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# AN INTRODUCTION TO SEMI-ALGEBRAIC SETS

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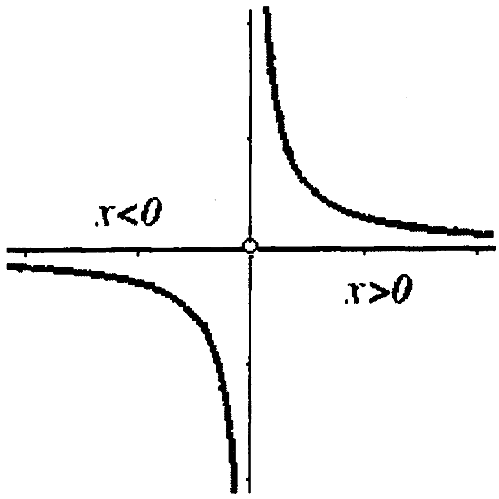
Roughly speaking, Algebraic Geometry on a field $\mathbb{K}$ studies **algebraic sets** in $\mathbb{K}^n$ i.e. the sets of the form $\{x \in \mathbb{K}^n : P_1(x) = \cdots = P_k(x) = 0\}$, where $P_i$ are polynomials with coefficients in $\mathbb{K}$. One of the difficulties when studying real algebraic sets is that the field of real numbers $\mathbb{R}$ is not algebraically closed, e.g. the number of zeros (counted with multiplicity) of a real polynomial can be not equal to its degree. Besides, though the class of real algebraic sets is closed under taking finite unions and intersections, it is not closed under taking complement. Moreover, in general, images of algebraic sets by polynomial functions and their connected components are not algebraic sets. For example, the equation $xy - 1 = 0$ defines a hyperbola in $\mathbb{R}^2$ consisting of the connected components: $\{(x,y) \in \mathbb{R}^2 : xy - 1 = 0, x > 0\}$ and $\{(x,y) \in \mathbb{R}^2 : xy - 1 = 0, x < 0\}$ , and its image under the projection on $Ox$ coordinate is two intervals. These sets are given by equations and inequalities, but they can not be given by equations only.




This lecture deals with the class of **semi-algebraic sets** which are those defined by Boolean combination of equalities and inequalities of real polynomials. This class has a very interesting property: it is stable under projection (Tarski-Seidenberg's Theorem). Moreover, a semi-algebraic set has only finitely many connected components, and each of the components is also semi-algebraic (Łojasiewicz's Theorem). These fundamental properties create great conveniences in studying semi-algebraic sets. Note that $\mathbb{R}$ is an ordered field. One can construct semi-algebraic sets in a general real closed field (see the excellent book by Bochnak-Coste-Roy cited in the references).

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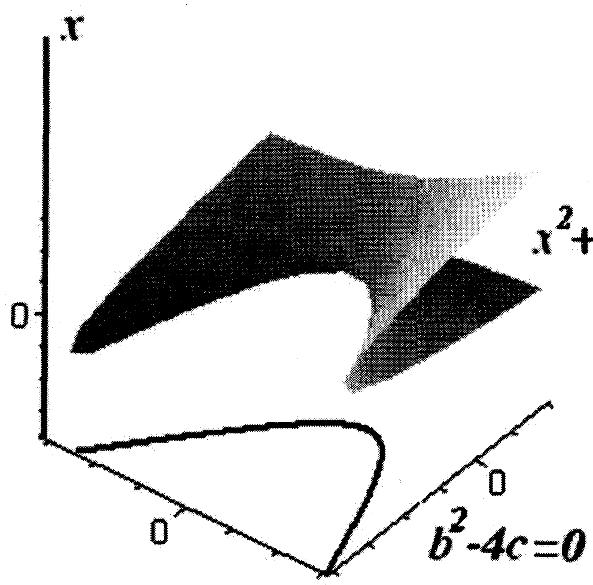
## 1. Lecture 1

In this lecture we will investigate some of the most basic properties of semi-algebraic sets. Deeper properties (e.g. stratification, the curve selection, the Łojasiewicz inequalities, triangulation, ... ) will be studied in the next lectures.

**1.1. Definition** The class of **semi-algebraic sets** in $\mathbb{R}^n$ is the smallest class of subsets of $\mathbb{R}^n$ satisfying the following properties:
(SA1) It contains all sets of the form $\{x \in \mathbb{R}^n : P(x) > 0\}$ , $P \in \mathbb{R}[X_1, \cdots, X_n]$.
(SA2) It is stable under taking finite unions, finite intersections and complements.
A mapping $f : X \to \mathbb{R}^m$ is called semi-algebraic if its graph is a semi-algebraic set.

