数理解析研究所講究録
第 1761 巻 2011 年 177-190

Integrable Structure of Nonlinear Waves Built Around the Casimir

Zensho Yoshida
Graduate School of Frontier Sciences, The University of Tokyo

Abstract

We formulate a nonlinear Beltrami wave equations that describe amplitude and pitch modulations of one-dimensional Alfvén waves propagating on a dispersive nonilnear plasma. The well-known fact that the ideal Alfvén wave can propagate on a homogeneous ambient magnetic field with conserving an arbitrary wave shape of any amplitude is explained by invoking the Casimirs stemming from a "topological defect" (or, a kernel) in the Poisson bracket operator of the ideal magnetohydrodynamic (MHD) system. Including the Hall term, however, the Alfvén waves are affected by the dispersive effect, and the aforementioned simplicity of the ideal Alfvén waves is greatly lost; an arbitrary wave can no longer propagate with a constant shape. Yet, we observe an "integrable" structure in the nonlinear modulation (induced by a compressible motion) of the Beltrami waves pertaining to the Casimirs.

1 Introduction

The word "vortex" means primarily a circulating, rotating, distorted, or, sometimes, shearing *mode* of some vector field (fluid velocity, electromagnetic field, etc.) which is measured by the "curl" derivative (or the exterior derivative of 1-form in general dimension). In some particular situation, however, we may view a vortex as a *matter* (or, a *particle*) with a certain sustaining identity; we may "quantize" a vortex (we are not speaking of quantum-mechanical effects; we consider quantization in a more general context). Because of the fundamental nonlinearity of the fluid or plasma system, it is, of course, not easy to separate a vortex from other part of the system, other coupled fields, and other scale hierarchies, thus the quantization of a vortex is not as simple as the quantization of waves in a linear system: Vortexes in a fluid or plasma may exhibit totally chaotic behavior.

There is yet a possibility to describe a vortex, in a rather simple system, as a "quantum" which carries a fixed "charge" —in an ideal fluid or plasma system, which can be formulated as a Hamiltonian system [1], the *helicity=Casimir* conserves as a constant of motion, giving an identity to the vortex. The Casimir pertains to the topological defect of the Lie-Poisson bracket

$$[F,G] := \langle \partial_u F(u), \mathscr{J} \partial_u G(u) \rangle,$$

or the kernel of the symplectic operator $\mathcal{J}$; we call a functional $C(u)$ a Casimir, if $[C,G] = 0$ for all G. If the evolution equation is written in a Hamiltonian form

$$\frac{d}{dt}u = \mathcal{J}(u)\partial_u H(u) \tag{1}$$

($H(u)$ is the Hamiltonian), the transformation $H(u) \to H_\mu(u) = H(u) - \mu C(u)$ (μ is a constant) does not change the dynamics, thus the critical points satisfying

$$\partial_u H_\mu(u) = \partial_u[H(u) - \mu C(u)] = 0 \tag{2}$$

will give fixed points. The combination of the energy $H(u)$ and the helicity $C(u)$ in (2) produces an interesting vortex structure: Since both of these functionals are quadratic (as to be shown in the following example), (2) reads as an eigenvalue problem determining the *quantized vortex*.

Let us see how the helicity can produce interesting structures and phenomena in an ideal MHD plasma. Denoting by n the number density, $\boldsymbol{V}$ the fluid velocity, $\boldsymbol{B}$ the magnetic field, m the ion mass, h the molar enthalpy (which is related to the thermal energy $\mathcal{E}$ by $h = \partial(n\mathcal{E})/\partial n$), the governing equations are

$$\left\{ \begin{array}{l} \partial_t n = -\nabla\cdot(\boldsymbol{V}n), \\ \partial_t \boldsymbol{V} = -(\nabla\times\boldsymbol{V})\times\boldsymbol{V} - \nabla(h+V^2/2) + n^{-1}(\nabla\times\boldsymbol{B})\times\boldsymbol{B}, \\ \partial_t \boldsymbol{B} = \nabla\times(\boldsymbol{V}\times\boldsymbol{B}). \end{array} \right. \tag{3}$$

The variables are normalized in the standard Alfvén units [energy densities (thermal h_j and kinetic V^2) are normalized by the magnetic energy density $B_0^2/(\mu_0 n_0)$]. The state variables are $\boldsymbol{u} = {}^t(n, m\boldsymbol{V}, \boldsymbol{B})$. We define

$$H = \int \left\{ n\left[\frac{V^2}{2} + \mathcal{E}(n)\right] + \frac{B^2}{2} \right\} dx, \tag{4}$$

$$\mathcal{J} = \begin{pmatrix} 0 & -\nabla\cdot & 0 \\ -\nabla & -n^{-1}(\nabla\times\boldsymbol{V})\times & n^{-1}(\nabla\times\circ)\times\boldsymbol{B} \\ 0 & \nabla\times[\circ\times n^{-1}\boldsymbol{B}] & 0 \end{pmatrix}. \tag{5}$$

Then, the corresponding Hamilton's equation (1) reproduces the MHD equations (3). We find three independent Casimirs:

$$C_1 = \int \boldsymbol{A}\cdot\boldsymbol{B}dx, \tag{6}$$

$$C_2 = \int \boldsymbol{V}\cdot\boldsymbol{B}dx, \tag{7}$$

$$C_3 = \int n dx. \tag{8}$$

We call C_1 the magnetic helicity and C_2 the cross helicity; C_3 is the total particle number.

