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HOLOMORPHIC INDEX AND DOMAINS AND RANGES OF PARABOLIC RENORMALIZATIONS

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Dedicated to Ushiki Shigehiro's 60th birthday

1. Introduction

In the study of dynamical systems, it is often important to study the behavior of orbits of "most" points. For complex rational maps in one variable, it is well-known that the orbit of almost every point in the Julia set accumulates to the post-critical set.

Shishikura and the author [IS] introduced a class of maps $\mathcal{F}_1$ with a parabolic fixed point, which is invariant under parabolic and near-parabolic renormalization operators. This allows us to construct a decreasing sequence of neighborhoods of the fixed point containing the post-critical set of a given infinitely renormalizable quadratic polynomials in the sense of near-parabolic renormalization [BC]. Therefore, almost all points in the Julia set eventually enters those neighborhoods and never escapes.

This type of argument is often used in proving no invariant line field for infinitely renormalizable maps. To do this, we need to pullback (parts of) those neighborhoods univalently near a given point with some properties holding for almost every point in the Julia set. To get such injective pullbacks, it is important to know how an orbit enters to those neighborhoods.

Our result, combined with Shishikura's construction of such neighborhoods for perturbations of maps in $\mathcal{F}_1$, gives a control of such entrance to neighborhoods:

Theorem 1.1. *If the parameter η in the definition of $\mathcal{F}_1$ satisfies $6 < \eta < 13$, then the parabolic renormalization operator $\mathcal{R}_0 : \mathcal{F}_1 \to \mathcal{F}_1$ defined in* [IS] *satisfies that for any $f \in \mathcal{F}_1$, the domain* $\operatorname{Dom}\mathcal{R}(f)$ *and the range* $\operatorname{Range}\mathcal{R}(f)$ *satisfies*

$$\operatorname{Dom}\mathcal{R}_0(f) \Subset \operatorname{Range}\mathcal{R}_0(f).$$

Since the near-parabolic renormalization operator $\mathcal{R}_\alpha$ for small α with $|\arg\alpha| < \frac{\pi}{4}$ or $|\arg\alpha| < -\frac{\pi}{4}$, depends continuously on α and $\mathcal{R}_\alpha \to \mathcal{R}_0$ as $\alpha \to 0$, we immediately have the following:

Corollary 1.2. *Under the same assumption, the same conclusion holds for near-parabolic renormalization operator $\mathcal{R}_\alpha : \mathcal{F}_1 \to \mathcal{F}_1$ when α is sufficiently small.*

For $f \in \mathcal{F}_1$ and an appropriate $\alpha \in \mathbb{R}/\mathbb{Q}$ (precisely, the coefficients of the continued fraction of α are uniformly bounded from below by some universal constant), $f_\alpha = e^{2\pi i\alpha} f$ for $f \in \mathcal{F}_1$ has an irrationally indifferent fixed point at 0. Shishikura constructed an abstract Riemann surface $\mathcal{S}_n$ and a univalent map $\iota_n : \mathcal{S}_n \to \mathbb{C}$ such that the image $\Omega_n^* = \iota_n(\mathcal{S}_n)$ is a punctured neighborhood of 0 containing the post-critical set. Furthermore, $\mathcal{S}_n$ is constructed by cutting and gluing a lot of copies of the domain and the range of the n-th near parabolic renormalization of f_α in an appropriate way. Therefore by the above corollary, we have $f_\alpha(\Omega_n) \supset \Omega_n$ and the "entrance"

$$f_\alpha^{-1}(f_\alpha(\Omega_n)) \cap \Omega_{n-1} \setminus \Omega_n$$

is contained in $N_n \setminus N_n'$, where $N_n \supset N_n'$ are small neighborhoods of the critical point, defined in terms of the abstract Riemann surfaces $\mathcal{S}_{n-1}$ and $\mathcal{S}_n$.

2. Dynamics near a parabolic fixed point

2.1. **Fatou coordinates.** Let $f(z) = z + a_2z^2 + O(z^3)$ be a germ of holomorphic maps at the origin. We only consider the case $a_2 \neq 0$. By changing the coordinate by $z = -\frac{1}{a_2 w}$, it is conjugate to

$$F(w) = w + 1 + \frac{b_1}{z} + O\left(\frac{1}{z^2}\right). \tag{2.1}$$

The value $1 - b_1$ is called the *holomorphic index* of f at 0. For a general definition, see e.g., [M].

