数理解析研究所講究録
第 1762 巻 2011 年 114-119

Postcritical sets and saddle basic sets for Axiom A polynomial skew products on $\mathbb{C}^2$

東京工芸大学 中根静男 (Shizuo Nakane)

Tokyo Polytechnic University

1 Introduction

We consider Axiom A regular polynomial skew products on $\mathbb{C}^2$. It is of the form : $f(z,w)=(p(z),q(z,w))$, where $p(z)=z^d+\cdots$ and $q_z(w)=q(z,w)=w^d+\cdots$ are polynomials of degree $d\geq 2$. Then its k-th iterate is expressed by :

$$f^{\circ k}(z,w)=(p^{\circ k}(z),q_{p^{k-1}(z)}\circ\cdots\circ q_z(w))=:(p^{\circ k}(z),Q_z^{\circ k}(w)).$$

Hence it preserves the family of *fibers* $\{z\}\times\mathbb{C}$ and this makes it possible to study its dynamics more precisely. Let K be the set of points with bounded orbits and set $K_z:=\{w\in\mathbb{C};(z,w)\in K\}$ and $K_{J_p}:=K\cap(J_p\times\mathbb{C})$. The *fiber Julia set* J_z is the boundary of K_z.

Let Ω be the set of *non-wandering points* for f. Then f is said to be *Axiom A* if Ω is compact, hyperbolic and periodic points are dense in Ω. For polynomial skew products, Jonsson [J2] has shown that f is Axiom A if and only if the following three conditions are satisfied :

(a) p is hyperbolic,
(b) f is vertically expanding over J_p,
(c) f is vertically expanding over $A_p:=\{$attracting periodic points of $p\}$.

Here f is *vertically expanding over* $Z\subset\mathbb{C}$ with $p(Z)\subset Z$ if there exist $\lambda>1$ and $C>0$ such that $|(Q_z^{\circ k})'(w)|\geq C\lambda^k$ holds for any $z\in Z,w\in J_z$ and $k\geq 0$.

We are interested in the dynamics of f on $J_p\times\mathbb{C}$ because the dynamics outside $J_p\times\mathbb{C}$ is fairly simple. Consider the critical set

$$C_{J_p}=\{(z,w)\in J_p\times\mathbb{C};q_z'(w)=0\}$$

over the *base Julia set* J_p. Let μ be the ergodic measure of maximal entropy for f (see Fornaess and Sibony [FS1]). Its support J_2 is called the *second Julia set* of f. Let $D_{J_p}:=\overline{\cup_{n\geq 1}f^{\circ n}(C_{J_p})}$ be the *postcritical set* of C_{J_p}. Jonsson [J2] has shown that

(d) $J_2=\overline{\cup_{z\in J_p}\{z\}\times J_z}$,
(e) the condition (b) $\Longleftrightarrow D_{J_p}\cap J_2=\emptyset$,
(f) J_2 is the closure of the set of repelling periodic points of f.

By the Birkhoff ergodic theorem, μ-a.e. x has a dense orbit in J_2. Especially, $J_2 = supp\,\mu$ is transitive. Hence J_2 coincides with the *basic set* of unstable dimension two. See also [FS2].

For any subset X in $\mathbb{C}^2$, its accumulation set is defined by

$$A(X) = \cap_{N \geq 0} \overline{\cup_{n \geq N} f^{\circ n}(X)}.$$

DeMarco & Hruska [DH1] defined the *pointwise* and *component-wise* accumulation sets of C_{J_p} respectively by

$$A_{pt}(C_{J_p}) = \overline{\cup_{x \in C_{J_p}} A(x)} \quad \text{and} \quad A_{cc}(C_{J_p}) = \overline{\cup_{C \in \mathcal{C}(C_{J_p})} A(C)},$$

where $\mathcal{C}(C_{J_p})$ denotes the collection of connected components of C_{J_p}. It follows from the definition that

$$A_{pt}(C_{J_p}) \subset A_{cc}(C_{J_p}) \subset A(C_{J_p}).$$

It also follows that $A_{pt}(C_{J_p}) = A_{cc}(C_{J_p})$ if J_p is a Cantor set and $A_{cc}(C_{J_p}) = A(C_{J_p})$ if J_p is connected.

