) with (2.29) and (2.30) is characterized by the $2N$ complex parameters p_j and ξ_{j0} $(j = 1, 2, ..., N)$. It produces in general the complex-valued solutions. The real-valued solutions are obtainable if one imposes certain conditions on these parameters. Actually, there arise various type of solutions depending on values of the parameters. These solutions include kinks, antikinks and breathers. Among them, we consider following three types:

Type 1: Kink solution

First, let p_j and ξ_{j0} $(j = 1, 2, ..., N)$ be real quantities. Then $f' = g^*$ and $g' = f^*$ and (2.27) becomes

$$u(y, \tau) = \mathrm{i}\ \ln \frac{f^* g^*}{fg}, \qquad x(y, \tau) = y - \tau + \mathrm{i}\ \ln \frac{f^* g}{fg^*} + y_0. \tag{2.32a, b}$$

Type 2: Breather solution

We put $N = 2M$ where M is a positive integer, and specify the parameters p_j and $\xi_{j,0}$ $(j = 1, 2, ..., 2M)$ as

$$p_{2j-1} = p_{2j}^*, \qquad \xi_{2j-1,0} = \xi_{2j,0}^*, \quad (j = 1, 2, ..., M). \tag{2.33}$$

It turns out that $f' = g^*$ and $g' = f^*$. Then, the solution can be written in the same form as (2.32).

Type 3: Kink-breather solution

Let $N = 2M + M'$ where M and M' are positive integers. In addition to the parameterization given by (2.33), the $2M'$ parameters $p_j(> 0)$ and ξ_{j0} $(j = 2M + 1, 2M + 2, ..., 2M + M')$ are chosen to be real. Then, the parameteric solution (2.32) represents the solution describing the interaction among M breathers and M' kinks. The antikink-breather solution can be constructed similarly.

For the above three types of solutions, ϕ from (2.18b) and u_x from (2.28) can be given explicitly in terms of the tau functions f, g and their complex conjugate as

$$\phi = \ln \frac{g^* g}{f^* f}, \tag{2.34}$$

$$u_x = \frac{(g^* g)^2 - (f^* f)^2}{(g^* g)^2 + (f^* f)^2}. \tag{2.35}$$

Note that (2.34) provides real solutions of equation (2.11).

3. Properties of solutions

3.1. 1-soliton solutions

The tau-functions for the 1-soliton solutions are given by (2.29) and (2.30) with $N = 1$:

$$f = 1 + \mathrm{i}e^{\xi_1 + d_1}, \qquad g = 1 + \mathrm{i}e^{\xi_1 - d_1}, \tag{3.1a, b}$$

$$\xi_1 = p_1 y + \frac{\tau}{p_1} + \xi_{10}, \qquad e^{d_1} = \sqrt{\frac{1 + \mathrm{i}p_1}{1 - \mathrm{i}p_1}}. \tag{3.1c}$$

The real parameters p_1 and ξ_{10} are related to the amplitude and phase of the soliton, respectively and ξ_1 is the phase variable characterizing the solution. The parametric representation of the solution (2.32) can be written in the form

$$u = 2\ \tan^{-1}\left(\sqrt{1 + p_1^2}\ \sinh\ \xi_1\right) + \pi, \tag{3.2a}$$

$$x = y - \tau + 2\tan^{-1}(p_1 \tanh \xi_1) + 2\tan^{-1} p_1 + y_0. \tag{3.2b}$$

Figure 1 shows a typical profile of the kink solution as a function of X together with the corresponding profile of $v \equiv u_x$.

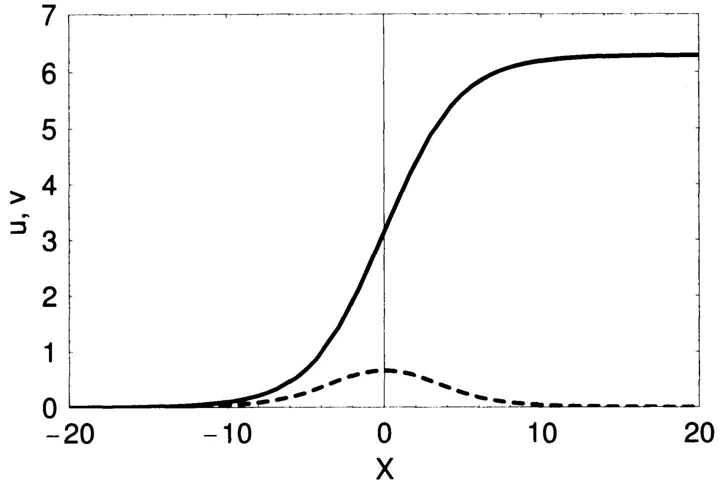


Figure 1 The profile of a kink u (solid line) and corresponding profile of $v \equiv u_X$ (broken line). The parameter p_1 is set to 0.4 and the parameter y_0 is chosen such that the center position of u_X is at $X = 0$. Here, $X = x + c_1 t + x_0$, $c_1 = 1/p_1^2 + 1$.

3.2. 2-soliton solutions

The tau-functions for the 2-soliton solutions read from (2.29) and (2.30) with $N = 2$ in the form

$$f = 1 + \mathrm{i}\left(\mathrm{e}^{\xi_1 + d_1} + \mathrm{e}^{\xi_2 + d_2}\right) - \delta \mathrm{e}^{\xi_1 + \xi_2 + d_1 + d_2}, \quad g = 1 + \mathrm{i}\left(\mathrm{e}^{\xi_1 - d_1} + \mathrm{e}^{\xi_2 - d_2}\right) - \delta \mathrm{e}^{\xi_1 + \xi_2 - d_1 - d_2}, \tag{3.3a, b}$$

$$\xi_j = p_j y + \frac{\tau}{p_j} + \xi_{j0}, \qquad \mathrm{e}^{d_j} = \sqrt{\frac{1 + \mathrm{i}p_j}{1 - \mathrm{i}p_j}} \quad (j = 1, 2), \qquad \delta = \frac{(p_1 - p_2)^2}{(p_1 + p_2)^2}. \tag{3.3c}$$

The parametric solution (2.32) with (3.3) represents three types of solutions, depending on values of the parameters p_j and ξ_{0j} $(j = 1, 2)$, i.e., kink-kink, kink-antikink and breather solutions.