Theorem 2.1. *For a holomorphic map of the form* (2.1) *and for any* $0 < k < 1$, *There exist some* $L > 0$ *and conformal maps*

$$\Phi_{attr} : \{|\operatorname{Im} w| > L - k \operatorname{Re} w\} \to \mathbb{C}$$
$$\Phi_{rep} : \{|\operatorname{Im} w| > L + k \operatorname{Re} w\} \to \mathbb{C}$$

such that $\Phi_*(F(w)) = \Phi_*(w) + 1$ $(* = attr,\ rep)$ *where both sides are defined. They are unique up to post-composition by translation.*

Furthermore, they have expansions

$$\Phi_{attr}(w) = w - b_1 \log w + c_{attr} + O(1),$$
$$\Phi_{rep}(w) = w - b_1 \log w + c_{rep} + O(1).$$

Here we let the branches of the logarithm coincide in $\{\operatorname{Im} w > |\operatorname{Re} w| + L\}$. Observe that they differ by $2\pi i$ in $\{-\operatorname{Im} w > |\operatorname{Re} w| + L\}$.

2.2. **Horn maps.** By Theorem 2.1, there exists $L > 0$ such that both Φ_{attr} and Φ_{rep} are defined on $\{w \in \mathbb{C};\ |\operatorname{Im} w| > L,\ 0 < \operatorname{Re} w < 2\}$.

Definition. Define maps $E_f : \Phi_{\mathrm{rep}}(\{|\operatorname{Im} w| > L,\ 0 < \operatorname{Re} w < 2\}) \to \mathbb{C}$ by

$$E_f = \Phi_{\mathrm{attr}} \circ \Phi_{\mathrm{rep}}^{-1}.$$

Lemma 2.2. *The E_f defined above satisfies the following:*

(1) $E_f(w+1) = E_f(w) + 1$.
(2) $c_\pm = \lim_{\operatorname{Im} w \to \pm\infty} E_f(w) - w$ *exists.*
(3) *When F has the form* (2.1), *then* $c_+ - c_- = 2\pi i b_1$.

Observe that the condition (3) comes from the difference of the branches of the logarithm.

Since Φ_{attr} and Φ_{rep} are unique only up to addition by constants, we can normalize them so that $E_f(w) - w \to 0$ as $\operatorname{Im} w \to \infty$; in other words, we may assume $c_+ = 0$. Therefore we have $E_f(w) - w \to -2\pi i b_1$ as $\operatorname{Im} w \to -\infty$.

Let $\operatorname{Exp}^\sharp(w) = \exp(2\pi i w)$ and $\operatorname{Exp}^\flat(w) = \exp(-2\pi i w)$. For a map F of the form (2.1), let

$$\begin{aligned} \mathcal{R}^\sharp(F) &= \operatorname{Exp}^\sharp \circ E_f \circ (\operatorname{Exp}^\sharp)^{-1}, \\ \mathcal{R}^\flat(F) &= \exp(-4\pi^2 b_1) \operatorname{Exp}^\flat \circ E_f \circ (\operatorname{Exp}^\flat)^{-1}. \end{aligned} \tag{2.2}$$

By Lemma 2.2 (1), $\mathcal{R}^*(F)$ are holomorphic maps defined on $\{0 < |z| < e^{-2\pi L}\}$ for $* = \sharp, \flat$. Furthermore, by Lemma 2.2 (2), 0 is a removable singularity and $\mathcal{R}^*(F)$ can be extended holomorphically so that 0 is a fixed point of multiplier 1 for both $\mathcal{R}^\sharp(F)$ and $\mathcal{R}^\flat(F)$ because of the normalization.

Although $\mathcal{R}^\sharp(F)$ and $\mathcal{R}^\flat(F)$ are not dynamical systems, we consider those as ("geometric limits" of) dynamical systems. So as a germ at the origin, $\mathcal{R}^\sharp(F)$ and $\mathcal{R}^\flat(F)$ are determined uniquely up to linear conjugacy, by the uniqueness of the Fatou coordinates.

3. Parabolic renormalization

Let $P(z) = z(1+z)^2$ and let U be a domain containing 0. Consider a family

$$\mathcal{F}_1(U) = \left\{ f = P \circ \varphi^{-1} : \varphi(U) \to \mathbb{C} \;\middle|\; \begin{array}{c} \varphi : U \to \mathbb{C} : \text{univalent}, \\ \varphi(0) = 0,\ \varphi'(0) = 1 \end{array} \right\}.$$

Then Shishikura and the author [IS] proved the following:

Theorem 3.1 (Inou-Shishikura). *There exist domains $V' \Supset V \ni 0$ such that for any $f \in \mathcal{F}_1(V)$, its parabolic renormalizations $\mathcal{R}^\sharp(f)$ and $\mathcal{R}^\flat(f)$ can be normalized and extended to holomorphic maps in $\mathcal{F}_1(V')$.*

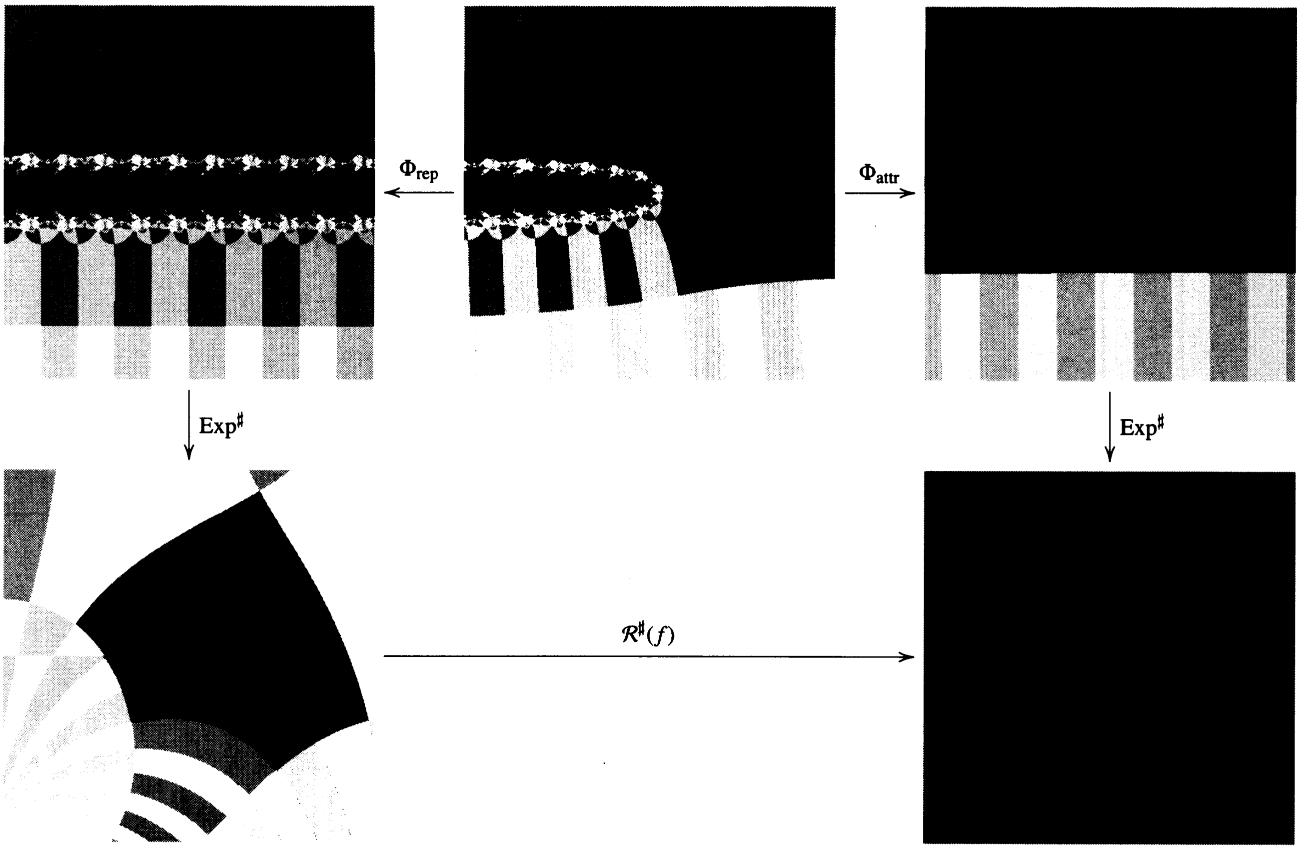

FIGURE 1. Parabolic renormalization $\mathcal{R}^\sharp(f)$ for $\eta = 2$. Since $e^{4\pi} \approx 286751$ is too large, some parts in the renormalized pictures are too large or too small to see.

By abuse of notation, we also denote those extensions by $\mathcal{R}^*(f)$. The figure 1 shows how to define the parabolic renormalization operator $\mathcal{R}^\sharp$: $\mathcal{F}_1(V) \to \mathcal{F}_1(V')$.