The generalized fixed-point equation (2) with these three Casimirs reads as

$$\nabla\times\boldsymbol{B} - \mu_1\boldsymbol{B} - \mu_2\nabla\times\boldsymbol{V} = 0, \tag{9}$$

$$n\boldsymbol{V} - \mu_2\boldsymbol{B} = 0, \tag{10}$$

$$V^2/2 + h - \mu_3 = 0. \tag{11}$$

Notice that (11) is Bernoulli's relation. To simplify the analysis, let us consider the solutions with $n = 1$. Then, (10) becomes a linear equation. Combining (9) and (10), we obtain

$$(1-\mu_2^2)\nabla\times\boldsymbol{B}-\mu_1\boldsymbol{B}=0. \tag{12}$$

For $\mu_2 \neq \pm 1$, we obtain the *Beltrami vortex* characterized as the eigenfunctions of curl: Denoting $\lambda = \mu_1/(1-\mu_2^2)$,

$$\nabla\times\boldsymbol{B}=\lambda\boldsymbol{B}, \quad \boldsymbol{V}=\mu_2\boldsymbol{B}. \tag{13}$$

An interesting situation is created by $\mu_2 = \pm 1$: $\boldsymbol{B}$ can be *arbitrary* and $\boldsymbol{V} = \pm\boldsymbol{B}$ ($\mu_1 = 0$). This (infinite dimension) set of stationary solutions can be connected to *Alfvén waves*: Let us write this static solution as

$$\boldsymbol{B}=\boldsymbol{B}_0+\tilde{\boldsymbol{B}}=\boldsymbol{e}_z+\tilde{\boldsymbol{B}}, \tag{14}$$

where $\boldsymbol{e}_z = \nabla z$ is the unit vector parallel to the coordinate z. We interpret that $\boldsymbol{B}_0$ is the homogeneous ambient magnetic field. The coupled flow velocity is, then,

$$\boldsymbol{V}=\boldsymbol{V}_0+\tilde{\boldsymbol{V}}=\pm(\boldsymbol{e}_z+\tilde{\boldsymbol{B}}). \tag{15}$$

Galilean boost $z \to \zeta = z \mp t$ yields a "propagating wave" with wave fields $\tilde{\boldsymbol{B}}(x,y,\zeta)$ and $\tilde{\boldsymbol{V}}(x,y,\zeta) = \pm\tilde{\boldsymbol{B}}(x,y,\zeta)$ on the ambient magnetic field $\boldsymbol{B}_0 = \boldsymbol{e}_z$, which solves the fully nonlinear equations (3) on the frame (x,y,ζ): In fact, substituting (14) and (15) into (3), we obtain

$$\left\{\begin{array}{l}(\partial_t+\boldsymbol{V}_0\cdot\nabla)n=-\nabla\cdot(\tilde{\boldsymbol{V}}n),\\ (\partial_t+\boldsymbol{V}_0\cdot\nabla)\tilde{\boldsymbol{V}}=-(\nabla\times\tilde{\boldsymbol{V}})\times\tilde{\boldsymbol{V}}-\nabla(h+\tilde{V}^2/2)+n^{-1}(\nabla\times\boldsymbol{B})\times\boldsymbol{B},\\ (\partial_t+\boldsymbol{V}_0\cdot\nabla)\boldsymbol{B}=\nabla\times(\tilde{\boldsymbol{V}}\times\boldsymbol{B}).\end{array}\right. \tag{16}$$

For a boosted quantity $f(\tau,\zeta)$ (with $\tau = t$ and $\zeta = z - V_0 t = z \mp t$), we may write $(\partial_t + \boldsymbol{V}_0 \cdot \nabla) = \partial_\tau$. Therefore, the foregoing static solution appears as a propagating wave on the boosted frame, which solves (3) with transforming $t \to \tau = t$, $z \to \zeta = z \mp t$, and $\boldsymbol{V} \to \tilde{\boldsymbol{V}}$.

Since $\tilde{\boldsymbol{B}}$ is arbitrary, perturbations of any shape and any amplitude propagate, with conserving the wave form, at the constant velocity ± 1 (the Alfvén velocity) in the direction of $\boldsymbol{B}_0 = \boldsymbol{e}_z$ —this is the well-know non-dispersive property of the nonlinear Alfvén waves on a homogeneous ambient magnetic field.

Foregoing analysis elucidates the fundamental relation between the topological defect of the MHD system and the strikingly robust property of the nonlinear Alfvén waves; the Alfvén wave is the "quantized vortex" at the singularity ($\mu_2 = \pm 1$) of the criticality condition (2).

In the present paper, we will analyze the Hall-MHD equations which includes the (nonlinear) dispersive effect. Despite the dispersion, we will find that nonlinear propagating waves exist; they stem in the topological defect of the Hall-MHD system. We will study an integrable structure in the perturbation (nonlinear modulation) of the "quantized" (Beltrami) waves. A non-constant n will play an essential role in the nonlinear modulation (which we neglected in the foregoing discussion).

2 Model of Hall MHD

2.1 Hall MHD system

We consider a Hall MHD plasma governed by

$$\partial_t \boldsymbol{P} - \boldsymbol{V} \times (\nabla \times \boldsymbol{P}) = -\delta_i \nabla \left(\phi + h_i + V^2/2\right), \tag{17}$$

$$\partial_t \boldsymbol{A} - \boldsymbol{V}_e \times (\nabla \times \boldsymbol{A}) = -\delta_i \nabla (\phi - h_e), \tag{18}$$

$$\partial_t n + \nabla \cdot (\boldsymbol{V} n) = 0, \tag{19}$$

where $\boldsymbol{P} = \delta_i \boldsymbol{V} + \boldsymbol{A}$, $\boldsymbol{V}_e = \boldsymbol{V} - \delta_i n^{-1} \nabla \times (\nabla \times \boldsymbol{A})$, $h_i(n)$ and $h_e(n)$ are the ion and electron enthalpy. The variables are normalized in the standard Alfvén units. The ion skin depth $\delta_i = (c/\omega_{pi})/L$ (L is the system size) is a small scale parameter.