3.2.1. Kink-kink solution

If we specify p_1 and p_2 be positive and ξ_{01} and ξ_{02} be real, then the kink-kink solution is obtained. The solution represents the so-called 4π kink. In figure 2a-c, we depict a typical profile of $v (\equiv u_x)$ instead of u for three different times. It represents the interaction of two solitons with the amplitudes $A_1 = 0.38$ and $A_2 = 0.75$.

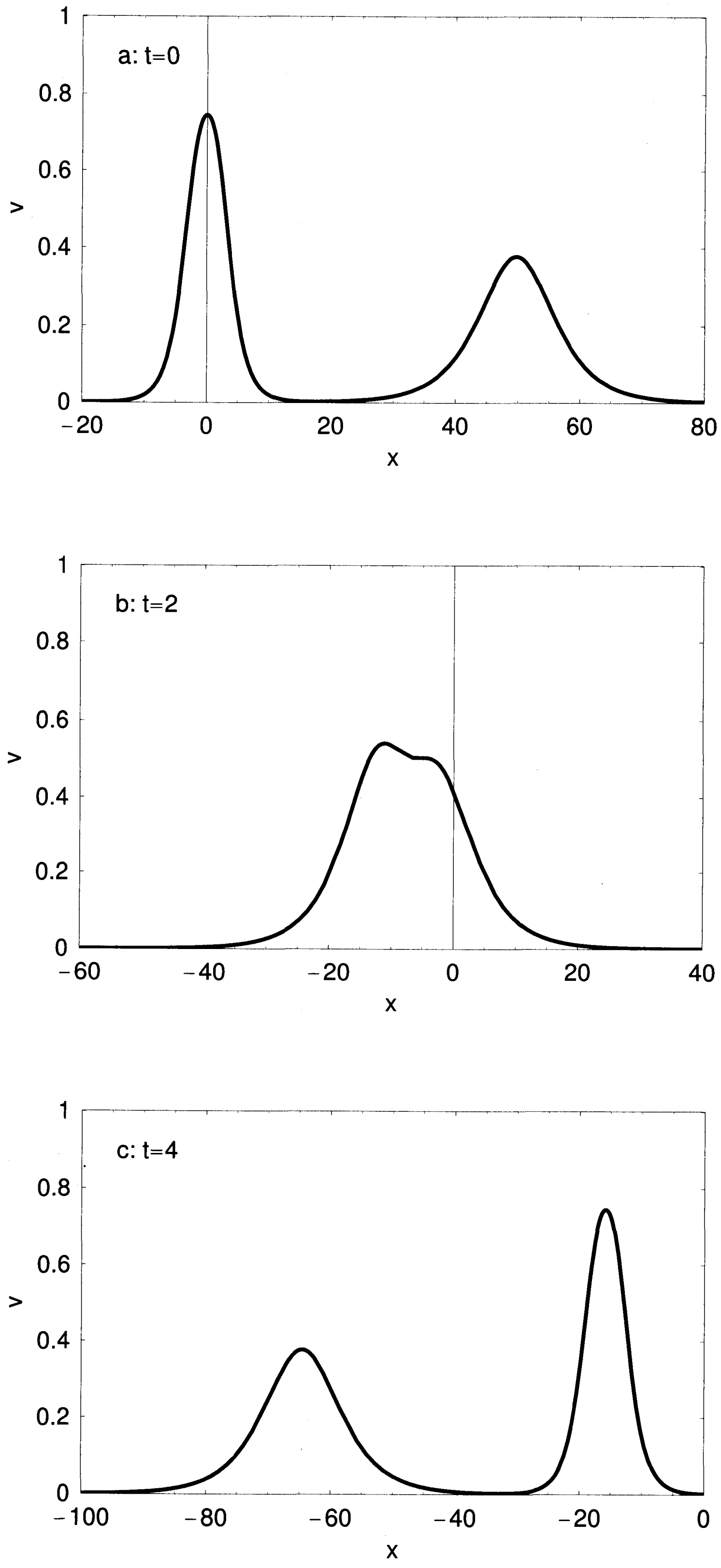


Figure 2 a-c The profile of a two-soliton solution $v \equiv u_x$ for three different times, a: $t = 0$, b: $t = 2$, c: $t = 4$. The parameters are chosen as $p_1 = 0.2$, $p_2 = 0.5$, $\xi_{10} = -8$, $\xi_{20} = 0$.

The formula for the phase shift arising from the interaction of two solitons is given as follows:

$$\Delta_1 = -\frac{1}{p_1}\ln\left(\frac{p_1-p_2}{p_1+p_2}\right)^2 + 4\tan^{-1}p_2, \qquad \Delta_2 = \frac{1}{p_2}\ln\left(\frac{p_1-p_2}{p_1+p_2}\right)^2 - 4\tan^{-1}p_1. \tag{3.4a, b}$$

It can be verified from (3.4) that $\Delta_1 > 0$ and $\Delta_2 < 0$ for $0 < p_1 < p_2$. In the present example, formula (3.4) yields $\Delta_1 = 10.3$ and $\Delta_2 = -4.2$.