The domain V' in the theorem is defined as follows: Fix $\eta > 0$. Let K be the component of $P^{-1}\left(\overline{\mathbb{D}(0, \frac{4}{27}e^{-2\pi\eta})}\right)$ containing -1, where $\mathbb{D}(c, r)$ be the open disk of radius r centered at c. Let

$$V' = P^{-1}\left(\mathbb{D}\left(0, \frac{4}{27}e^{2\pi\eta}\right)\right) \setminus ((-\infty), -1] \cup K).$$

In [IS], we gave a proof of the theorem when $\eta = 2$, but the proof works for any $2 \leq \eta \leq 13$ (see [IS, § 5.N (a)]. Note that V' becomes bigger and $\mathcal{F}_1(V')$ becomes smaller as η increases).

Let

$$Q(z) = z\frac{\left(1+\frac{1}{z}\right)^6}{\left(1-\frac{1}{z}\right)^4}, \quad \psi_1(z) = -\frac{4z}{(1+z)^2}, \quad \psi_0(z) = -\frac{4}{z}.$$

Then we have $Q = \psi_0^{-1} \circ P \circ \psi_1$.

By this coordinate change, we can identify $\mathcal{F}_1(V)$ (precisely speaking for $V \subset \mathbb{C} \setminus (-\infty, -1])$ with

$$\mathcal{F}_1^Q(V_Q) = \left\{ F = Q \circ \varphi^{-1} : \varphi(V_Q) \to \hat{\mathbb{C}} \,\middle|\, \begin{array}{l} \varphi : V_Q \to \mathbb{C} \setminus 0 : \text{ univalent,} \\ \frac{\varphi(z)}{z} \to 1 \ (z \to \infty) \end{array} \right\}.$$

where $V_Q = \psi_1^{-1}(V) \cap \mathbb{C} \setminus \overline{\mathbb{D}}$. Namely, the map

$$\mathcal{F}_1(V) \ni f \mapsto F = \psi_0^{-1} \circ F \circ \psi_0 \in \mathcal{F}_1^Q(V^Q)$$

is a bijection.

The domain V in Theorem 3.1 is defined as follows: Let

$$V_Q = \left\{ x + iy \in \mathbb{C};\ \left(\frac{x - x_E}{a_E}\right)^2 + \left(\frac{y}{b_E}\right)^2 > 1 \right\},$$

where $x_E = -0.18$, $a_E = 1.24$, $b_E = 1.04$ and let $V = \psi_1(V_Q) \cup \{0\}$. Then we have $V \Subset V' \Subset \mathbb{C} \setminus (-\infty, -1]$ (see [IS, Proposition 5.2]).

By Koebe 1/4 theorem, we can give a lower estimate of the size of the domain of definition of $f \in \mathcal{F}_1(V)$.

Lemma 3.2. *Let $f \in \mathcal{F}_1(V)$. Then the domain $\varphi(V)$ of definition of f contains the disk $\mathbb{D}(0, r_0)$ where $r_0 = \frac{1}{1.14}$.*

The following theorem follows from the construction of $\mathcal{R}^*(f)$ for $f \in \mathcal{F}_1(V)$:

Theorem 3.3. *Let V and V' be as in Theorem 3.1. For $f \in \mathcal{F}_1(V)$, let $F = \psi_0^{-1} \circ f \circ \psi_0 \in \mathcal{F}_1^Q(V_Q)$ has the expansion*

$$F(z) = z + 10 - c_0 + \frac{49 - c_1}{z} + O(z^{-2}).$$

near $z = \infty$. Then we have the following:

(1) *$\mathcal{R}^*(f) \in \mathcal{F}_1(V')$. In particular, $\mathcal{R}^*(f) = \varphi_* \circ P$ for some univalent maps $\varphi_* : V' \to \mathbb{C}$.*

(2) *Let $D_* = \varphi_*(V')$ and let*

$$\tilde{D}_\flat = \left\{ z \in \hat{\mathbb{C}};\ \left(\frac{4}{27}\right)^2 \frac{\exp(4\pi^2 b_1)}{z} \in D_\flat \right\},$$

where $b_1 = \frac{49-c_1}{(10-c_0)^2}$. Then $D_\sharp$ and $\tilde{D}_\flat$ are disjoint.

In fact, the first part is the main result of [IS]. Moreover, as in Figure 1, $\mathcal{R}^\sharp(f)$ and $\mathcal{R}^\flat(f)$ are constructed by taking appropriate backward images of (subsets of) a fundamental domain of the attracting Fatou coordinate of f. Since we take different backward branches of the same fundamental domain to construct $\mathcal{R}^\sharp(f)$ and $\mathcal{R}^\flat(f)$, their domains are necessarily disjoint. Therefore, the second statement just follows by checking the difference between $\mathrm{Exp}^\sharp \circ \Phi_{\mathrm{rep}}$ and $\mathrm{Exp}^\flat \circ \Phi_{\mathrm{rep}}$ and see how the domain for $c\mathcal{R}^\flat(f)$ is mapped by $\mathrm{Exp}^\sharp \circ \Phi_{\mathrm{rep}}$. By the equations (2.2), the difference can be written in terms of the holomorphic index.