**1.2. Example.**
1.2.1. Every real algebraic set is semi-algebraic. Moreover, in the real field, $P_1 = \cdots = P_k = 0 \Leftrightarrow P_1^2 + \cdots + P_k^2 = 0$, and hence every algebraic subset in $\mathbb{R}^n$ is of the form $\{x \in \mathbb{R}^n : P(x) = 0\}$, where $P$ is a polynomial.
1.2.2. A semi-algebraic set in $\mathbb{R}$ is a finite union of points and open intervals.
1.2.3. Let $f(b,c,x) = x^2 + bx + c$. The set of the values of $(b,c)$ in $\mathbb{R}^2$ such that $f$ has a real solution is the projection of the set $\{(x,b,c) : f(b,c,x) = 0\}$ onto the plane $(b,c)$. It is the semi-algebraic set $\{(b,c) : b^2 - 4c \geq 0\}$.




1.2.4. Polynomial functions are semi-algebraic.
1.2.5. The function $\xi : \{(b,c) : b^2 - 4c > 0\} \to \mathbb{R}$, $\xi(b,c) = \frac{1}{2}(b + \sqrt{b^2 - 4c})$ is semi-algebraic because its graph is given by: $\{(b,c,x) : x^2 + bx + c = 0, b^2 - 4c > 0, x > \frac{b}{2}\}$.
1.2.6. The following sets are not semi-algebraic:

$$\{(x,y) \in \mathbb{R}^2 : y = \sin x \},\ \{(x,y) \in \mathbb{R}^2 : y = nx,\ n \in \mathbb{N} \},\ \{(x,y) \in \mathbb{R}^2 : y = [x] \}.$$

**Exercise:** Let $f : X \to \mathbb{R}$ be a semi-algebraic function.
1.2.7. Prove that if $f(x) \neq 0$, for all $x \in X$, then $1/f$ is semi-algebraic.
1.2.8. Prove that if $f \geq 0$, then $\sqrt{f}$ is semi-algebraic.

**1.3. Proposition.** *A subset of $\mathbb{R}^n$ is semi-algebraic if and only if it can be represented as a finite union of sets of the form:*

$$\{x \in \mathbb{R}^n : f(x) = 0,\ g_1(x) > 0, \cdots,\ g_m(x) > 0\ \},\quad f, g_i \in \mathbb{R}[X_1, \cdots, X_n].$$

*Proof:* The class of sets of the above form satisfies (SA1) and (SA2), and it is contained in the class of semi-algebraic sets. □

**Exercise:**
1.3.1. The class of **constructible sets in** $\mathbb{C}^n$, by definition, is the smallest Boolean algebra of subsets of $\mathbb{C}^n$ which contains all complex algebraic sets.
Prove that $X \subset \mathbb{C}^n$ is constructible if and only if $X = \bigcup_{i=1}^{p} V_i \setminus W_i$, where $V_i, W_i$ are algebraic sets.
1.3.2. Prove that if we identify $\mathbb{C} \equiv \mathbb{R}^2$, then every constructible subset of $\mathbb{C}^n$ is semi-algebraic in $\mathbb{R}^{2n}$.
1.3.3. Prove that $f = (f_1, \cdots, f_m) : X \to \mathbb{R}^m$ is semi-algebraic if and only if $f_i$ is semi-algebraic for all $i \in \{1, \cdots, m\}$.
1.3.4. Let $f, g : X \to \mathbb{R}$ be semi-algebraic functions. Prove that the functions $|f|$, $\max(f, g)$, $\min(f, g)$ are semi-algebraic.
1.3.5. Prove that every semi-algebraic set $X$ in $R^n$ can be represented as the image $p(A)$ of an algebraic set $A \subset \mathbb{R}^n \times \mathbb{R}^p$ under projection $\pi : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$.
1.3.6. Let $f : [0, r) \to \mathbb{R}$ be a semi-algebraic function. Prove that there is a polynomial $P(X, Y) \neq 0$, such that $P(x, f(x)) = 0$, for all $x \in [0, r)$.

Most of the basic properties of semi-algebraic sets are implied from the following two theorems:

**1.4. Theorem** (Tarski-Seidenberg). *The image of a semi-algebraic subset of $\mathbb{R}^n \times \mathbb{R}^k$ under the natural projection $\mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ is a semi-algebraic set.*

**1.5. Theorem** (Łojasiewicz). *The number of connected components of a semi-algebraic set is finite, and each of the components is also semi-algebraic.*

First, we consider the relationship between semi-algebraic-sets and the formulas.

**1.6. Definition.** A **first-order formula** (of the language of ordered fields with parameters in $\mathbb{R}$) is constructed according to the following rules:

- If $P \in \mathbb{R}[X_1, \cdots, X_n]$, then $P \star 0$, where $\star \in \{=, >, <\}$, is a formula.
- If $\phi$ and $\psi$ are formulas, then their conjunction $\phi \wedge \psi$ , their disjunction $\phi \vee \psi$, and the negation $\neg\phi$ are formulas.
- If $\phi$ is a formula and $x$ is a variable ranging over $\mathbb{R}$, then $\exists x, \phi$ and $\forall x, \phi$ are formulas.