3.2.2. Breather solution

The breather solution can be constructed following the parameterization given by (2.33). For $M = 1$, let

$$p_1 = a + ib, \qquad p_2 = a - ib = p_1^*, \qquad (a > 0,\ b > 0), \tag{3.5a}$$

$$\xi_{10} = \lambda + i\mu, \qquad \xi_{20} = \lambda - i\mu = \xi_{10}^*. \tag{3.5b}$$

Then, f and g from (2.29) and (2.30) become

$$f = 1 + i(e^{\xi_1+d_1} + e^{\xi_1^*-d_1^*}) + \left(\frac{b}{a}\right)^2 e^{\xi_1+\xi_1^*+d_1-d_1^*}, \tag{3.6a}$$

$$g = 1 + i(e^{\xi_1-d_1} + e^{\xi_1^*+d_1^*}) + \left(\frac{b}{a}\right)^2 e^{\xi_1+\xi_1^*-d_1+d_1^*}, \tag{3.6b}$$

where

$$\xi_1 = \theta + i\chi, \tag{3.6c}$$

$$\theta = a\left(y + \frac{1}{a^2+b^2}\tau\right) + \lambda, \tag{3.6d}$$

$$\chi = b\left(y - \frac{1}{a^2+b^2}\tau\right) + \mu, \tag{3.6e}$$

$$e^{d_1} = \sqrt{\frac{1-a^2-b^2+2ia}{a^2+(1-b)^2}} \equiv \alpha\, e^{i\beta}. \tag{3.6f}$$

Figure 3 shows a profile of u for three different times.

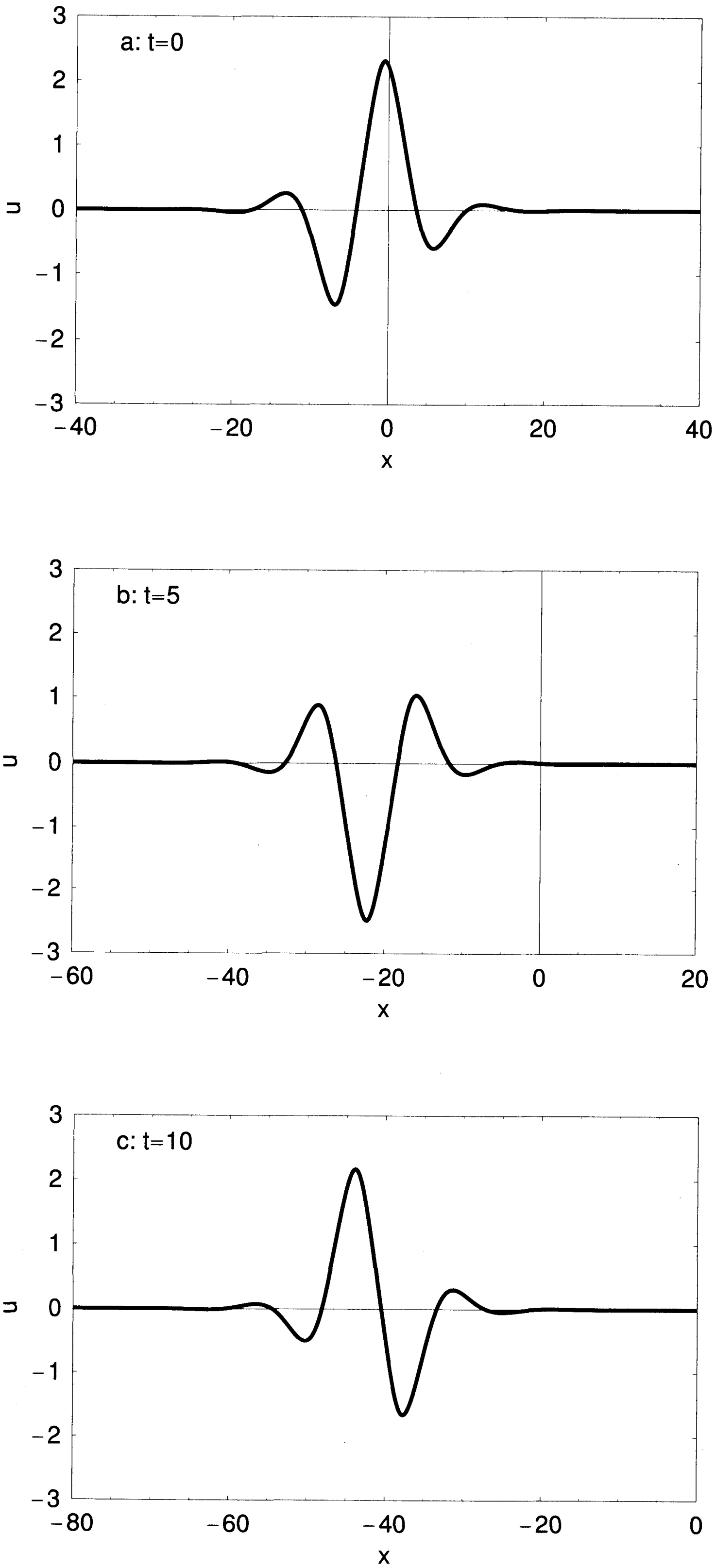


Figure 3 a-c The profile of a breather solution for three different times, a: $t = 0$, b: $t = 5$, c: $t = 10$. The parameters are chosen as $p_1 = 0.3 + 0.5\,\mathrm{i}$, $p_2 = p_1^* = 0.3 - 0.5\,\mathrm{i}$, $\xi_{10} = \xi_{20}^* = 0$.

