(A certain surgery construction of contact 3-manifolds)

(Jiro ADACHI
Department of Mathematics,
Hokkaido University

1   Introduction

The construction and classification of contact manifolds is still a basic problem in differential topology. It was shown by Weinstein [W2] and Eliashberg [E] that the manifold obtained from a contact manifold by a certain handle surgery carries a contact structure. In order to define surgery of contact manifolds, they use symplectic and Stein handles. In this note, a new construction method, contact round surgery, is introduced. In other words, symplectic round handle is introduced. Round handle itself has some geometric meanings. Therefore, it may have some meaning in contact topology to consider round surgery of contact manifolds. An attempt to apply round handle theory to contact topology is in [Ad2].

In this note, round surgery of contact 3-manifolds is introduced. In other words, 4-dimensional symplectic round handles are constructed. The cases of general dimensions are discussed in [Ad3]. The situation is rather complicated in that case.

We should remark that a 4-dimensional symplectic round handle appeared in [Ga]. That symplectic round handle is different from the round handle defined in this paper. Although Gay's symplectic round handle is not exact, the symplectic round handle in this paper is exact. Because

there is a global Liouville vector field on the round handle in this paper. Unfortunately, no application of symplectic round handle is given in [Ga].

These days, Weinstein's contact surgery is one of important tools in contact topology. In terms of contact Dehn surgery, Weinstein's contact surgery along a Legendrian knot is a (-1)-surgery. Attaching the symplectic handle on the concave end of the trivial cobordism corresponds to $(+1)$-surgery. Ding and Geiges [DGe] proved that every closed orientable contact manifold is constructed by these two operations. And now, these are good tools to deal with Heegaard Floer homology (see [OS]). Then the contact round surgery may also be a good tool in contact topology.

The result is described as follows.

Theorem A. *Let M be a convex contact type subset of the boundary of a 4-dimensional symplectic manifold (W, ω) with respect to a Liouville vector field X defied near $M \subset W$. Let $L \subset M$ be an Legendrian link with two components with respect to the contact structure induced on M from X and ω. Then the Liouville vector field X and the symplectic structure ω extend to the manifold obtained from W by attaching a round handle of index k along $\tilde{S}^k$, and the modified boundary is also convex.*

We should remark that the framing of the attachment is restricted to the so called contact framing of Legendrian knots. As a corollary, we obtain the following.

Corollary B. *Let (M, ξ) be a contact 3-manifold, and $L \subset (M, \xi)$ a Legendrian link with two components. Then the manifold obtained from M by a round surgery along L with respect to contact framing of L has a contact structure. If (M, ξ) is strongly symplectically fillable, then the obtained contact manifold is also strongly symplectically fillable.*

2   Round handle and Round surgery

Round handle was introduced by Asimov [As] to study the Morse-Smale flow (see also [M]). As well as for such a study, round handle itself is a

useful tool for the study of manifolds and some structures on manifolds.

Round handle and round handle decomposition are defined as follows. Let M be a manifold of dimension n with boundary $\partial M \neq \emptyset$.

Definition. A *round handle* of dimension n and index k attached to M is defied as a pair

$$R_k = \left(D^k \times D^{n-k-1} \times S^1, f\right)$$

of a product of an $(n-1)$-dimensional disk $D^k \times D^{n-k-1}$ with a circle and an attaching embedding $f\colon \partial_- \left(D^k \times D^{n-k-1} \times S^1\right) \to \partial M$, where $\partial_- \left(D^k \times D^{n-k-1} \times S^1\right) := \partial D^k \times D^{n-k-1} \times S^1$ is the attaching region.

Round handles are used to study flow manifolds. Flow manifold is defined as follows. Let $(M, \partial_- M)$ be a pair of a manifold M with a specific union $\partial_- M$ of connected components of the boundary ∂M. The pair $(M, \partial_- M)$ is called a *flow manifold* if there exists a non-singular vector field on M which looks inward on $\partial_- M$ and outward on $\partial_+ M :=$ $\partial M \backslash \partial_- M$. The following property of flow manifolds is proved by Asimov.

Theorem 2.1 (Asimov, [As]). *Let $(M, \partial_- M)$ be a compact flow manifold whose dimension is greater than 3. Then, M has a round handle decomposition.*

By using round handles in stead of ordinary handles, round surgery is defined. In other words, a round surgery corresponds to attaching a round handle to a cobordism. Let M be a manifold of dimension n. A *round surgery* of index k is defined as the operation removing an embedded int $\left(\partial D^k \times D^{n-k} \times S^1\right)$ from M and regluing $D^k \times \partial D^{n-k} \times S^1$ by the identity mapping of $\partial D^k \times \partial D^{n-k} \times S^1$.

In this paper we construct certain symplectic structures on these round handles.