4. Proof of the theorem

First recall that $V \Supset V'$. Hence for $f = P \circ \varphi^{-1} \in \mathcal{F}_1(V)$, its lower parabolic renormalization $\mathcal{R}^\flat(f) \in \mathcal{F}_1(V')$ has the form $Q \circ (\varphi_\flat)^{-1}$ where $\varphi_\flat : V' \to \mathbb{C}$ is an univalent map. Therefore, $\mathcal{R}^\flat(f)|_{\varphi_\flat(V)} : \varphi_\flat(V) \to \mathbb{C} \in \mathcal{F}_1(V)$ is an element of $\mathcal{F}_1(V)$. In particular, the domain $D_\flat$ of $\mathcal{R}^\flat(f)$ contains $\overline{\mathbb{D}}(0, r_0)$ by Lemma 3.2. Hence it follows that $\tilde{D}_\flat$ contains $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}(0, r_1)$ where

$$r_1 = \left(\frac{4}{27}\right)^2 \frac{\exp(4\pi^2 \operatorname{Re} b_1)}{r_0}.$$

Therefore, by Theorem 3.3, we have the following:

Lemma 4.1. *If $f \in \mathcal{F}_1(V)$ satisfies the inequality*

$$r_1 < \frac{4}{27} e^{2\pi\eta}, \tag{4.1}$$

then the domain $D_\sharp$ = Dom($\mathcal{R}^*(f)$) *of $\mathcal{R}(f)$ is contained in the range* Range($\mathcal{R}^*(f)$).

We need to check that the inequality (4.1) holds. To do this end, we first give an estimate for b_1.

Lemma 4.2. *Let $\varphi : V^Q \to \mathbb{C} \setminus 0$ be a univalent map of the form*

$$\varphi(\zeta) = \zeta + c_0 + \sum_{n=1}^{\infty} \frac{c_n}{\zeta^n}. \tag{4.2}$$

Then the following hold:

(1) $|c_0 - c_{00}| < c_{01,max}$ *where* $c_{00} = 0.18 (= -x_E)$ *and* $c_{01,max} = 2.28 (= 2e_1)$.

(2) $|c_1| \le (1.42)^2$.

Proof. (1) is already proved in [IS, Lemma 5.22].

Since $V^Q \subset \mathbb{C} \setminus \overline{\mathbb{D}}(0, 1.42)$, it follows that a map $\hat{\varphi}(z) = \frac{1}{1.42}\varphi(1.42z)$ is a univalent map defined on $\mathbb{C} \setminus \overline{\mathbb{D}}$ having the form

$$\hat{\varphi}(z) = z + \frac{c_0}{1.42} + \frac{c_1}{(1.42)^2 z} + O(z^{-2}).$$

Therefore, (2) follows from the area theorem. □

Corollary 4.3. *Let* $F = Q \circ \phi^{-1} \in \mathcal{F}_1^Q(V^Q)$. *If* φ *is of the form* (4.2), *then*

$$\operatorname{Re} b_1 \le \frac{49 + (1.42)^2}{(10 - 2.46)^2}.$$

In particular,

$$r_1 \le 1.14\left(\frac{4}{27}\right)^2 \exp\left(4\pi^2 \frac{49 + (1.42)^2}{(10 - 2.46)^2}\right)$$

for any $f \in \mathcal{F}_1(V)$.

Proof. It directly follows from Lemma 4.2 and $b_0 = \frac{49 - c_1}{(10 - c_0)^2}$. □

Therefore, we have proved the following:

Theorem 4.4. *If* η $(\eta \le 13)$ *satisfies*

$$\eta > \frac{1}{2\pi}\log\left(\frac{4 \times 1.14}{27}\right) + 2\pi \frac{49 + (1.42)^2}{(10 - 2.46)^2},$$

then the domain of $\mathcal{R}^*(f) \in \mathcal{F}_1(V')$ *is contained in the range for any* $f \in \mathcal{F}_1(V)$.

Rigorous numerical computation shows that the right hand side lies in the interval

$$[5.355225163872033, 5.355225163872037].$$

In particular, the theorem follows when $\eta \ge 6$.

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