3.3. N-soliton solutions

3.3.1. N-kink solution

Let the velocity of the jth kink be $c_j = (1/p_j^2)+1$ $(p_j > 0)$ and order the magnitude of the velocity of each kink as $c_1 > c_2 > ... > c_N$. We observe the interaction of N kinks in a moving frame with a constant velocity c_n. We take the limit $t \to -\infty$ with the phase variable ξ_n being fixed. Then

$$u \sim 2 \tan^{-1}\left[\sqrt{1+p_n^2}\,\sinh\left(\xi_n + \delta_n^{(-)}\right)\right] + \pi, \tag{3.7a}$$

$$x \sim y-\tau+2 \tan^{-1}\left[p_n \tanh\left(\xi_n + \delta_n^{(-)}\right)\right]+4 \sum_{j=n+1}^{N} \tan^{-1} p_j+2 \tan^{-1} p_n+y_0. \tag{3.7b}$$

As $t \to +\infty$, the expressions corresponding to (3.7) are given by

$$u \sim 2\tan^{-1}\left[\sqrt{1+p_n^2}\,\sinh\left(\xi_n + \delta_n^{(+)}\right)\right] + \pi, \tag{3.8a}$$

$$x \sim y-\tau+2 \tan^{-1}\left[p_n \tanh\left(\xi_n + \delta_n^{(+)}\right)\right]+4\sum_{j=1}^{n-1} \tan^{-1} p_j+2 \tan^{-1} p_n+y_0. \tag{3.8b}$$

where

$$\delta_n^{(+)} = \sum_{j=1}^{n-1} \ln\left(\frac{p_n - p_j}{p_n + p_j}\right)^2, \tag{3.8c}$$

$$\delta_n^{(-)} = \sum_{j=n+1}^{N} \ln\left(\frac{p_n - p_j}{p_n + p_j}\right)^2, \tag{3.8d}$$

$$\delta_n = \prod_{n+1\le j<k\le N} \left(\frac{p_j - p_k}{p_j + p_k}\right)^2. \tag{3.8e}$$

Let x_c be the center position of the nth kink in the (x,t) coordinate system. As $t \to -\infty$

$$x_c + c_n t + x_{n0} \sim -\frac{1}{p_n}\delta_n^{(-)} + 4 \sum_{j=n+1}^{N} \tan^{-1} p_j + y_0, \tag{3.9}$$

where $x_{n0} = \xi_{n0}/p_n - 2 \tan^{-1} p_n$. As $t \to +\infty$, on the other hand, the corresponding expression turns out to be

$$x_c + c_n t + x_{n0} \sim -\frac{1}{p_n}\delta_n^{(+)} + 4\sum_{j=1}^{n-1} \tan^{-1} p_j + y_0. \tag{3.10}$$

If we take into account the fact that all kinks propagate to the left, we can define the phase shift of the nth kink as

$$\Delta_n = x_c(t \to -\infty) - x_c(t \to +\infty). \tag{3.11}$$

Using (3.8c), (3.8d), (3.9) and (3.10), we find that

$$\Delta_n = \frac{1}{p_n}\left\{\sum_{j=1}^{n-1}\ln\left(\frac{p_n-p_j}{p_n+p_j}\right)^2 - \sum_{j=n+1}^{N}\ln\left(\frac{p_n-p_j}{p_n+p_j}\right)^2\right\}$$

$$+4\sum_{j=n+1}^{N}\tan^{-1}p_j - 4\sum_{j=1}^{n-1}\tan^{-1}p_j, \quad (n=1,2,...,N). \tag{3.12}$$

3.3.2. M-breather solution

We specify the parameters in (2.29) and (2.30) for the tau-functions f and g as

$$p_{2j-1} = p_{2j}^* \equiv a_j + \mathrm{i}b_j, \quad a_j > 0, \quad b_j > 0, \quad (j=1,2,...,M), \tag{3.13a}$$

$$\xi_{2j-1,0} = \xi_{2j,0}^* \equiv \lambda_j + \mathrm{i}\mu_j, \quad (j=1,2,...,M). \tag{3.13b}$$

Then, the phase variables ξ_{2j-1} and ξ_{2j} are written as

$$\xi_{2j-1} = \theta_j + \mathrm{i}\chi_j, \quad (j=1,2,...,M), \tag{3.14a}$$

$$\xi_{2j} = \theta_j - \mathrm{i}\chi_j, \quad (j=1,2,...,M), \tag{3.14b}$$

with the real phase variables

$$\theta_j = a_j(y + c_j\tau) + \lambda_j, \quad (j=1,2,...,M), \tag{3.14c}$$

$$\chi_j = b_j(y - c_j\tau) + \mu_j, \quad (j=1,2,...,M), \tag{3.14d}$$

$$c_j = \frac{1}{a_j^2+b_j^2}, \quad (j=1,2,...,M). \tag{3.14e}$$

The parametric solution (2.32) with (3.13) and (3.14) describes multiple collisions of M breathers.

3.3.3 Kink-breather solution

We take a 3-soliton solution with parameters p_j and ξ_{0j} $(j = 1,2,3)$. If one impose the conditions that $p_2 = p_1^*, \xi_{02} = \xi_{01}^*$ as already specified for the breather solution and $p_3(>0)$, ξ_{03} real for the kink solution, then the expression of u would represent a solution describing the interaction between a kink and a breather. The tau functions f and g now become

$$f = 1 + \mathrm{i}\left(s_1 e^{\xi_1} + \frac{1}{s_1^*}e^{\xi_1^*} + s_3 e^{\xi_3}\right) + \left(\frac{b}{a}\right)^2 \frac{s_1}{s_1^*}e^{\xi_1+\xi_1^*}$$