3 Symplectic round handle

3.1 Liouville vector fields and symplectizations

A contact manifold appears as a hypersurface in a symplectic manifold. This relation is given by the so called Liouville vector field. A vector field X on a symplectic manifold (M, ω) is called a *Liouville vector field* if the Lie derivative along it preserves the symplectic structure: $L_X \omega = \omega$. The following property is well known (see [W1] for example).

Lemma 3.1. *Let X be a Liouville vector field on a symplectic manifold (W, ω). If $M \subset W$ is a hypersurface transverse to X, then the pullback $i^*(X \lrcorner \omega)$ is a contact form on M, where $i \colon M \hookrightarrow W$ is the inclusion mapping.*

Such hypersurface $M \subset (W, \omega)$ transverse to X is said to be of *contact type*. When a contact type hypersurface is a boundary of a symplectic manifold, it is said to be *convex* (resp. *concave*) if the Liouville vector field looks outward (resp. inward) there.

From another point of view, the induced contact structure depends more on the Liouville vector field than on the hypersurface. The following lemma implies this property (see [W2]).

Lemma 3.2. *Let X be a Liouville vector field on a symplectic manifold (W, ω). And let $M_j \subset W$, $j = 0, 1$, be hypersurfaces with the inclusion mappings $i_j \colon M_j \hookrightarrow W$. Assume that there exists a diffeomorphism $f \colon M_0 \to M_1$ following the integral curves of X. Then f is contactomorphic with respect to the induced contact structures $\xi_j = \ker(i_j^*(X \lrcorner \omega))$, $j = 0, 1$.*

One of the most typical examples is the symplectization of a contact manifold. In other words, any contact manifold with a contact form is realized as a contact type hypersurface in some symplectic manifold. Let (M, ξ) be a contact manifold with a contact form α on M. The 2-form $\omega := d(e^t \alpha) = e^t(dt \wedge \alpha + d\alpha)$ is a symplectic structure on $M \times \mathbb{R}$, where

t is a coordinate of $\mathbb{R}$. The symplectic manifold $(M \times \mathbb{R}, \omega)$ is called the *symplectization* of (M, α). The Liouville vector field on $(M \times \mathbb{R}, \omega)$ is $X = \partial/\partial t$, which is transverse to $M = M \times \{0\} \subset M \times \mathbb{R}$, and the induced contact on $M = M \times \{0\}$ is α. This implies that (M, α) is realized as a contact type hypersurface in the symplectization $(M \times \mathbb{R}, \omega = d(e^t \alpha))$.

On the other hand, a tubular neighborhood of any contact type hypersurface is symplectomorphic to the tubular neighborhood of the hypersurface in its symplectization (see [Ad1], [Ge] for example).

Lemma 3.3. *Let M be a compact contact type hypersurface in a symplectic manifold (W, ω) with a Liouville vector field X. Let α denote the contact form $i^*(X \lrcorner \omega)$ induced on M, where $i \colon M \hookrightarrow W$ is the inclusion. Then there exists a local symplectomorphism between neighborhoods of $M = M \times \{0\} \subset (M \times \mathbb{R}, d(e^t \alpha))$ and $M \subset (W, \omega)$ which maps $\partial/\partial t$ to X, and M to M identically.*

This lemma is important in the attaching procedure of symplectic round handles in Subsection 3.3.

3.2 The model symplectic round handle

The model round handle is taken as a subset in $\mathbb{R}^3 \times S^1$ with some symplectic structure whose attaching region is of concave contact type and the belt region is of convex contact type. Note that like Weinstein's $2n$-dimensional symplectic handle is defined for index $k = 1, 2, \ldots, n$ (see [W2]), the 4-dimensional symplectic round handle is defined only for index $k = 1$.

First of all, we need the following symplectic structure and Liouville vector field on $\mathbb{R}^3 \times S^1$. The standard symplectic structure ω_0 on $\mathbb{R}^3 \times S^1$ is given as

$$\omega_0 := (dp \wedge dq) + dz \wedge d\phi,$$

where (p, q, z, ϕ) are coordinates of $\mathbb{R}^3 \times S^1 = \mathbb{R}^2 \times \mathbb{R} \times S^1$. Set a vector

field X_1 on $\mathbb{R}^{2n-1} \times S^1$ as

$$X_1 := \left(-q\frac{\partial}{\partial q} + 2p\frac{\partial}{\partial p}\right) + z\frac{\partial}{\partial z} \tag{3.1}$$

(see the dotted curves and arrows in Figure 1). It is the Liouville vector field for the symplectic structure ω_0. Note that the vector field X_k is the gradient vector field of the function

$$f_1(p, q, z, \phi) := \left(-\frac{1}{2}q^2 + p^2\right) + \frac{1}{2}z^2 \tag{3.2}$$

with respect to the standard Euclidean metric. It is a Morse-like function whose critical loci are not isolated points but circles.