The formulas obtained by using only the first and the second rules are called **quantifier-free formulas**.

We use the relations between logical notations and boolean algebras: Let $x, y$ be variables ranging over nonempty sets $X, Y$, and let $\phi(x,y)$ and $\psi(x,y)$ be first-order formulas on $(x,y) \in X \times Y$ defining sets

$\Phi = \{(x,y) \in X \times Y : \phi(x,y)\}$, and $\Psi = \{(x,y) \in X \times Y : \psi(x,y)\}$.

Then

$\phi(x,y) \vee \psi(x,y)$ defines $\Phi \cup \Psi$,
$\phi(x,y) \wedge \psi(x,y)$ defines $\Phi \cap \Psi$,
$\neg\phi(x,y)$ defines $X \times Y \setminus \Phi$,
$\exists x \phi(x,y)$ defines $\pi_Y(\Phi)$, where $\pi_Y : X \times Y \to Y$ is the natural projection,
$\forall x \phi(x,y)$ defines $Y \setminus \pi_Y(X \times Y \setminus \Phi)$.

From these relations, we have:
$X \subset \mathbb{R}^n$ is semi-algebraic if and only if there is a quantifier-free formula $\Phi(x_1, \cdots, x_n)$ such that

$$X = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : \Phi(x)\ \}$$

The Tarski-Seidenberg theorem has the following logical formulation:

**1.4'. Theorem** (Tarski-Seidenberg). *For every first-order formula $\Phi(x_1, \cdots, x_n)$, there exists a quantifier-free formula $\Psi(x_1, \cdots, x_n)$, such that the following formula is always true in $\mathbb{R}$:*

$$\forall x_1, \cdots, x_n (\Phi(x_1, \cdots, x_n) \ \Leftrightarrow \ \Psi(x_1, \cdots, x_n)).$$

*In particular, the set $\{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : \Phi(x)\ \}$ is semi-algebraic.*

For example, the formula $\Phi = (\exists x, x^2 + bx + c = 0) \wedge (\exists y, y^2 + by + c = 0) \wedge \neg(x = y)$ is equivalent to the qualifier-free formula $\Psi = (b^2 - 4c > 0)$.

Before proving the theorems, we give some applications of the Tarski-Seidenberg theorem.

**1.7. Proposition** (Elementary properties).
(i) *The closure, the interior, and the boundary of a semi-algebraic set are semi-algebraic.*
(ii) *Images and inverse images of semi-algebraic sets under semi-algebraic maps are semi-algebraic.*
(ii) *Compositions of semi-algebraic maps are semi-algebraic.*

*Proof:* If $A$ is a semi-algebraic subset of $\mathbb{R}^n$, then its closure is

$$\overline{A} = \{x \in \mathbb{R}^n : \forall \epsilon, \epsilon > 0, \exists y (y \in A) \wedge (\sum_{i=1}^{n} (x_i - y_i)^2 < \epsilon^2)\},$$

where $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots y_n)$. By the Tarski-Seidenberg theorem, $\overline{A}$ is semi-algebraic. The interior and the boundary of $A$ can be expressed by $\mathrm{int}(A) = \mathbb{R}^n \setminus \overline{\mathbb{R}^n \setminus A}$ and $\mathrm{bd}(A) = \overline{A} \cap \overline{\mathbb{R}^n \setminus A}$, so they are semi-algebraic.

Let $f : X \to Y$ be a semi-algebraic function and $A \subset X, B \subset Y$ be semi-algebraic subsets. Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the natural projections. Then $f(A) = \pi_Y(f \cap A \times Y)$ and $f^{-1}(B) = \pi_X(f \cap X \times B)$. So they are semi-algebraic.
Let $f : X \to Y, g : Y \to Z$ be semi-algebraic maps. Then $g \circ f = \pi(f \times Z \cap X \times g)$, where $\pi : X \times Y \times Z \to X \times Z$ defined by $\pi(x, y, z) = (x, z)$. So $g \circ f$ is semi-algebraic. □

**Exercise:** Use Tarski-Seidenberg's Theorem 1.4' to do the following:
1.7.1. Let $n \in \mathbb{N}, k \leq n$, and $i_1, \cdots, i_k \in \{1, \cdots, n\}$. Denote $\Gamma_{i_1 \cdots i_k} =$

$$\{(a_0, \cdots, a_n) \in \mathbb{R}^n : a_0 + \cdots + a_n T^n \text{ has } k \text{ zeros with mutiplicities } i_1, \cdots, i_k\}.$$