$$-\delta_{13}s_1 s_3 e^{\xi_1+\xi_3} - \delta_{13}^*\frac{s_3}{s_1^*}e^{\xi_1^*+\xi_3} + \mathrm{i}\left(\frac{b}{a}\right)^2\frac{s_1 s_3}{s_1^*}\delta_{13}\delta_{13}^* e^{\xi_1+\xi_1^*+\xi_3}, \tag{3.15a}$$

$$g = 1 + \mathrm{i}\left(\frac{1}{s_1}\mathrm{e}^{\xi_1} + s_1^*\mathrm{e}^{\xi_1^*} + \frac{1}{s_3}\mathrm{e}^{\xi_3}\right) + \left(\frac{b}{a}\right)^2 \frac{s_1^*}{s_1}\mathrm{e}^{\xi_1+\xi_1^*}$$

$$-\frac{\delta_{13}}{s_1 s_3}\mathrm{e}^{\xi_1+\xi_3} - \delta_{13}^*\frac{s_1^*}{s_3}\mathrm{e}^{\xi_1^*+\xi_3} + \mathrm{i}\left(\frac{b}{a}\right)^2 \frac{s_1^*}{s_1 s_3}\delta_{13}\delta_{13}^*\mathrm{e}^{\xi_1+\xi_1^*+\xi_3}. \tag{3.15b}$$

where

$$s_1 = \mathrm{e}^{d_1} = \sqrt{\frac{1-b+\mathrm{i}a}{1-b-\mathrm{i}a}} = \frac{1}{s_2^*}, \qquad s_3 = \sqrt{\frac{1+\mathrm{i}p_3}{1-\mathrm{i}p_3}}, \qquad \delta_{13} = \left(\frac{a-p_3+\mathrm{i}b}{a+p_3+\mathrm{i}b}\right)^2 = \delta_{23}^*. \tag{3.15c}$$

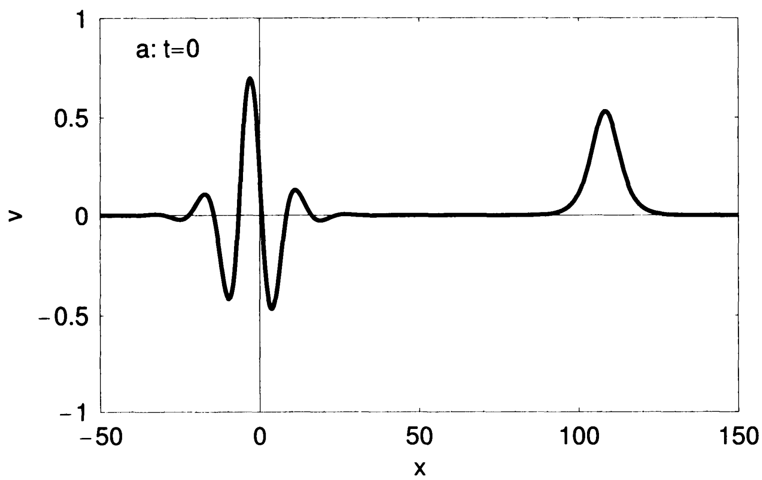
Figure 4a-c shows a typical profile of $v \equiv u_x$ for three different times. We see that the soliton overtakes the breather whereby it suffers a phase shift. Actually, one has for $p_3^2 < a^2 + b^2$

$$\Delta = \frac{2}{p_3}\ln\frac{(p_3+a)^2+b^2}{(p_3-a)^2+b^2} + 4\tan^{-1}\frac{2a}{1-a^2-b^2}, \tag{3.16a}$$

and for $a^2 + b^2 < p_3^2$

$$\Delta = -\frac{2}{p_3}\ln\frac{(p_3+a)^2+b^2}{(p_3-a)^2+b^2} - 4\tan^{-1}\frac{2a}{1-a^2-b^2}. \tag{3.16b}$$

In the present example, formula (3.16a) gives $\Delta = 7.7$.