Now, the model symplectic round handle is defined as follows. In order to define it, we use two functions f_1 above and

$$g_1(p, q, z, \phi) := -Aq^2 + B\left(p^2 + z^2\right) \tag{3.3}$$

on $\mathbb{R}^3 \times S^1$, where A, B are arbitrary positive constants. The 4-dimensional model symplectic round handle is defined as a domain in the symplectic space $\left(\mathbb{R}^3 \times S^1, \omega_0\right)$ above bounded by the following two hypersurfaces:

$$W_- := \left\{x = (p, q, z, \phi) \in \mathbb{R}^3 \times S^1 \mid f_1(x) = -1\right\},$$
$$V_c := \left\{x = (p, q, z, \phi) \in \mathbb{R}^3 \times S^1 \mid g_1(x) = c\right\}, \quad c > 0.$$

These hypersurfaces do intersect and bound a domain if B/A is sufficiently large.

Definition. The 4-dimensional *model symplectic round handle* R_1^0 of index 1 is defined as

$$R_1^0 := \left\{x = (p, q, z, \phi) \in \mathbb{R}^3 \times S^1 \mid f_1(x) \geq -1, \; g_1(x) \leq c\right\} \tag{3.4}$$

with the symplectic structure ω_0 (see Figure 1).

This R_k^0 is homeomorphic to $D^k \times D^{n-k-1} \times S^1$.

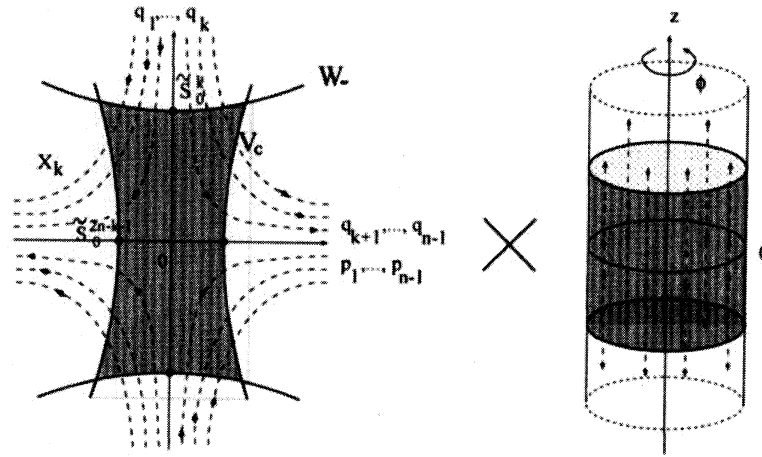The model symplectic round handle has the following properties.

Figure 1: Model round handle

Lemma 3.4. *Let* $\left(R_1^0, \omega_0\right)$ *be a model symplectic round handle.*

(1) *The attaching region* $\partial_- R_1^0 \subset R_1^0$ *is transverse to the Liouville vector field* X_1. *At* $\partial_- R_1^0$, X_1 *looks inward.*

(2) *The belt region* $\partial_+ R_1^0 \subset R_1^0$ *is transverse to the Liouville vector field* X_1. *At* $\partial_+ R_1^0$, X_1 *looks outward.*

(3) *The attaching core* $\tilde{S}_0^1 \cong S^0 \times S^1$ *is a Legendrian link with two components in the contact manifold* $\partial_- R_1^0 \subset W_-$.

(4) R_1^0 *can be taken so that its attaching region is contained in an arbitrary small neighborhood of* $\tilde{S}_0^1$ *in* W_-.

3.3 Attaching the model symplectic round handles

We attach the model symplectic round handle defined in the previous subsection to a symplectic manifold in this subsection. The attachment implies Theorem A.

We attach the model symplectic round handle (R_1^0, ω_0) to the following setup. Let (W, ω) be a symplectic manifold with boundary, and $M \subset \partial W$ convex components with respect to a Liouville vector field X defined near $M \subset W$. Let α denote a contact form $i^*(X \lrcorner \omega)$ on M, where $i \colon M \hookrightarrow W$ is the inclusion mapping, and ξ the contact structure $\ker \alpha$. We are

going to attach (R_1^0, ω_0) along an Legendrian link $L \subset (M, \xi)$ with two components.

We attach the model symplectic round handle (R_1^0, ω_0) as follows. It is well known that Legendrian knots have a unique standard contact tubular neighborhood. Therefore, there exists a local strict contactomorphism $\varphi \colon \left(U(\tilde{S}_0^1, W_-), \alpha_1 \right) \to (U(L, M), \alpha)$ between suitable neighborhoods of $\tilde{S}_0^k \subset \partial_- R_k^0 \subset W_-$ and $L \subset M$ which satisfies $\varphi(\tilde{S}_0^k) = L$. We may suppose $\partial_- R_k^0 \subset U(\tilde{S}_0^k, W_-) \subset W_-$ since $\partial_- R_k^0$ can be taken arbitrarily close to $\tilde{S}_0^k$ from the construction of the model symplectic round handle (see Lemma 3.4). Then, from Lemma 3.3, the contactomorphism extends to a symplectomorphism of neighborhoods. By this symplectomorphism, two symplectic manifolds (R_k^0, ω_0) and (W, ω) are glued symplectically.