Remark 1. Subtracting (18) from (17) yields the

$$\partial_t \boldsymbol{V} - \boldsymbol{V} \times (\nabla \times \boldsymbol{V}) + n^{-1} (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = -\nabla (h + V^2/2), \tag{20}$$

where $h = h_i + h_e$. On the other hand, the curl of (18) yields

$$\partial_t \boldsymbol{B} - \nabla \times [(\boldsymbol{V} - \delta_i n^{-1} \nabla \times \boldsymbol{B}) \times \boldsymbol{B}] = 0. \tag{21}$$

The Hall term $\delta_i n^{-1} \nabla \times \boldsymbol{B}$ acts as a singular perturbation connecting different (smaller) scale hierarchies, and yielding dispersive effect [2].

Remark 2. For ion acoustic waves, it is often assumed that $h_j \approx 0$ (cold ions to avoid ion Landau damping) and $\nabla h \approx \nabla h_e = T_e \nabla \log n_e = \nabla \phi$, i.e., the Boltzmann distribution $n_e = e^{\phi/T_e}$ with a constant electron temperature T_e (in the normalized unit, $n_e T_e$ is the half of the electron beta ratio). Then, we replace h on the right-hand side of (20) by ϕ, and involve the Poisson equation

$$\nabla^2 \phi = e^{\phi/T_e} - n. \tag{22}$$

Let us cast the Hall MHD system (19), (17) and (21) in a Hamiltonian form. The state variables are $\boldsymbol{u} = {}^t(n, \boldsymbol{P}, \boldsymbol{B})$. We define

$$H = \int \left\{ n \left[\frac{(\boldsymbol{P} - \boldsymbol{A})^2}{2\delta_i^2} + \phi + \mathscr{E}(n) \right] + \frac{B^2}{2} \right\} dx, \tag{23}$$

$$\mathscr{J} = \delta_i \begin{pmatrix} 0 & -\nabla \cdot & 0 \\ -\nabla & -n^{-1} (\nabla \times \boldsymbol{P}) \times & 0 \\ 0 & 0 & \nabla \times [(\boldsymbol{B}/n) \times (\nabla \times \circ)] \end{pmatrix}. \tag{24}$$

Then, we have

$$\partial_{\boldsymbol{u}} H = \begin{pmatrix} \partial_n H \\ \partial_{\boldsymbol{P}} H \\ \partial_{\boldsymbol{B}} H \end{pmatrix} = \begin{pmatrix} V^2/2 + \phi + h \\ n\boldsymbol{V}/\delta_i \\ \boldsymbol{B} - \mathrm{curl}^{-1} (n\boldsymbol{V}/\delta_i) \end{pmatrix},$$

and Hamilton's equation (1) is equivalent to the system (19), (17) and (21). The symplectic operator $\mathcal{J}$ has three independent Casimirs: the magnetic helicity (6), the total particle number (8) and, in the pace of the cross helicity (7), the ion canonical helicity

$$C_2' = \int \boldsymbol{P}\cdot(\nabla\times\boldsymbol{P})dx, \tag{25}$$

The generalized fixed-point equation (2) with these three Casimirs reads as

$$\begin{aligned} \nabla\times\boldsymbol{B} - n\boldsymbol{V}/\delta_i - \mu_1\boldsymbol{B} &= 0, && (26)\\ n\boldsymbol{V}/\delta_i - \mu_2(\nabla\times\boldsymbol{V}/\delta_i + \boldsymbol{B}) &= 0, && (27)\\ V^2/2 + \phi + h - \mu_3 &= 0. && (28) \end{aligned}$$

In the next subsection, we will derive the same set of equations, the Beltrami-Bernoulli conditions, from a more succinct consideration [3, 4].

2.2 Beltrami-Bernoulli solutions

We may write the momentum equations (17) and (18) in a symmetric form

$$\partial_t\boldsymbol{P}_j - \boldsymbol{U}_j\times\boldsymbol{\Omega}_j = -\nabla\varphi_j \quad (j=i,e) \tag{29}$$

with defining the canonical momenta ($\boldsymbol{P}_j$), vortices ($\boldsymbol{\Omega}_j = \nabla\times\boldsymbol{P}_j$), flows ($\boldsymbol{U}_j$) and energy densities ($\varphi_j$) of the ion ($j=i$) and electron ($j=e$) fluids as

$$\begin{array}{ll} \boldsymbol{P}_i = \boldsymbol{P} = \delta_i\boldsymbol{V} + \boldsymbol{A} & \boldsymbol{P}_e = \boldsymbol{A} \\ \boldsymbol{\Omega}_i = \delta_i\nabla\times\boldsymbol{V} + \boldsymbol{B}, & \boldsymbol{\Omega}_e = \boldsymbol{B}, \\ \boldsymbol{U}_i = \boldsymbol{V}, & \boldsymbol{U}_e = \boldsymbol{V} - \delta_i n^{-1}\nabla\times\boldsymbol{B}, \\ \varphi_i = \delta_i\left(\phi + h_i + V^2/2\right), & \varphi_e = \delta_i\left(\phi - h_e\right). \end{array} \tag{30}$$

Taking the curl of (29), we obtain a symmetric vortex dynamic system

$$\partial_t\boldsymbol{\Omega}_j - \nabla\times(\boldsymbol{U}_j\times\boldsymbol{\Omega}_j) = 0 \quad (j=i,e). \tag{31}$$

The *Beltrami condition* demands the generators of the vortex dynamics to vanish under the relation

$$\boldsymbol{U}_j = \mu_j\boldsymbol{\Omega}_j \quad (j=i,e), \tag{32}$$

where μ_j $(j=i,e)$ are certain constants. This system of equations is nothing but the generalized fixed-point equations (26)-(27). Solving this set of equations for $\boldsymbol{V}$ and $\boldsymbol{B}$, we obtain *Beltrami fields*. To satisfy the equilibrium condition, the Beltrami condition demands the energy densities φ_j $(j=i,e)$ to satisfy the *Bernoulli conditions*

$$\nabla\varphi_j = 0 \quad (j=i,e). \tag{33}$$

Adding φ_e and φ_i, (33) yields (28).