Prove that $\Gamma_{i_1 \cdots i_k}$ is a semi-algebraic set..
1.7.2. Let $f : A \to \mathbb{R}$ be a definable function and $p \in \mathbb{N}$. Prove that the set $C^p(f) = \{x \in A : f \text{ is of class } C^p \text{ at } x\}$ is definable, and the partial derivatives $\partial f / \partial x_i$ are definable functions on $C^p(f)$.
1.7.3. Let $f : \mathbb{R} \to \mathbb{R}$ be a semi-algebraic function. Prove that there exists a partition $-\infty = a_0 < a_1 < \cdots < a_n = +\infty$ such that on each interval $(a_i, a_{i+1})$ the function is either constant, or strictly monotone and continuous. As a consequence, the limits $\lim_{x \to a^+} f(x), \lim_{x \to a^-} f(x)$ exist in $\mathbb{R} \cup \{\pm\infty\}$, for all $a \in \mathbb{R} \cup \{\pm\infty\}$.
1.7.4. Let $f : A \to \mathbb{R}$ be a definable function. Suppose that $f$ is bounded from below. Let $g : A \to \mathbb{R}^m$ be a definable mapping. Prove that the function $\varphi : g(A) \to \mathbb{R}$, defined by $\varphi(y) = \inf\limits_{x \in g^{-1}(y)} f(x)$, is a definable function.

Tarski (1931, see [T]) stated and proved Theorem 1.4 in logic form (the real closed field $\mathbb{R}$ admits quantifier elimination). Later, Seidenberg (1954, see [S]) proved the theorem by using Sturm's sequences, which proved to be of great interest to other mathematicians. Here we give Łojasiewicz's proof (1964, see [Ł]), which is based on the cylindrical decomposition theorem and hence gives rather precise information on semi-algebraic sets.

**1.8. Thom's Lemma.** *Let $f_1, \cdots, f_k \in \mathbb{R}[T]$ be a finite family of polynomials which is stable under differentiation, i.e. if $f_i' \neq 0$ then $f_i' \in \{f_1, \cdots, f_k\}$.*
*For $s : \{1, \cdots, k\} \to \{<, =, >\}$, put $A_s = \{t \in \mathbb{R} : f_i(t)\ s(i)\ 0,\ i = 1, \cdots, k\}$. Then $A_s$ is connected, i.e. empty, a point, or an interval.*

*Proof:* By induction on $k$. It is trivial for $k = 0$. Suppose the lemma is true for $k - 1$ $(k > 0)$. Order $f_1, \cdots, f_k$ such that $\deg(f_k) = \max\{\deg(f_i) : i = 1, \cdots, k\}$. Let $A' = \{t : f_i(t)\ s(i)\ 0, i = 1, \cdots, k-1\}$. By the inductive hypothesis $A'$ is empty, a point, or an interval. If $A'$ is empty or a point, so is $A_s = A' \cap \{t : f_k(t)\ s(k)\ 0\}$. If $A'$ is an interval, then $f_k'$ has a constant sign on $A'$ and hence $f_k$ is either strictly monotone or constant on $A'$. In each case one can easily check that $A_s$ is connected. □

**Exercise:** Find $f \in \mathbb{R}[T]$, such that $\{t \in \mathbb{R} : f(t) > 0\}$ is not connected.

**1.9. Lemma.** *Let $G(A, T) = A_0 + A_1 T + \cdots + A_d T^d \in \mathbb{Z}[A, T], A = (A_0, \cdots, A_d)$, be a general polynomial of degree $d$, and $e \in \{0, \cdots, d, \infty\}$. Then the set*

$$\{a = (a_0, \cdots, a_d) \in \mathbb{R}^{d+1} : G(a, T) \text{ has exactly } e \text{ distinct complex zeros }\}$$

*is a semi-algebraic set.*
*As a consequence, for every* $f \in \mathbb{R}[X_1, \cdots, X_n, T] = \mathbb{R}[X_1, \cdots, X_n][T]$,

$$f(X_1, \cdots, X_n, T) = a_0(X_1, \cdots, X_n) + a_1(X_1, \cdots, X_n)T + \cdots + a_d(X_1, \cdots, X_n)T^d,$$

*the set*

$$\{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : f(x, T) \text{ has exactly } e \text{ distinct complex zeros}\}$$

*is a semi-algebraic subset of* $\mathbb{R}^n$.

*Proof:* [1] The cases $d = 0$ or $e \in \{0, \infty\}$ are trivial.
Let $d > 0, e \in \{1, \cdots, d\}$, and $a = (a_0, \cdots, a_d) \in \mathbb{C}^{d+1}, a_d \neq 0$.
Let $g =$ degree of $\mathrm{GCD}(G(a,T), \frac{\partial G}{\partial T}(a,T))$ in $\mathbb{C}[T]$.
Then the number of distinct complex zeros of $G(a,T)$ is $d - g$, and the degree of $\mathrm{LCM}(G(a,T), \frac{\partial G}{\partial T}(a,T))$ is $2d - g - 1$.
Hence the condition is that $G(a,T)$ has at most $e$ distinct zeros, which is equivalent to $d - g \leq e$, that is, to $2d - g - 1 \leq d + e - 1$. The last condition is equivalent to the condition:
(*) There exist $q(x,T) = x_0 + x_1 T + \cdots + x_{e-1}T^{e-1}$ and $r(x,T) = x_e + x_{e+1}T + \cdots + x_{2e}T^e$, with $x = (x_0, \cdots, x_{2e}) \in \mathbb{C}^{2e+1} \setminus 0$, such that