Let Λ be the closure of the set of saddle periodic points in $J_p \times \mathbb{C}$. It decomposes into a disjoint union of *saddle basic sets* : $\Lambda = \sqcup_{i=1}^m \Lambda_i$. Put $\Lambda_z = \{w \in \mathbb{C}; (z,w) \in \Lambda\}$. The *stable* and *unstable sets* of Λ, the *local stabe* and *local unstable manifolds* of $x \in \Lambda$ and $\hat{x} \in \hat{\Lambda}$ are respectively defined by

$$\begin{aligned}
W^s(\Lambda) &= \{y \in \mathbb{C}^2; f^{\circ k}(y) \to \Lambda\}, \\
W^u(\Lambda) &= \{y \in \mathbb{C}^2; \exists \text{ backward orbit } \hat{y} = (y_{-k}) \text{ tending to } \Lambda\}, \\
W^s_\delta(x) &= \{y \in \mathbb{C}^2; ||f^{\circ k}(y) - f^{\circ k}(x)|| < \delta, \forall k \geq 0\}, \\
W^s_\delta(\hat{x}) &= \{y \in \mathbb{C}^2; \exists \hat{y} \text{ s.t. } ||y_{-k} - x_{-k}|| < \delta, \forall k \geq 0\}.
\end{aligned}$$

On Λ, f is contracting in the fiber direction and

$$W^s_\delta(x) \subset \{z\} \times \mathbb{C}, \;\; x = (z,w) \in \Lambda.$$

Theorem A. ([DH1])

$$A_{pt}(C_{J_p}) = \Lambda, \quad A(C_{J_p}) = W^u(\Lambda) \cap (J_p \times \mathbb{C}).$$

Theorem B. ([DH1, DH2])

$$A_{cc}(C_{J_p}) = A_{pt}(C_{J_p}) \Longrightarrow \forall C \in \mathcal{C}(C_{J_p}),\; C \cap K = \emptyset \text{ or } C \subset K. \tag{1}$$

Theorem C. ([DH1, DH2])

$$A(C_{J_p}) = A_{pt}(C_{J_p}) \iff \textit{the map } z \mapsto \Lambda_z \textit{ is continuous in } J_p. \tag{2}$$

Under the assumption $W^u(\Lambda) \cap W^s(\Lambda) = \Lambda$,

$$A(C_{J_p}) = A_{pt}(C_{J_p}) \iff \textit{the map } z \mapsto K_z \textit{ is continuous in } J_p. \tag{3}$$

As for the assumption in the above theorem, we have

Lemma 1. $W^u(\Lambda) \cap W^s(\Lambda) = \Lambda \iff W^u(\Lambda_i) \cap W^s(\Lambda_j) = \emptyset$ *if* $i \neq j$.

Sumi [S] gives an example of Axiom A polynomial skew product which does not satisfy the condition in the above lemma. See the last section. It is also (incorrectly) described as Example 5.10 in [DH1]. See also [DH2].

We define a relation $\succ$ among saddle basic sets by

$$\Lambda_i \succ \Lambda_j \quad \text{if} \quad (W^u(\Lambda_i) \setminus \Lambda_i) \cap (W^s(\Lambda_j) \setminus \Lambda_j) \neq \emptyset.$$

A *cycle* is a chain of basic sets :

$$\Lambda_{i_1} \succ \Lambda_{i_2} \succ \cdots \succ \Lambda_{i_n} = \Lambda_{i_1}.$$

For Axiom A open endomorphisms, there is no trivial cycle because $W^u(\Lambda_i) \cap W^s(\Lambda_i) = \Lambda_i$ holds for any i. See [J2], Proposition A.4. Jonsson has also shown that, for Axiom A polynomial skew products on $\mathbb{C}^2$, the non-wandering set Ω coincides with the *chain recurrent set* $\mathcal{R}$. This leads to the following lemma.

Lemma 2. ([J2], Corollary 8.14) *Axiom A polynomial skew products on* $\mathbb{C}^2$ *have no cycles.*

Set $\Lambda_0 := \emptyset$, $W^s(\Lambda_0) := (J_p \times \mathbb{C}) \setminus K$ and $C_i := C_{J_p} \cap W^s(\Lambda_i)$ $(0 \leq i \leq m)$. If we consider in $\mathbb{P}^2$, Λ_0 should be the superattracting fixed point $\{[0:1:0]\}$.

We will give characterizations of the equalities $A_{cc}(C_{J_p}) = A_{pt}(C_{J_p})$ and $A_{pt}(C_{J_p}) = A(C_{J_p})$ in terms of C_i.

Lemma 3. $C_{J_p} = \sqcup_{i=0}^m C_i$.

Note that $A(C_i) \supset A_{pt}(C_i) = \Lambda_i$ for any $i \geq 0$.

The author would like to thank Hiroki Sumi for helpful discussion on his example.