2.3 Linear Beltrami condition

In what follows, we set $\delta_i = 1$ by normalizing the length scale by the ion skin depth. The Beltrami condition demands $\boldsymbol{V}$ to be incompressible ($\nabla\cdot\boldsymbol{V} = 0$), and hence, a constant density n satisfies the static mass conservation law (19). With a constant n ($= 1$), the Beltrami conditions reduce into a linear system of equations

$$\boldsymbol{V} = \mu_i(\nabla\times\boldsymbol{V}+\boldsymbol{B}), \tag{34}$$

$$\boldsymbol{V}-\nabla\times\boldsymbol{B} = \mu_e\boldsymbol{B}. \tag{35}$$

Combining (34) and (35), we obtain an equation governing both $\boldsymbol{u} = \boldsymbol{B}$ and $\boldsymbol{V}$:

$$\nabla\times\nabla\times\boldsymbol{u}+(\mu_e-\mu_i^{-1})\nabla\times\boldsymbol{u}+(1-\mu_e/\mu_i)\boldsymbol{u} = 0, \tag{36}$$

which may be rewritten as

$$(\mathrm{curl}-\lambda_0)(\mathrm{curl}-\lambda_1)\boldsymbol{u} = 0, \tag{37}$$

where the "eigenvalues" λ_1 and λ_2 are determined by

$$\lambda_0+\lambda_1 = \mu_i^{-1}-\mu_e, \quad \lambda_0\lambda_1 = 1-\mu_e/\mu_i. \tag{38}$$

A general solution of (37) is given by a linear combination of two *Beltrami eigenfunctions* [3, 4] (eigenfunctions of the curl operator [5]): with $\boldsymbol{G}_\ell$ such that $(\mathrm{curl}-\lambda_\ell)\boldsymbol{G}_\ell = 0$ and arbitrary constants C_ℓ ($\ell = 0, 1$),

$$\boldsymbol{B} = C_0\boldsymbol{G}_0+C_1\boldsymbol{G}_1, \tag{39}$$

$$\boldsymbol{V} = C_0(\lambda_0+\mu_e)\boldsymbol{G}_0+C_1(\lambda_1+\mu_e)\boldsymbol{G}_1. \tag{40}$$

2.4 Beltrami waves (stationary waveform)

Here, we are interested in a special class of Beltrami solutions where one of the Beltrami eigenvalues is zero ($\lambda_0 = 0$), which implies that the corresponding Beltrami eigenfunction is a *harmonic field* (see Appendix A for the reason of choosing $\lambda_0 = 0$). In the entire space, a harmonic field is just a constant vector field. Assuming that this harmonic field is an "ambient field", the other component may be viewed as a "wave field" propagating on the ambient field. From (38), we see that this occurs when

$$\mu_e = \mu_i \; (= \mu). \tag{41}$$

Then, the other eigenvalue becomes $\lambda_1 = \mu^{-1}-\mu$.

Let us see how the wave component propagates. We set $\lambda_0 = 0$, $\boldsymbol{G}_0 = \boldsymbol{e}_z$ and $C_0 = 1$ (i.e., we normalize $\boldsymbol{B}$ by the ambient magnetic field). The corresponding ambient flow is $\boldsymbol{V}_0 = \mu\boldsymbol{e}_z$. Now, we Galilean-boost the coordinates:

$$(x,y,z) \to (x,y,\zeta) := (x,y,z-\mu t). \tag{42}$$

In this frame, the flow field appears as

$$\tilde{\boldsymbol{V}} = \boldsymbol{V}-\boldsymbol{V}_0 = C_1(\lambda_1+\mu_e)\boldsymbol{G}_1,$$

which is nothing but the wave component of $\boldsymbol{V}$ (we interpret that the original frame is moving with the wave, so that the wave component is static, while the matter moves at the velocity V_0). The phase velocity is given by μ that may be written as a function of the Beltrami eigenvalue $\lambda_1 = \mu^{-1} - \mu$:

$$\mu = \frac{1}{2}\left(\lambda_1 \pm \sqrt{\lambda_1^2 + 4}\right). \tag{43}$$

When λ_1 is viewed as the *wave number*, (43) agrees with the dispersion relation of the circularly polarized Alfvén waves. Indeed, the Beltrami eigenfunction corresponding to the eigenvalue λ_1 is

$$\boldsymbol{G}_1 = \begin{pmatrix} \sin(\lambda_1 \zeta) \\ \cos(\lambda_1 \zeta) \\ 0 \end{pmatrix}.$$

Because $V^2 = V_0^2 + \tilde{V}^2$ = constant, the Bernoulli conditions (33) are satisfied (on the rest frame) by $\nabla h_i = \nabla h_e = 0$ (consistent to the homogeneous density $n \equiv 1$) and $E_z := -\partial_t A_z - \partial_z \phi = 0$.

Notice that this solution may have any amplitude —it is an exact solution of the fully nonlinear system of equations. The reader is refereed to Ref. [6] for the application of Beltrami eigenfunctions in the description of circularly polarized waves. A more general eigenfunctions are given by three-dimensional ABC map. However, the corresponding solution does not satisfy the Bernoulli conditions, if we do not invoke the incompressible model to decouple the conservation law and the pressure terms.

In what follows, we consider a one-dimensional system with inhomogeneous density n, and discuss nonlinear modulation of the Beltrami waves.