$$G(a,T)q(x,T) = \frac{\partial G}{\partial T}(a,T)r(x,T)$$

This equality can be rewritten as

$$G(a,T)q(x,T) - \frac{\partial f}{\partial T}(a,T)r(x,T) = \beta_0(a,x) + \beta_1(a,x)T + \cdots + \beta_{d+e-1}(a,x)T^{d+e-1},$$

where $\beta = (\beta_0, \cdots, \beta_{d+e-1}) : \mathbb{C}^{d+1} \times \mathbb{C}^{2e+1} \to \mathbb{C}^{d+e}$ is a bilinear function.
So (*) is equivalent to the condition $\beta_0(a,x) = \cdots = \beta_{d+e-1}(a,x) = 0$ has nonzero solution $x \in \mathbb{C}^{2e+1}$. The last condition is equivalent to the vanishing of all $(2e+1)$-minor of the matrix of the linear map $\beta(a, \cdot)$. Note that each of the minors is a polynomial in $a_0, \cdots, a_d$ with coefficients in $\mathbb{Z}$. Therefore, for each $d' \leq d$, the set

$$M_e^{d'} = \{a \in \mathbb{R}^{d+1} : \deg G(a,T) = d', G(a,T) \text{ has at most } e \text{ distinct complex zeros}\}$$

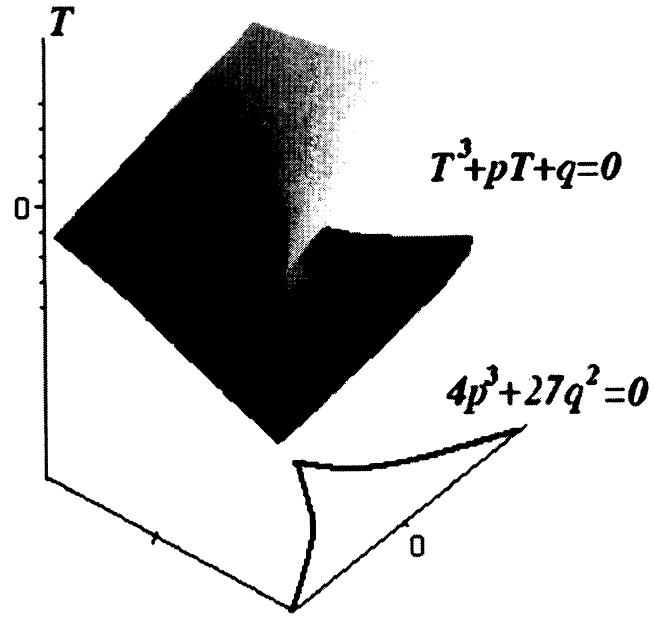
is the intersection of the set $\{a \in \mathbb{R}^{d+1} : a_d = \cdots = a_{d'+1} = 0, a_{d'} \neq 0\}$ with the zero set of certain polynomials in $\mathbb{Z}[A]$. So

$$\{a = (a_0, \cdots, a_d) \in \mathbb{R}^{d+1} : G(a,T) \text{ has exactly } e \text{ complex zeros}\} = \bigcup_{d'=0}^{d} M_e^{d'} \setminus M_{e-1}^{d'}$$

is a semi-algebraic set.
Since $f(x,T) = G(a_0(x), \cdots, a_d(x), T)$, the second part follows. □

**Exercise:** Use the method of proving the lemma to check:
1.9.1. The condition that $f(T) = T^2 + bT + c$ has $\leq 1$ zero is $b^2 - 4c = 0$.
1.9.2. The condition that $f(T) = T^3 + pT + q$ has $\leq 2$ zeros is $4p^3 + 27q^2 = 0$.

[1] Compare with a proof basing on resultants given at the end of the lecture.

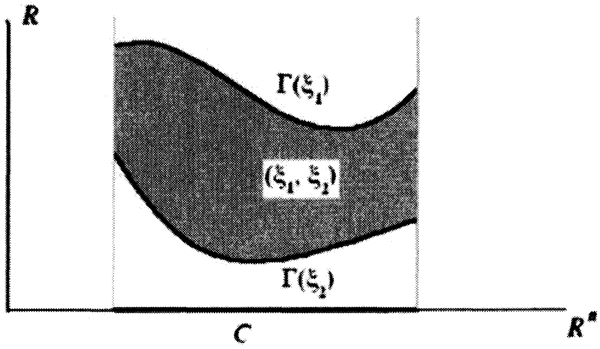



**1.10. Lemma.** *Let* $f = a_0 + \cdots + a_dT^d \in \mathbb{R}[X_1, \cdots, X_n][T]$ *and* $e \leq d$. *Let* $C$ *be a connected subset of* $\mathbb{R}^n$. *Suppose that* $f(x,T) \in \mathbb{R}[T]$ *has exactly e distinct complex zeros for each* $x \in C$. *Then the number of distinct real zeros of* $f(x,T)$ *is also constant as x ranges over* $C$. *If these zeros are ordered by* $\xi_1(x) < \cdots < \xi_r(x)$, *then the functions* $\xi_j : X \longrightarrow \mathbb{R}$ *are continuous.*