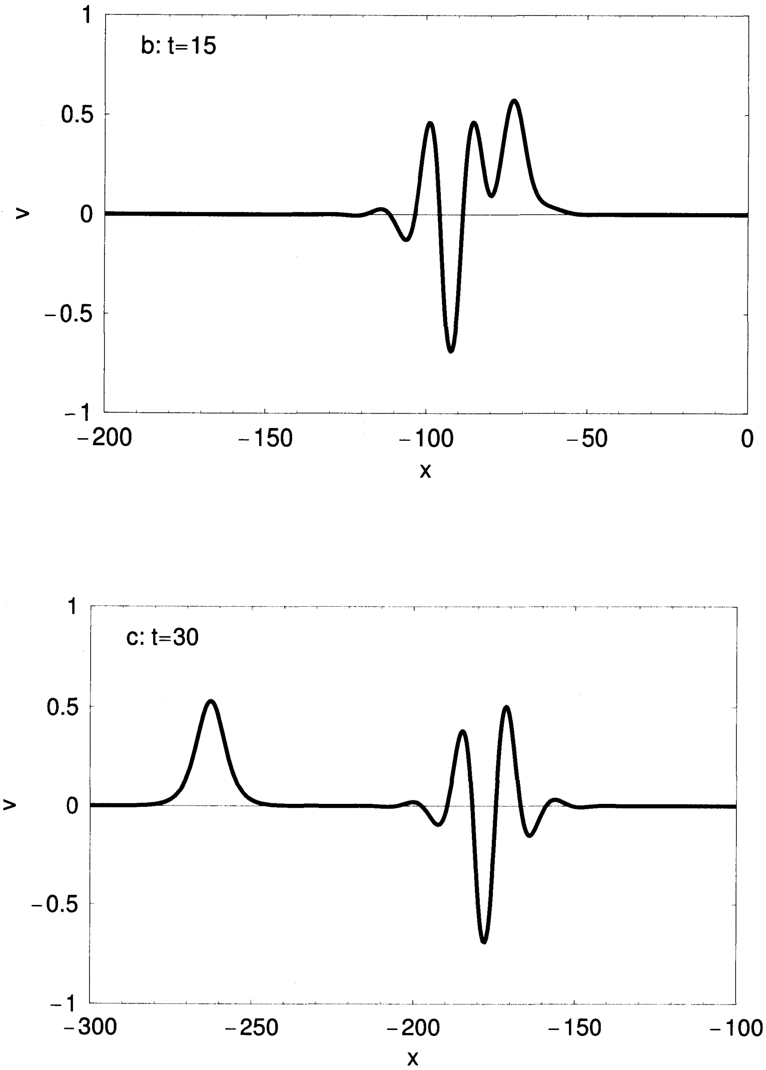


Figure 4 a-c The profile of $v \equiv u_x$ for three different times which represents the interaction between a soliton and a breather, a: $t = 0$, b: $t = 15$, c: $t = 30$. The parameters are chosen as $p_1 = 0.2 + 0.4\,\mathrm{i}$, $p_2 = p_1^* = 0.2 - 0.4\,\mathrm{i}$, $p_3 = 0.3$, $\xi_{10} = \xi_{20} = 0$, $\xi_{30} = -30$.

3.3.4 Breather-breather solution

The breather-breather (or 2-breather) solution is reduced from a 4-soliton solution. Figure 5a-c shows a typical profile of u for three different times. It represents a typical feature common to the interaction of solitons, i.e., each breather recovers its profile after collision.

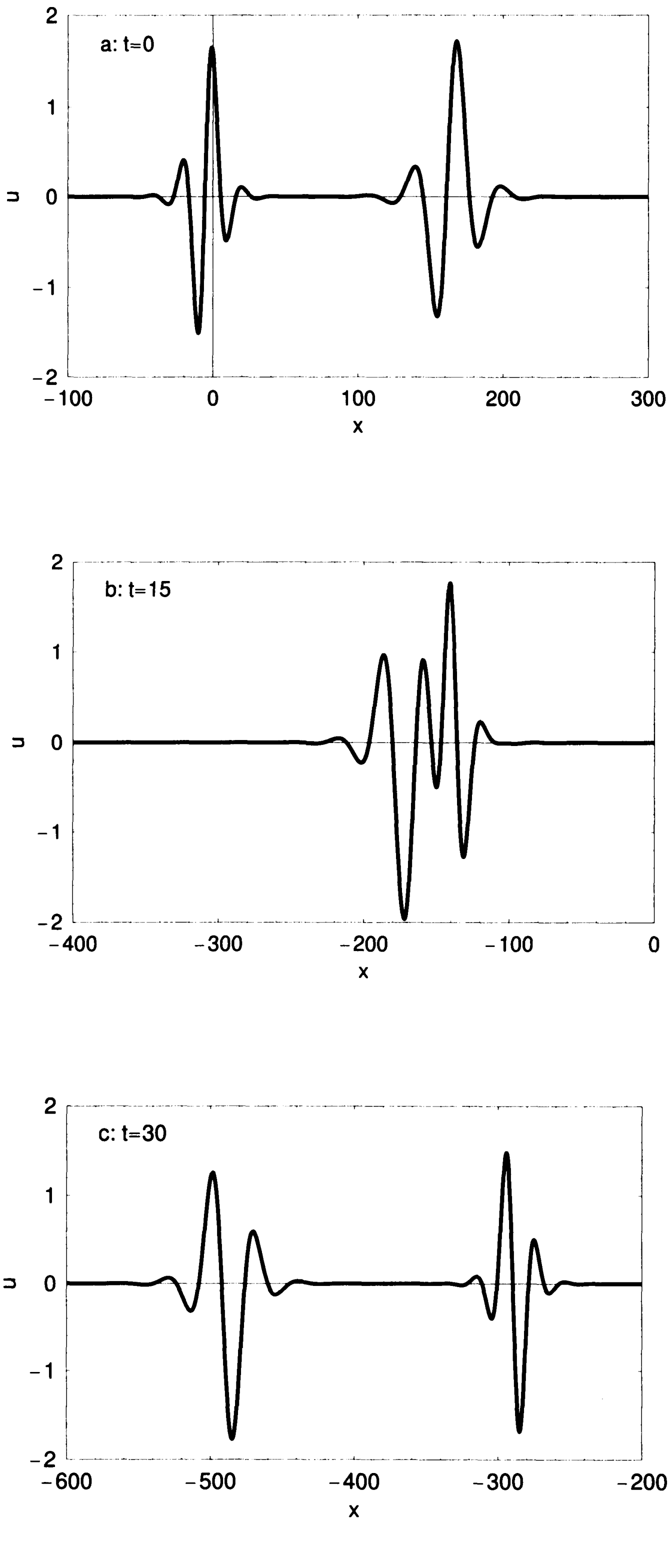


Figure 5a-c The profile of a breather-breather solution u for three different times, a: $t = 0$, b: $t = 15$, c: $t = 30$. The parameters are chosen as $p_1 = 0.1 + 0.2\,\mathrm{i}, p_2 = p_1^* = 0.1 - 0.2\,\mathrm{i}, p_3 = 0.15 + 0.3\,\mathrm{i}$, $p_4 = p_3^* = 0.15 - 0.3\,\mathrm{i}$, $\xi_{10} = \xi_{20}^* = -15$, $\xi_{30} = \xi_{40}^* = 0$.