3 Nonlinear Beltrami fields and modulated waves

3.1 Beltrami-Bernoulli conditions in 1D geometry

In this section, we will generalize the Beltrami-Bernoulli conditions to introduce compressibility, inhomogeneous density and nonlinear evolution of the wave field.

We consider a one-dimensional system where all fields are functions of only z (in the (x,y,z) Cartesian coordinates) and t (time). We also assume that the magnetic field may be written as

$$\boldsymbol{B} = \begin{pmatrix} B_x(z,t) \\ B_y(z,t) \\ B_0 \end{pmatrix} = \boldsymbol{B}_\perp(z,t) + B_0 \boldsymbol{e}_z, \tag{44}$$

where B_0 represents the ambient homogeneous magnetic field (normalizing $\boldsymbol{B}$ by this ambient magnetic field, we set $B_0 = 1$).

We generalize the Beltrami conditions (34)-(35) as

$$\boldsymbol{V} = \mu_i(\nabla \times \boldsymbol{V} + \boldsymbol{B}) + u\boldsymbol{e}_z, \tag{45}$$
$$\boldsymbol{V} - n^{-1}\nabla \times \boldsymbol{B} = \mu_e \boldsymbol{B} + u\boldsymbol{e}_z, \tag{46}$$

where $n(z,t)$ is an inhomogeneous density and $u(z,t)$ is a certain scalar function (μ_i and μ_e are constant numbers as before). Immediately, we find $\nabla \cdot \boldsymbol{V} = \partial_z u$, and hence, an inhomogeneous u allows compression of the flow.

In the one-dimensional geometry, the $\nabla\times$ does not have a z component. Hence, the z components of (45) and (46), respectively, read as $V_z = \mu_i + u$ and $V_z = \mu_e + u$, implying that

$$V_z = \mu + u \quad (\mu := \mu_i = \mu_e). \tag{47}$$

Remembering the discussions in Subsec. 2.3, we see that the magnetic field (44) consists of a harmonic (ambient) component $\boldsymbol{e}_z$ and the transverse wave component $\boldsymbol{B}_\perp$, and hence, we require (41).

Combining (45) and (46) yields

$$\nabla\times\nabla\times\boldsymbol{V}_\perp + (n\mu - \mu^{-1})\nabla\times\boldsymbol{V}_\perp = 0, \tag{48}$$

which is a modification of (36) with an inhomogeneous $n(z,t)$.

The scalar functions $u(z,t)$ and $n(z,t)$ bring about nonlinear evolution of the generalized Beltrami fields –plugging (45) and (46) into the momentum equations (29), we obtain

$$\partial_t \boldsymbol{P}_j - u\boldsymbol{e}_z \times \boldsymbol{\Omega}_j = -\nabla\varphi_j \quad (j = i,e). \tag{49}$$

The x-y components of (49) are equivalent to the vortex equation; taking the curl, we obtain

$$\partial_t \boldsymbol{\Omega}_j - \nabla\times(u\boldsymbol{e}_z\times\boldsymbol{\Omega}_j) = 0 \quad (j = i,e), \tag{50}$$

which imply that the vortices $\boldsymbol{\Omega}_j$ propagate with the velocity u in the direction of $\boldsymbol{e}_z$ (on the reference frame).

The z components of (49), both for the ions and electrons, read as "generalized Bernoulli conditions" (compare with (33)):

$$\partial_t P_z = -\partial_z\left(\phi + h_i + \frac{1}{2}V^2\right), \tag{51}$$

$$\partial_t A_z = -\partial_z(\phi - h_e). \tag{52}$$

Subtracting (52) from (51) yields

$$\partial_t V_z = -\partial_z\left(h + \frac{1}{2}V^2\right), \tag{53}$$

which may be rewritten as

$$\partial_t V_z + V_z\partial_z V_z = -\partial_z\left(h + \frac{1}{2}V_\perp^2\right), \tag{54}$$

where $\boldsymbol{V}_\perp$ must be determined by the Beltrami condition (48) that includes the unknown variable $n(z,t)$ that is governed by the mass conservation law

$$\partial_t n + \partial_z(V_z n) = 0. \tag{55}$$

In summary, our nonlinear system consists of the z and perpendicular components of the generalized Beltrami conditions (47) and (48), the generalized Bernoulli condition (54) and the mass conservation law (55).

As mentioned in *Remark 2*, one may replace h in (54) by ϕ and invoke the Poisson equation (22), instead of assuming a conventional barotropic relation $h = h(n)$.

3.2 Reductive perturbation

To simplify the system of equations, we invoke the reductive perturbation method, and reduce the number of dependent variables (they have a common wave form). Introducing a small parameter ε, We write the dependent variables as

$$\begin{aligned} n &= 1+\varepsilon n^{(1)}+\varepsilon^2 n^{(2)}+\cdots, &(56)\\ u &= 0+\varepsilon u^{(1)}+\varepsilon^2 u^{(2)}+\cdots, &(57)\\ V_z &= V_0+\varepsilon V_z^{(1)}+\varepsilon^2 V_z^{(2)}+\cdots, &(58)\\ \boldsymbol{V}_\perp &= 0+\varepsilon \boldsymbol{V}_\perp^{(1)}+\varepsilon^2 \boldsymbol{V}_\perp^{(2)}+\cdots, &(59) \end{aligned}$$

where V_0 is assumed to be a constant number. We assume $h=\phi=0+\varepsilon\phi^{(1)}+\varepsilon^2\phi^{(2)}+\cdots$. We also expand the independent variables as

$$\begin{aligned} \tilde{z} &= \varepsilon\zeta=\varepsilon(z-ct), &(60)\\ \tilde{t} &= \varepsilon^2 t, &(61) \end{aligned}$$

where c is a constant to be determined later. We note that our scaling is different from the one that derives the ion-acoustic KdV equation.