3.4 Contact round surgery

Now, we define the contact round surgery as follows. Then we obtain Corollary B almost directly. Let $(M, \xi = \ker \alpha)$ be a contact 3-manifold, and $L \subset (M, \xi)$ a Legendrian link with two components. In general, round surgery of an $(2n - 1)$-dimensional manifold M is defined by attaching a round handle of dimension $2n$ to $M \times [0, 1]$ (see Subsection 2 for precise definition). Then we need a symplectic structure on $M \times [0, 1]$. We take $M \times [0, 1]$ as a subset of the symplectization $(M \times \mathbb{R}, d(e^t \alpha))$ of the given contact manifold (M, ξ). The induced contact structure on $M \times \{i\}$ is ξ for both $i = 0, 1$. Regarding L as a Legendrian link in $(M \times \{1\}, \xi)$, we can attach a symplectic round handle of index 1 along L by Theorem A. Since the modified end is also convex from Theorem A, a contact structure is induced there. Thus we obtained a new contact manifold. We call this operation a *contact round surgery* (see Figure 2).

Like Weinstein's contact surgery, we can discuss the strong symplectic fillability by the contact round surgery. A contact manifold (M, ξ) is said to be *strongly symplectically fillable* if it is a convex contact type
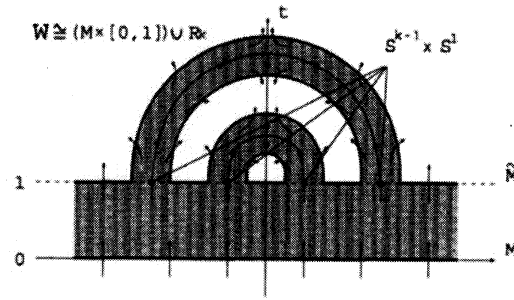
Figure 2: Contact round surgery

boundary of a compact symplectic manifold (W, ω) and the induced contact structure is also ξ. The manifold obtained from (M, ξ) by a contact round surgery and (M, ξ) itself have a symplectic cobordism $(\tilde{W}, \tilde{\omega})$ constructed by attaching a symplectic handle. Note that (M, ξ) is a convex boundary of (W, ω) and a concave end of $\tilde{W}, \tilde{\omega})$. Therefore, they are glued symplectically along (M, ξ). Then we obtain a symplectic filling of the surgered contact manifold.

References

[Ad1]   J. Adachi, *Liouville setup and contact cobordism*, Hokkaido Math. J. **25** (1996), 637–650.

[Ad2]   J. Adachi, *Round surgery and contact structures*, (preprint).

[Ad3]   J. Adachi, *Contact round surgery and symplectic round handlebodies*, (preprint).

[As]   D. Asimov, *Round handles and non-singular Morse-Smale flows*, Ann. of Math. (2) **102** (1975), 41–54.

[DGe]   F. Ding, H. Geiges, *A Legendrian surgery presentation of contact 3-manifolds*, Math. Proc. Cambridge Philos. Soc. **136** (2004), 583–598.

[E]   Ya. Eliashberg, *Topological characterization of Stein manifolds of dimension > 2*, Internat. J. Math. **1** (1990), 29–46.

[Ga]   D. Gay, *Four-dimensional symplectic cobordisms containing three-handles*, Geom. Topol. **10** (2006), 1749–1759.

[Ge]   H. Geiges, An introduction to contact topology, *Cambridge Studies in Advanced Mathematics* **109**, Cambridge University Press, Cambridge, 2008.

[M]   J. Morgan, *Nonsingular Morse-Smale flows on 3-dimensional manifolds*, Topology **18** (1979), 41–53.

[OS]   B. Ozbagci and A. Stipsicz, Surgery on contact 3-manifolds and Stein surfaces, *Bolyai Society Mathematical Studies*, **13**, Springer-Verlag, Berlin, 2004.

[W1]   A. Weinstein, *On the hypotheses of Rabinowitz' periodic orbit theorems*, J. Differential Equations **33** (1979), 353–358.

[W2]   A. Weinstein, *Contact surgery and symplectic handlebodies*, Hokkaido Math. J. **20** (1991), 241–251.

Department of Mathematics,
Hokkaido University,
Sapporo, 060-0810, Japan.

e-mail: j-adachi@math.sci.hokudai.ac.jp