4. Reduction to the short pulse and sG equations

We write the short pulse equation in the form

$$u_{tx} = u - \frac{\nu}{6}(u^3)_{xx}, \tag{4.1}$$

where $u = u(x,t)$ represents the magnitude of the electric field and ν is a real constant. The short pulse equation (4.1) with $\nu = -1$ was proposed as a model nonlinear equation describing the propagation of ultra-short optical pulses in nonlinear media [5]. Quite recently, equation (4.1) with $\nu = 1$

$$u_{tx} = u - \frac{1}{6}(u^3)_{xx}, \tag{4.2}$$

was shown to model the evolution of ultra-short pulses in the band gap of nonlinear metamaterials [6]. See [7] for a review on exact solutions of the short pulse equation and related topics.

4.1. Reduction to the short pulse equation

4.1.1. Scaling limit of the generalized sG equation

Let us first introduce new variables with bar according to the relations

$$\bar{u} = \frac{u}{\epsilon}, \qquad \bar{x} = \frac{1}{\epsilon}(x+t), \qquad \bar{y} = \frac{y}{\epsilon} \qquad \bar{y}_0 = \frac{y_0}{\epsilon}, \qquad \bar{t} = \epsilon t, \qquad \bar{\tau} = \epsilon\tau,$$

$$\bar{p}_j = \epsilon p_j, \qquad \bar{\xi}_{j0} = \xi_{j0}, \qquad (j = 1, 2, ..., N), \tag{4.3}$$

where ϵ is a small parameter and the quantities with bar are assumed to be order 1. Rewriting equation (1.2) in terms of the new variables and expanding $\sin \epsilon\bar{u}$ in an infinite series with respect to ϵ and comparing terms of order ϵ on both sides, we obtain equation (4.2) written by the new variables.

Under the scaling (4.3), expression (2.7) is invariant and hence we put $\bar{\phi} = \phi$ to give

$$\bar{u}_{\bar{y}} = \sinh \bar{\phi}. \tag{4.4}$$

Equation (2.9) then reduces to

$$\bar{\phi}_{\bar{\tau}} = \bar{u}. \tag{4.5}$$

Equations (2.10) and (2.11) now become

$$\frac{\bar{u}_{\bar{\tau}\bar{y}}}{\sqrt{1+\bar{u}_{\bar{y}}^2}} = \bar{u} \tag{4.6}$$

$$\bar{\phi}_{\bar{\tau}\bar{y}} = \sinh \bar{\phi}, \tag{4.7}$$

respectively. Equation (4.7) is known as the sinh-Gordon equation.

4.1.2. Scaling limit of the N-soliton solution

The expansion of the tau function f is given by

$$f = \sum_{\mu=0,1}\left(1 - \mathrm{i}\epsilon\sum_{j=1}^{N}\frac{\mu_j}{\bar{p}_j}\right)\exp\left[\sum_{j=1}^{N}\mu_j\left(\bar{\xi}_j + \pi\mathrm{i}\right) + \sum_{1\le j<k\le N}\mu_j\mu_k\bar{\gamma}_{jk}\right] + O(\epsilon^2)$$
$$= \bar{f} - \mathrm{i}\epsilon\bar{f}_{\bar{\tau}} + O(\epsilon^2), \tag{4.8a}$$

where

$$\bar{f} = \sum_{\mu=0,1}\exp\left[\sum_{j=1}^{N}\mu_j\left(\bar{\xi}_j + \pi\mathrm{i}\right) + \sum_{1\le j<k\le N}\mu_j\mu_k\bar{\gamma}_{jk}\right], \tag{4.8b}$$

$$\bar{\xi}_j = \bar{p}_j\bar{y} + \frac{\bar{\tau}}{\bar{p}_j} + \bar{\xi}_{j0}, \qquad (j = 1, 2, \ldots, N), \tag{4.8c}$$

$$\mathrm{e}^{\bar{\gamma}_{jk}} = \left(\frac{\bar{p}_j - \bar{p}_k}{\bar{p}_j + \bar{p}_k}\right)^2, \qquad (j, k = 1, 2, \ldots, N; j \neq k). \tag{4.8d}$$

Similarly

$$f' = \bar{g} - \mathrm{i}\epsilon\bar{g}_{\bar{\tau}} + O(\epsilon^2), \qquad g = \bar{g} + \mathrm{i}\epsilon\bar{g}_{\bar{\tau}} + O(\epsilon^2), \qquad g' = \bar{f} + \mathrm{i}\epsilon\bar{f}_{\bar{\tau}} + O(\epsilon^2), \tag{4.9a, b, c}$$

with

$$\bar{g} = \sum_{\mu=0,1}\exp\left[\sum_{j=1}^{N}\mu_j\bar{\xi}_j + \sum_{1\le j<k\le N}\mu_j\mu_k\bar{\gamma}_{jk}\right]. \tag{4.9d}$$

The parametric solution of the short pulse equation (4.2) in terms of the tau functions $\bar{f}$ and $\bar{g}$ is given as follows:

$$\bar{u} = 2\left(\ln\frac{\bar{g}}{\bar{f}}\right)_{\bar{\tau}}, \qquad \bar{x} = \bar{y} - 2(\ln\bar{f}\bar{g})_{\bar{\tau}} + \bar{y}_0. \tag{4.10a, b}$$