Using these variables in (47), we obtain

$$V_0=\mu, \quad V_z^{(1)}=u^{(1)}, \quad V_z^{(2)}=u^{(2)}. \tag{62}$$

The Beltrami equation (48) starts from the terms of the order of ε^2, which summarize as

$$(\mu^{-1}-\mu)\tilde{\nabla}\times\boldsymbol{V}_\perp^{(1)}=0. \tag{63}$$

To proceed with nontrivial $\boldsymbol{V}_\perp^{(1)}$, we satisfy (63) by choosing

$$\mu^{-1}-\mu=0 \quad \leftrightarrow \quad \mu=1. \tag{64}$$

By (62), $V_0=\mu=1$. From the order of ε^3, we obtain

$$\tilde{\nabla}\times\tilde{\nabla}\times\boldsymbol{V}_\perp^{(1)}+n^{(1)}\tilde{\nabla}\times\boldsymbol{V}_\perp^{(1)}=0. \tag{65}$$

Next, we examine the conservation law (55). From the oder of ε^2, we find

$$c'n^{(1)}=V_z^{(1)} \quad (c':=c-V_0=c-1), \tag{66}$$

and, from the oder of ε^3,

$$\partial_{\tilde{t}}n^{(1)}+\partial_{\tilde{z}}\left(n^{(1)}V_z^{(1)}+V_z^{(2)}-c'n^{(2)}\right)=0. \tag{67}$$

The Bernoulli condition (54) yields, from the oder of ε^2,

$$c'V_z^{(1)}=\phi^{(1)}, \tag{68}$$

and, from the oder of ε^3,

$$\partial_{\tilde{t}} V_z^{(1)} + V_z^{(1)} \partial_{\tilde{z}} V_z^{(1)} + \partial_{\tilde{z}} \left(-c' V_z^{(2)} + \phi^{(2)} + \frac{1}{2} |V_\perp^{(1)}|^2 \right) = 0. \tag{69}$$

Finally, the one-dimensional Poisson equation (indeed, it is just the charge-neutrality condition in this scaling) yields, from the oder of ε^2,

$$\frac{\phi^{(1)}}{T_e} = n^{(1)}, \tag{70}$$

and from the oder of ε^3,

$$\frac{\phi^{(2)}}{T_e} + \frac{1}{2} \left(\frac{\phi^{(1)}}{T_e} \right)^2 - n^{(2)} = 0. \tag{71}$$

To satisfy both (66), (68) and (70), we have to set

$$c' = \pm c_s := \sqrt{T_e} \quad \leftrightarrow \quad c = V_0 \pm c_s.$$

Now, (66), (68) and (62) deduce

$$V_z^{(1)} = u^{(1)} = \pm c_s n^{(1)} = \pm c_s^{-1} \phi^{(1)}. \tag{72}$$

Summing up the $\pm c_s$ multiple of (67), (69) and $-\partial_{\tilde{z}}$ of (71), and using (72), we obtain

$$\partial_{\tilde{t}} u^{(1)} + \partial_{\tilde{z}} \left[\frac{1}{2} (u^{(1)})^2 + \frac{1}{4} |V_\perp^{(1)}|^2 \right] = 0. \tag{73}$$

This evolution equation must be solved simultaneously with the Beltrami equation (65) that now reads as

$$\tilde{\nabla} \times \tilde{\nabla} \times V_\perp^{(1)} \pm c_s^{-1} u^{(1)} \tilde{\nabla} \times V_\perp^{(1)} = 0. \tag{74}$$

Remark 3. If we assume a simple barotropic relation $h = h(n)$ and write $dh = c_s^2(\varepsilon dn^{(1)} + \varepsilon^2 dn^{(2)} + \cdots)$ (physical meaning of c_s is different from that of the ion acoustic mode), the term $(u^{(1)})^2/2$ on the left-hand side of (73) is replaced by $(u^{(1)})^2$. All other relations are unchanged excepting that ϕ is no longer involved.

Remark 4. The present model of nonlinear dispersive Alfvén waves may be compared with other previously formulated models in the literature: At the same ordering of independent variables (60)-(61), but with a smaller parallel perturbations ($n^{(1)} = u^{(1)} = V_z^{(1)} = 0$), we may consider an envelope wave $\psi(\tilde{z}, \tilde{t})$ multiplying to the carrier wave of the form of $\exp i(kz - \omega t)$. Then, $\psi(\tilde{z}, \tilde{t})$ obeys a nonlinear Schrödinger equation [7, 8]. At larger amplitude modulations ($V_\perp = \varepsilon^{1/2} V_\perp^{(1)} + \varepsilon^{3/2} V_\perp^{(2)} + \cdots$) we obtain a differential nonlinear Schrödinger equation [9]. In comparison with these models, the present formulation assumes a longer wavelengths and lower frequency of the wave (we do not assume a carrier wave of a short-wavelength $\sim$ ion skin depth). In a different (larger) scale hierarchy, we obtain the conventional ion acoustic soliton that is produced by the dispersive effect due to a small charge non-neutrality: Instead of (60) and (61), we set

$$\begin{aligned} \tilde{z} &= \varepsilon^{1/2}(z-ct), && (75)\\ \tilde{t} &= \varepsilon^{3/2}t. && (76) \end{aligned}$$