*Proof:* Let $x_0 \in C$, and let $z_1, \cdots, z_e$ be the distinct zeros of $f(x_0, T)$. Take closed balls $B_i$ centered at $z_i$ in $\mathbb{C}$, such that $B_i \cap B_j = \emptyset$ for $i \neq j$ and $B_i \cap \mathbb{R} = \emptyset$ if $z_i \notin R$. By continuity of roots (Rouché's theorem), there exists a neighborhood $U$ of $x_0$ in $C$ such that for each $x \in U$ the ball $B_i$ contains at least one zero $\zeta_i(x)$ of $f(x,T)$. By the supposition, $\zeta_i(x)$ is the only zero of $f(x,T)$ in $B_i$. The graph of $\zeta_i$ on $U$ is $\{(x,t) \in U \times B_i : f(x,t) = 0\}$, hence this graph is closed in $U \times B_i$, in combination with the compactness of $B_i$ which implies that $\zeta_i$ is continuous on $U$. Since the coefficients of $f(x,T)$ are real, the set $\{\zeta_1(x), \cdots, \zeta_e(x)\}$ is closed under complex conjugation. Hence if $\zeta_i(x_0) \in \mathbb{R}$ then $\zeta_i(x) \in \mathbb{R}$ for all $x \in U$. This shows that the number of real zeros is locally constant. Since $C$ is connected, this number is constant and the real zeros must keep their order as $x$ runs through $C$. □

**Exercise:** Examine the lemma when $f(T) = T^2 + bT + c$, $(b,c) \in X = \mathbb{R}^2$.

Let $\xi_1, \xi_2 : C \longrightarrow \overline{\mathbb{R}}$, vi $\xi_1 < \xi_2$. Denote
$\Gamma(\xi_1) = \{(x,t) : t = \xi_1(x)\}$ (the graph),
$(\xi_1, \xi_2) = \{(x,t) : x \in C,\ \xi_1(x) < t < \xi_2(x)\}$ (the band).




**1.11. Theorem** (Cylindrical decomposition - Łojasiewicz).
*Let $f_1, \cdots, f_p \in \mathbb{R}[X_1, \cdots, X_n][T]$, $X = (X_1, \cdots, X_n)$. Then there exist an augmentation $f_1, \cdots f_p, f_{p+1}, \cdots, f_{p+q} \in \mathbb{R}[X][T]$ and a partition of $\mathbb{R}^n$ into finitely many semi-algebraic sets $S_1, \cdots, S_k$ such that for each connected component $C$ of each $S_i$ there are continuous functions*

$$-\infty = \xi_{C,0} < \xi_{C,1} < \cdots < \xi_{C,r(C)} < \xi_{C,r(C)+1} = +\infty$$

*on $C$ satisfying the following two properties:*
(i) *Each $f_i$ $(1 \leq i \leq p+q)$ has a constant sign on each $\Gamma(\xi_{C,j})$ $(1 \leq j \leq r(C))$ and on each $(\xi_{C,j}, \xi_{C,j+1})$ $(0 \leq j \leq r(C))$.*
(ii) *Each of the set $\Gamma(\xi_{C,j})$, $(\xi_{C,j}, \xi_{C,j+1})$ is of the form*

$$\{(x,t) \in C \times \mathbb{R} : f_i(x,t)\ s(i)\ 0,\ i = 1, \cdots, p+q\},$$

*for a suitable $s : \{1, \cdots, p+q\} \longrightarrow \{<, =, >\}$.*

*Proof:* Let $d = \max\{\deg_T(f_i), i = 1, \cdots, p\}$.
Augment $f_1, \cdots, f_p$ to $\{f_1, \cdots, f_{p+q}\} = \{\dfrac{\partial^\nu f_i}{\partial T^\nu} : 1 \leq i \leq p, 0 \leq \nu \leq d\}$.
For each $\Delta \subset \{1, \cdots, p\} \times \{0, \cdots, d\}$, and $e \in \{0, \cdots, pd^2\} \cup \{\infty\}$, put

$$f_\Delta(T) = \prod_{(i,\nu)\in\Delta} \frac{\partial^\nu f_i}{\partial T^\nu} \in \mathbb{R}[X][T], \text{ and}$$

$$A_{\Delta,e} = \{x \in \mathbb{R}^n : f_\Delta(x,T) \text{ has exactly } e \text{ complex zeros}\}.$$