4.2. Reduction to the sG equation

If we introduce the following new scaled variables

$$\bar{u} = u, \qquad \bar{x} = \epsilon x, \qquad \bar{y} = \epsilon y, \qquad \bar{t} = \frac{t}{\epsilon}, \qquad \bar{\tau} = \frac{\tau}{\epsilon},$$
$$\bar{p}_j = \frac{p_j}{\epsilon}, \qquad \bar{\xi}_{j0} = \xi_{j0}, \; (j = 1, 2, \ldots, N), \tag{4.11}$$

then in the limit of $\epsilon \to 0$, we can deduce the generalized sG equation (1.2) to the sG equation

$$\bar{u}_{\bar{t}\bar{x}} = \sin\bar{u}. \tag{4.12}$$

The scaling limit of (2.27b) now leads to the expression $\bar{y} = \bar{x}$ which, combined with the obvious relation $\bar{\tau} = \bar{t}$, yields the limiting form of the tau functions (2.29) and (2.30)

$$f = \bar{f}, \qquad f' = \bar{f}', \qquad g = \bar{f}, \qquad g' = \bar{f}', \tag{4.13a}$$

where

$$\bar{f} = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j \left(\bar{\xi}_j + \frac{\pi}{2}\mathrm{i}\right) + \sum_{1\le j<k\le N} \mu_j \mu_k \bar{\gamma}_{jk}\right], \tag{4.13b}$$

$$\bar{f}' = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j \left(\bar{\xi}_j - \frac{\pi}{2}\mathrm{i}\right) + \sum_{1\le j<k\le N} \mu_j \mu_k \bar{\gamma}_{jk}\right], \tag{4.13c}$$

$$\bar{\xi}_j = \bar{p}_j \bar{x} + \frac{\bar{t}}{\bar{p}_j} + \bar{\xi}_{j0}, \qquad (j = 1, 2, ..., N), \tag{4.13d}$$

$$\mathrm{e}^{\bar{\gamma}_{jk}} = \left(\frac{\bar{p}_j - \bar{p}_k}{\bar{p}_j + \bar{p}_k}\right)^2, \qquad (j, k = 1, 2, ..., N; j \neq k). \tag{4.13e}$$

The parametric solution (2.27) with the tau functions (2.29) and (2.30) reduces to the usual form of the N-soliton solution of the sG equation i.e.,

$$\bar{u}(\bar{x}, \bar{t}) = 2\mathrm{i} \ln \frac{\bar{f}'}{\bar{f}}. \tag{4.14}$$

5. Conservation laws

First, let

$$\sigma = u - \mathrm{i} \sinh^{-1} u_y. \tag{5.1}$$

By direct substitution, we find the relation

$$\sigma_{\tau y} - \sin \sigma = \left\{(1 + u_y^2)^{\frac{1}{2}} - \mathrm{i}\frac{\partial}{\partial y}\right\}\left\{\frac{u_{\tau y}}{(1 + u_y^2)^{\frac{1}{2}}} - \sin u\right\}. \tag{5.2}$$

Thus, if u is a solution of equation (2.10), then σ given by (5.1) satisfies the sG equation (2.13a). First, note that the sG equation (2.13a) admits local conservation laws of the form

$$P_{n,\tau} = Q_{n,y}, \qquad (n = 0, 1, 2, ...), \tag{5.3}$$

where P_n and Q_n are polynomials of σ and its y-derivatives. Rewriting this relation in terms of the original variables x and t by (2.4) and using equation (2.2), we can recast (5.3) to the form

$$(rP_n)_t = (rP_n \cos u + Q_n)_x. \tag{5.4}$$

The quantities

$$I_n = \int_{-\infty}^{\infty} rP_n dx, \qquad (n = 0, 1, 2, ...), \tag{5.5}$$

then become the conservation laws of equation (1.2) upon substitution of (5.1). We present the first three of them. The corresponding P_n for the sG equation may be written as

$$P_0 = 1 - \cos\,\sigma, \qquad P_1 = \frac{1}{2}\sigma_y^2, \qquad P_2 = \frac{1}{4}\sigma_y^4 - \sigma_{yy}^2. \tag{5.6}$$

It follows from (5.5), (5.6) and the relations $r_x = -u_x u_{xx}/r, (u_x/r)_x = u_{xx}/r^3$ which stem from (2.1) that

$$I_0 = \int_{-\infty}^{\infty} (r - \cos\,u)dx, \tag{5.7a}$$

$$I_1 = \frac{1}{2}\int_{-\infty}^{\infty} \left(\frac{u_x^2}{r} - \frac{u_{xx}^2}{r^5}\right) dx, \tag{5.7b}$$

$$I_2 = \int_{-\infty}^{\infty} \left[\frac{1}{4}\frac{u_x^4}{r^3} + \frac{3}{2}\frac{u_{xx}^2}{r^5} + \frac{1}{r^7}\left(u_{xxx}^2 - \frac{5}{2}u_{xx}^2\right) + \frac{7u_{xx}^4}{r^9} - \frac{35}{4}\frac{u_{xx}^4}{r^{11}}\right] dx. \tag{5.7c}$$

The conservation laws generated by the procedure outlined above reduce to those of the short pulse and sG equations in the scaling limits described in section 4. In particular, the first three conservation laws of the short pulse equation (4.2) read

$$I_0 = \int_{-\infty}^{\infty} (r - 1)dx, \tag{5.8a}$$

$$I_1 = -\frac{1}{2}\int_{-\infty}^{\infty} \frac{u_{xx}^2}{r^5} dx, \tag{5.8b}$$

$$I_2 = \int_{-\infty}^{\infty} \left(\frac{u_{xxx}^2}{r^7} + \frac{7u_{xx}^4}{r^9} - \frac{35}{4}\frac{u_{xx}^4}{r^{11}}\right) dx. \tag{5.8c}$$

6. Conclusion

1. We have developed a systematic procedure for solving the generalized sG equation (1.2). The structure of solutions was found to differ substantially from that of the generalized sG equation (1.1) with $\nu = -1$

2. We have obtained three types of solutions, i.e., kink, breather and kink-breather solutions and investigated their properties.

3. We have shown that the generalized sG equation reduces to the short pulse and sG equations in appropriate scaling limits.

4. We have obtained an infinite number of conservation laws by using a novel Bäcklund transformation connecting solutions of the sG and generalized sG equations.

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