Then, the dispersive term $\partial_{\tilde{z}}\phi^{(1)}$ and the nonlinear term $(\phi^{(2)})^2$ make a balance in the Poisson equation, to yield an additional term $-\partial_{\tilde{z}}^2\phi^{(1)}$ on the left-had side of (71). Other relations (66)-(70) are unchanged. For a totally electrostatic mode ($V_\perp = 0$), we obtain the well-known KdV equation by adding $\partial_{\tilde{z}}^3\phi^{(1)}$ on the left-hand side of (73). To couple a transverse (electromagnetic) component $V_\perp = 0$ to this KdV equation, we need to assume a smaller $n^{(1)}$ in the Beltrami equation (48): To match the scaling (75), we assume

$$n\mu - \mu^{-1} = \varepsilon^{1/2}\lambda_0 + \varepsilon^{3/2}\mu n^{(1)} + \cdots,$$

inlying that $\mu - \mu^{-1}$ and $n^{(j)}$ ($j = 1,2,\cdots$) are restricted to be of the order of $\varepsilon^{1/2}$. Then, we obtain, from the order of ε^2, $\tilde{\nabla}\times\tilde{\nabla}\times V_\perp^{(1)} \pm c_s^{-1}\lambda_0\tilde{\nabla}\times V_\perp^{(1)} = 0$, which yields a homogeneous $|V_\perp^{(1)}|^2$ (modulation of the transverse component is separated to the smaller scale hierarchy).

3.3 Hamilton-Jacobi equation

The model (73)-(74) is a new type of nonlinear evolution equation that has an interesting Hamiltonian structure.

In what follows, we will simplify the notation with omitting $^{(1)}$ on the dependent variables and $\tilde{}$ on the independent variables.

Let us define an *action* $S(z,t)$ and *Hamiltonian* $H(u,z,t)$ by [1]

$$\begin{aligned} u(z,t) &= \partial_z S \quad \text{(momentum)}, && (77)\\ H(u,z,t) &= \frac{1}{2}u^2 + \frac{1}{4}|V_\perp|^2(z,t). && (78) \end{aligned}$$

Integrating (73) with respect to z, we obtain a Hamilton-Jacobi equation

$$\partial_t S + H(\partial_z S, z, t) = 0. \tag{79}$$

The potential energy $|V_\perp|^2(z,t)/4$ included in the Hamiltonian (78) must be determined by solving the Bernoulli condition (74) as a "potential equation", and there, the $S(z,t)$ appears as the *eikonal* of the vorticity field. Denoting $\Omega = \nabla\times V_\perp$, the Beltrami equation (74) is written as $\nabla\times\Omega + c_s^{-1}u\Omega = 0$ (in what follows, c_s absorbs the $\pm$ sign), which is solved by

$$\Omega = \Re\, We^{iS/c_s}\begin{pmatrix} 1 \\ -i \end{pmatrix}, \tag{80}$$

---

[1] In view the Bernoulli condition (53), we find that this Hamiltonian is the perturbation part of the total energy density.

where W is a constant. Obviously, we have the *enstrophy conservation*:

$$|\Omega|^2 = |\nabla \times V_\perp|^2 = |W|^2. \tag{81}$$

Using (80), we may formally write the potential energy as

$$\frac{1}{4}|V_\perp|^2 = \frac{1}{4}|\mathrm{curl}^{-1}\Omega|^2 = \frac{1}{4}|W \int e^{iS/c_s} dz|^2.$$

4 Conclusion

As reviewed in Introduction, the ideal Alfvén wave can have an arbitrary waveform — undetermined solutions occur at the singularity (the point where the determining differential equation degenerates) of the Beltrami equation. The Hall MHD system includes a singular perturbation [2], which removes the singularity, and thus, the Alfvén waves no longer have an arbitrary waveform.

We have derived a system of equations which describes the nonlinear modulation of one-dimensional Alfén waves propagating on a Hall MHD plasma. The trivial solution (i.e., non-modulated, homogeneous-velocity propagation) is the Galilean-boosted Beltrami vortex that is the kernel of the generator of the system. The Casimirs quantize the vortex structure; μ_1, μ_2 (scaling the helicities) and μ_3 (scaling energy) are the quantum numbers. A compressional motion and the corresponding density perturbation cause the nonlinear modulation of the wave; an integrable system of equations governs a small but finite amplitude wave stemming in the vicinity of the kernel of the generator.

Appendix A: Beltrami fields and Alfvén waves

Taking the curl of (17) and (18), we obtain a set of canonical vortex equations: denoting $\boldsymbol{\Omega} = \nabla \times \boldsymbol{P}$ and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$,

$$\partial_t \boldsymbol{\Omega} - \nabla \times (\boldsymbol{V} \times \boldsymbol{\Omega}) = 0, \tag{82}$$

$$\partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{V}_e \times \boldsymbol{B}) = 0. \tag{83}$$

We add a homogeneous ambient magnetic field $\boldsymbol{B}_0 = B_0 \boldsymbol{e}_z$, which does not change the flows $\boldsymbol{V}$ and $\boldsymbol{V}_e$. Writing $\boldsymbol{\Omega}' = \boldsymbol{\Omega} - \boldsymbol{B}_0$ and $\boldsymbol{B}' = \boldsymbol{B} - \boldsymbol{B}_0$, (82) and (83) translate as

$$\partial_t \boldsymbol{\Omega}' - \partial_z (B_0 \boldsymbol{V}) - \nabla \times (\boldsymbol{V} \times \boldsymbol{\Omega}') = 0, \tag{84}$$

$$\partial_t \boldsymbol{B}' - \partial_z (B_0 \boldsymbol{V}_e) - \nabla \times (\boldsymbol{V}_e \times \boldsymbol{B}') = 0, \tag{85}$$

where we have assumed $\nabla \cdot \boldsymbol{V} = \nabla \cdot \boldsymbol{V}_e = 0$. Otherwise, we have to add $\boldsymbol{B}_0(\nabla \cdot \boldsymbol{V})$ and $\boldsymbol{B}_0(\nabla \cdot \boldsymbol{V}_e)$ on the left-hand sides of (84) and (85), respectively.