By Lemma 1.9, $A_{\Delta,e}$ is a semi-algebraic set. For a given $\Delta$ the family $\{A_{\Delta,e} : e \text{ varies }\}$ forms a partition of $\mathbb{R}^n$. Since the class of semi-algebraic sets is a boolean algebra we can find a partition (the intersection of the partitions) $\mathbb{R}^n = S_1 \cup \cdots \cup S_k$, where each $S_i$ is semi-algebraic such that each set $A_{\Delta,e}$ is a union of the $S_i'$s.
We will prove that $f_1, \cdots, f_{p+q}$ and $S_1, \cdots, S_k$ satisfy the conclusion of the theorem.
For each connected component $C$ of $S_i$ put

$$\Delta(C) = \{(i,\nu) : \frac{\partial^\nu f_i}{\partial T^\nu} \not\equiv 0 \text{ on } C \times \mathbb{R}\}.$$

By Lemma 1.10, there exist continuous functions $\xi_{C,1} < \cdots < \xi_{C,r(C)}$ on $C$ such that $\{(x,t) \in C \times \mathbb{R} : f_{\Delta(C)} = 0\} = \Gamma(\xi_{C,1}) \cup \cdots \cup \Gamma(\xi_{C,r(C)})$.
*Check* (i): If $(i,\nu) \notin \Delta(C)$ then $\dfrac{\partial^\nu f_i}{\partial T^\nu} \equiv 0$ on the sets given in (i).
If $(i,\nu) \in \Delta(C)$, then $C \subset A_{\{(i,\nu)\},e}$, for certain $e \in \{0, \cdots, d\} \cup \{\infty\}$ and the number

complex zeros of $\frac{\partial^\nu f_i}{\partial T^\nu}(x,T)$ is independent of $x \in C$. Since $\frac{\partial^\nu f_i}{\partial T^\nu}$ is a factor of $f_{\Delta(C)}$, by Lemma 1.9, the zeros of $\frac{\partial^\nu f_i}{\partial T^\nu}(x,T)$, for $x \in C$, must be among the $\xi_{C,j}(x)'s$ . Since $C$ is connected, (i) is checked.

*Check* (ii): Let $B$ be one of the sets in (i). By (i), $\epsilon(i,\nu) = \mathrm{sign}(\frac{\partial^\nu f_i}{\partial T^\nu}\mid_B)$ is well-defined. Put

$$B' = \{(x,t) \in C \times \mathbb{R} : \mathrm{sign}(\frac{\partial^\nu f_i}{\partial T^\nu}(x,t) = \epsilon(i,\nu),\ 1 \le i \le p, 0 \le \nu \le d\}.$$

Clearly $B \subset B'$. If $B \neq B'$ then exist $(x,t') \in B' \setminus B$, $(x,t) \in B$ (say $t < t'$). Thom's lemma 1.8 implies that $\{r \in \mathbb{R} : (x,r) \in B'\}$ is connected, so $\{x\} \times [t,t'] \subset B'$. Since $(x,t) \in B$, $(x,t') \notin B$, $f_{\Delta(C)}$ must change sign on $\{x\} \times [t,t']$. But $f_{\Delta(C)}$ is a product of $\frac{\partial^\nu f_i}{\partial T^\nu}$, so $f_{\Delta(C)}$ cannot change sign on $B'$, contradiction. Therefore $B = B'$. □

**Exercise:**

1.11.1. Contract the augment family of polynomials and the partition of $\mathbb{R}^2 = \{(b,c)\}$ satisfying the theorem for $f(b,c,T) = T^2 + bT + c$.

1.11.2. Contract the augment family of polynomials and the partition of $\mathbb{R}^2 = \{(p,q)\}$ satisfying the theorem for $f(p,q,T) = T^3 + pT + q$.

**1.12. Proof of Theorems 1.4 and 1.5:** It is sufficient to prove the followings:

$(\text{T-S})_n$ *If $S \subset \mathbb{R}^n \times \mathbb{R}$ is a semi-algebraic set, then $\pi(S)$ is semi-algebraic.*

$(\text{L})_n$ *If $S \subset \mathbb{R}^n \times \mathbb{R}$ is a semi-algebraic set, then the number of the connected components of $S$ is finite, and each of the components is also semi-algebraic.*

*Proof:* By induction on $n$. It is trivial when $n = 0$.