2 Results

Theorem 1.

$$A_{cc}(C_{J_p}) = A_{pt}(C_{J_p}) \iff \forall C \in \mathcal{C}(C_{J_p}),\ 0 \leq \exists i \leq m \ \textit{ such that } C \subset C_i. \quad (4)$$

In terms of C_i, the condition in (1) is expressed by

$$\forall C \in \mathcal{C}(C_{J_p}), \quad C \subset C_0 \text{ or } C \subset \cup_{i=1}^{m} C_i.$$

Hence, if $m = 1$, that is, Λ itself is a basic set, then the condition in (4) coincides with that in (1). In general, the condition in (4) is stronger than that in (1).

We have another characterization of $A_{pt}(C_{J_p}) = A(C_{J_p})$ in terms of C_i.

Theorem 2. *For any $i \geq 0$, we have*

$$A(C_i) = \Lambda_i \iff C_i \textit{ is closed }. \tag{5}$$

Consequently we have

$$A_{pt}(C_{J_p}) = A(C_{J_p}) \iff C_i \textit{ is closed for any } i \geq 0.$$

As for the condition in (3), we have

Theorem 3. *The following three conditions are equivalent to each other.*
(a) C_0 is closed,
(b) $A(C_{J_p}) = W^u(\Lambda) \cap W^s(\Lambda)$,
(c) the map $z \mapsto K_z$ is continuous in J_p.

As a corollary, we get the following.

Corollary 1. $W^u(\Lambda) \cap W^s(\Lambda) = \Lambda \iff C_i$ *is closed for any* $i \geq 1$.

As for the condition in (2), we have the following.

Theorem 4. *For each $j \geq 1$,*

$$\begin{aligned} C_j \textit{ is open in } C_{J_p} &\iff W^u(\Lambda_j) \cap (J_p \times \mathbb{C}) = \Lambda_j \\ &\iff z \mapsto \Lambda_{j,z} \textit{ is continuous in } J_p. \end{aligned}$$

Consequently,

$$\begin{aligned} \forall j \geq 1, C_j \textit{ is open in } C_{J_p} &\iff W^u(\Lambda) \cap (J_p \times \mathbb{C}) = \Lambda \\ &\iff z \mapsto \Lambda_z \textit{ is continuous in } J_p. \end{aligned}$$

Recall that $C_0 = C_{J_p} \setminus K$ is always open in C_{J_p}.

3 Sumi's example

Sumi [S] considers the following example.

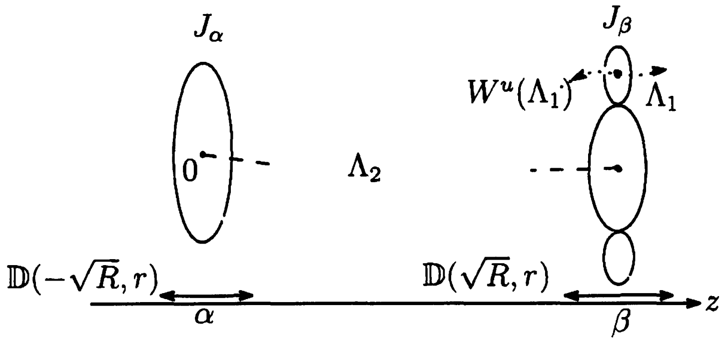
$$f(z,w) = \left((z^2 - R)^{\circ n}, w^{2^n} + \frac{z+\sqrt{R}}{2\sqrt{R}} t_{n,\epsilon}(w)\right).$$

Here $R \gg 1, 0 < \epsilon \ll 1$, n is even and large, and

$$t_{n,\epsilon}(w) = ((w-\epsilon)^2 - 1 + \epsilon)^{\circ n} - w^{2^n}.$$

Let $\alpha < 0$ and $\beta > 0$ be the fixed points of $z^2 - R$. It satisfies the following.

- J_p is a Cantor set in $\mathbb{D}(-\sqrt{R}, r) \cup \mathbb{D}(\sqrt{R}, r)$ for some r.
- J_α is a quasicircle, while J_β is a basilica.
- $\Lambda = \Lambda_1 \sqcup \Lambda_2$, where $\Lambda_1 \subset \{\beta\} \times \mathbb{C}$ is a single point.
- $C_{J_p} \subset K$, i.e. $C_0 = \emptyset$, hence $z \mapsto K_z$ is continuous in J_p.
- $C_1 \subset \{\beta\} \times \mathbb{C}$ is a finite set.
- $C_2 = C_{J_p} \setminus C_1$ is open in C_{J_p} and $\overline{C_2} \supset C_1$.
- $W^u(\Lambda_1) \cap W^s(\Lambda_2) \setminus \Lambda \neq \emptyset$, i.e. $\Lambda_1 \succ \Lambda_2$.
- The map $z \mapsto \Lambda_{2,z}$ is continuous in J_p.
- $A_{pt}(C_{J_p}) = A_{cc}(C_{J_p}) \neq A(C_{J_p})$.



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