Now we seek a propagating wave solution that may be written as $f(x,y,z,t) = \tilde{f}(x,y,\tilde{z},t)$ with $\tilde{z} = z - ct$. Then, (84) and (85) transform into

$$\partial_t \tilde{\boldsymbol{\Omega}}' - \partial_{\tilde{z}} (B_0 \tilde{\boldsymbol{V}} + c\tilde{\boldsymbol{\Omega}}') - \tilde{\nabla} \times (\tilde{\boldsymbol{V}} \times \tilde{\boldsymbol{\Omega}}') = 0, \tag{86}$$

$$\partial_t \tilde{\boldsymbol{B}}' - \partial_{\tilde{z}} (B_0 \tilde{\boldsymbol{V}}_e + c\tilde{\boldsymbol{B}}') - \tilde{\nabla} \times (\tilde{\boldsymbol{V}}_e \times \tilde{\boldsymbol{B}}') = 0. \tag{87}$$

Here after, we omit ˜ to simplify the notation.

The Beltrami wave solutions (stationary solutions in the moving frame) are given by

$$V = \mu \Omega' \qquad (\mu^{-1}V = \delta_i \nabla \times V + B'), \tag{88}$$

$$V_e = \mu B' \qquad (V - \delta_i n^{-1} \nabla \times B' = \mu B'), \tag{89}$$

where $\mu = -c/B_0$.

From (88), the Beltrami wave must be incompressible ($\nabla \cdot V = 0$). A constant n is, then, consistent to the mass conservation law (19), and it also simplifies (89). Let us first calculate the Beltrami equations. Combining (88) and (89), we obtain (denoting $\delta_i \nabla = \mathrm{curl}$)

$$\mathrm{curl}(n^{-1}\mathrm{curl}B) + (\mu - \mu^{-1}n^{-1})\mathrm{curl}B = 0. \tag{90}$$

Since n is assumed to be constant, (90) simplifies

$$\mathrm{curl}\left[\mathrm{curl} + (n\mu - \mu^{-1})\right] B = 0,$$

which has general solutions of the form of

$$B = C_0 G_0 + C_\lambda G_\lambda, \quad V = \mu C_0 G_0 + n^{-1}\mu^{-1} C_\lambda G_\lambda.$$

with $\mathrm{curl}G_0 = 0$ and $\mathrm{curl}G_\lambda = \lambda G_\lambda$ ($\lambda = \mu^{-1} - n\mu$) and arbitrary constants C_0 and C_λ. The first component (harmonic field) yields a "Doppler shift" of the Alfvén wave: Adding $B = C_0 e_z$, for instance, yields a change of the ambient field $B_0 = B_0 e_0 \to (B_0 - C_0)e_z$, which results in the change of the propagation velocity by $-cC_0/B_0 = C_0\mu$.

Let us examine the Bernoulli condition in this constant-n situation. De-curling (86) and (87), we obtain (omitting ˜)

$$\partial_t P' + (B_0 e_z \times V - c\partial_z P') - V \times \Omega' = -\nabla\phi - \nabla h_i - \frac{1}{2}\nabla V^2, \tag{91}$$

$$\partial_t A' + (B_0 e_z \times V_e - c\partial_z A') - V_e \times B' = -\nabla\phi + \nabla h_e. \tag{92}$$

For the above-mentioned Beltrami waves with constant n, we may set $\partial_t = 0$, $V \times \Omega' = 0$, $V_e \times B' = 0$, $\nabla h_i = \nabla h_e = 0$, and $(B_0 e_z \times V - c\partial_z P') = \nabla \psi_i$, $(B_0 e_z \times V_e - c\partial_z A') = \nabla \psi_e$ with some scalar ψ_i and ψ_e. Hence, the Bernoulli condition reads as

$$\nabla \psi_i = -\nabla\phi - \frac{1}{2}\nabla V^2, \tag{93}$$

$$\nabla \psi_e = -\nabla\phi. \tag{94}$$

Subtracting (93) from (94), and remembering the definition of ψ_i and ψ_e, as well as using the Beltrami conditions (88) and (89), we obtain

$$\begin{aligned} \nabla\psi_e - \nabla\psi_i &= \frac{1}{2}\nabla V^2 \\ &= B_0 e_z \times (-\delta_i n^{-1} \nabla \times B') + c\partial_z \delta_i V \\ &= -\delta_i \mu B_0 \left[e_e \times (\nabla \times V) + \partial_z V\right] \\ &= -\delta_i \mu B_0 \nabla V_z. \end{aligned} \tag{95}$$

If the fields are one dimensional (functions of only z), the right-hand side becomes $\partial_z V_z \equiv 0$. Hence, the Beltrami wave must have a homogeneous energy density V^2 = constant, which implies that only a single Beltrami wave may propagate.

References

[1] P. J. Morrison, Rev. Mod. Phys. **70** (1998), 467.

[2] Z. Yoshida, S. M. Mahajan, and S. Ohsaki, Phys. Plasmas **11** (2004), 3660.

[3] S. M. Mahajan and Z. Yoshida, Phys. Rev. Lett. **81** (1998), 4863.

[4] Z. Yoshida and S. M. Mahajan, J. Math. Phys. **40** (1999), 5080.

[5] Z, Yoshida and Y. Giga, Math Z. **204** (1990), 235.

[6] Z. Yoshida, J. Plasma Phys. **45** (1991), 481.

[7] T. Taniuti and H. Washimi, Phys. Rev. Lett. **21** (1968), 209.

[8] A. Hasegawa, Phys. Fluids **15** (1972), 870.

[9] K. Mio, T. Ogino, K. Minami, and S. Takeda, J. Phys. Soc. Jpn. **41** (1976), 256.