Suppose $(\text{T-S})_{n-1}$ and $(\text{L})_{n-1}$. Let $S \subset \mathbb{R}^n \times \mathbb{R}$ be a semi-algebraic described by equalities and inequalities of $f_1, \cdots, f_p \in \mathbb{R}[X_1, \cdots, X_n][X_{n+1}]$. Now apply the cylindrical decomposition theorem 1.11. There exist an augmentation of this family and a partition $\mathbb{R}^n = \bigcup_i S_i = \bigcup_i \bigcup_j C_{ij}$, where $S_i$ is semi-algebraic and $C_{ij}$ is a connected component of $S_i$. By $(\text{L})_{n-1}$, the number of the $C'_{ij}s$ is finite and $C_{ij}$ is semi-algebraic. Therefore, $\mathbb{R}^n \times \mathbb{R}$ is partitioned into graphs and bands of continuous functions on the $C'_{ij}s$, which are connected semi-algebraic sets. Since $S$ is a union of these sets, $\pi(S) = \cup\{C_{ij} : C_{ij} \times \mathbb{R} \cap S \neq \emptyset\}$ is semi-algebraic, i.e. $(\text{T-S})_n$, and $S$ satisfies $(\text{L})_n$. □

**Exercise:** The following exercises are related to **resultants** (ref. [BR]).

Let $A$ be a factorial commutative ring. Let

$P(T) = a_0 + \cdots + a_p T^p \in A[T]$, $a_p \neq 0$,

$Q(T) = b_0 + \cdots + b_q T^q \in A[T],\ b_q \neq 0.$

For $0 \leq k \leq \min(p,q)$, **the $k$-nd Sylvester's matrix** of $P, Q$ is defined by:

$$M_k(P,Q) \quad = \quad \begin{array}{l} \left.\begin{pmatrix} a_0 & \cdots & 0 & b_0 & \cdots & 0 \\ \vdots & \ddots & & \vdots & \ddots & \\ & & a_0 & & & b_0 \\ a_p & & \vdots & b_q & & \vdots \\ & \ddots & & & \ddots & \\ 0 & & a_p & 0 & & b_q \end{pmatrix}\right\} {\scriptstyle p+q-k} \\ \quad \underbrace{\qquad\qquad\quad}_{q-k} \ \underbrace{\qquad\qquad\quad}_{p-k} \end{array}$$

1.12.1. Prove that the following conditions are equivalent:

(a) The degree of GCD$(P,Q)$ is $\geq k+1$.

(b) $P, Q$ have $\geq k+1$ common zeros (counted with multiplicity) in the algebraic closure $\overline{A}$.

(c) Every $(p+q-2k)$-minor of $M_k(P.Q)$ vanishes.

1.12.2. From the above exercise, prove that the condition is that $P, Q$ have $k$ distinct zeros in $\overline{A}$, which is the condition given by equalities and inequalities of certain polynomials in $\mathbb{Z}[a_0, \cdots, a_p, b_0, \cdots, b_q]$.

1.12.3. When $A = \mathbb{C}$, prove that $P$ has exactly $k$ zeros if and only if the degree of GCD$(P, P')$ is $p-k$.

This implies Lemma 1.9.

1.12.4. The **resultant** of $P, Q$ is defined by $\operatorname{Res}(P,Q) = \det(M_0(P,Q))$. Therefore,

$$\operatorname{Res}(P,Q) = 0 \ \Leftrightarrow \ P, Q \ \text{ having GCD of degree } > 0.$$

1.12.5. The **discriminant** of $P$ is defined by $\operatorname{Disc}(P)$=$\operatorname{Res}(P,P') = \det(M_0(P,Q))$. When $A = \mathbb{C}$, we have

$$\operatorname{Disc}(P) = 0 \ \Leftrightarrow \ P \ \text{ having zeros of multiplicity} > 0$$

1.12.6. Compute the discriminants of polynomials of degree 2, 3.

**Seminar:** Sturm's theorem and Tarski-Seidenberg's theorem. (ref. [BCR] or [C]).

## References

[BR] R. Benedetti and J-J. Risler, *Real Algebraic and Semi-algebraic Sets,* Hermann, 1990.

[BCR] J. Bochnak, M. Coste and M.-F. Roy, *Géométrie algébrique réel,* Springer-Verlarg, Berlin, 1987.

[C] M. Coste, *An introduction to semialgebraic geometry,* Universita di Pisa, Dottorato di recerca in Matematica, Instituti editoriali e poligrafigi internazionali, Pisa-Roma, 2000.

[D] L. van den Dries, *Tame Topology and o-minimal Structures,* LMS Lecture Note Serries, 248, Cambridge University Press, Cambridge, 1998.

[L] S. Łojasiewicz, *Ensembles Semi-Analytiques,* IHES, Bures-sur-Yvette, 1965.

[LZ] S. Łojasiewicz and M.A. Zurro, *Una introducción a la geometria semi -y subanalitica,* Univesidad de Valladolid, 1993.

[S] S. Seidenberg, *A new decision method for elementary algebra,* Ann. of Math. 60(1954), 365-374.

[T] A. Tarski *A decision method for elementary algebra and geometry,* 2nd ed. University of California Press, Berkeley and Los Angeles, Calif. , 1951.

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