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ON IMMERSED ORIENTED SURFACES AND THEIR PLANE PROJECTIONS

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1. Introduction

Throughout the report, all manifolds and maps are differentiable of class C^∞. Let M be a closed connected surface, f and $g : M \to \mathbb{R}^3$ immersions. Let $F : M \times [0,1] \to \mathbb{R}^3$ be a homotopy between f and g. That is, $F|M \times \{0\} = f$ and $F|M \times \{1\} = g$ hold. We call F a regular homotopy between f and g if a level preserving map $(F \times \mathrm{id}) : M \times [0,1] \to \mathbb{R}^3 \times [0,1]$ defined by $(F \times \mathrm{id})(x,t) = (F(x,t),t)$ is an immersion. James and Thomas [4] proved that the space of immersions $f : M \to \mathbb{R}^3$ has $2^{\dim H_1(M;\mathbb{Z}_2)}$ connected components. Pinkall [6] associated to any immersion $f : M \to \mathbb{R}^3$ a $\mathbb{Z}_4$-valued quadratic form $q_f : H_1(M;\mathbb{Z}_2) \to \mathbb{Z}_4$ and proved that two immersions f and $g : M \to \mathbb{R}^3$ are regularly homotopic if and only if $q_f = q_g$ holds.

Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be an orthogonal projection and $\tilde{f} : M \to \mathbb{R}^2$ a stable map. If a stable map $\tilde{f} : M \to \mathbb{R}^2$ has an immersion $f : M \to \mathbb{R}^3$ such that $\pi \circ f = \tilde{f}$, we call that $\tilde{f}$ has an immersion lift f and that f is an immersion lift over $\tilde{f}$. Let f and $g : M \to \mathbb{R}^3$ be immersion lifts over $\tilde{f} : M \to \mathbb{R}^2$. If there exists a regular homotopy $F : M \times [0,1] \to \mathbb{R}^3$ between f and g which satisfies that $\pi \circ (F|M \times \{t\}) = \tilde{f}$ for any $t \in [0,1]$, then we call that f and g are $\tilde{f}$-regularly homotopic.

In this report, when M is a closed connected oriented surface, we study $\tilde{f}$-regular homotopy classes for a fixed stable map $\tilde{f} : M \to \mathbb{R}^2$ which has an immersion lift.

This report is organized as follows. In Section 2, we give the definition and see properties of a stable map $\tilde{f} : M \to \mathbb{R}^2$. In Section 3, we restate Haefliger's theorem [3] by putting sings on the apparent contour of $\tilde{f}$. In Section 4, we determine $\tilde{f}$-regular homotopy classes for a fixed stable map $\tilde{f}$ which has an immersion lift. In Section 5, we give the definitions of generic homotopy and regular homotopy lift. By using the method obtained in the previous section, we introduce a necessary and sufficient condition for existence of regular homotopy lift over the given generic homotopy. As an application, we construct a generic homotopy whose regular homotopy lift corresponds to a sphere eversion.

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2. Stable map

In this section, we give the definition and see properties of a stable map.

Let $\tilde{f} : M \to \mathbb{R}^2$ be a smooth map of a closed connected surface M into the plane. We denote the set of such maps by $C^\infty(M, \mathbb{R}^2)$, which is equipped with the Whitney C^∞-topology. A smooth map $\tilde{f}$ is said to be a stable map if in $C^\infty(M, \mathbb{R}^2)$, there exists an open neighborhood U of $\tilde{f}$ such that for any $\tilde{g} \in U$, $\tilde{g}$ is C^∞ right-left equivalent to $\tilde{f}$, i.e., there exist two diffeomorphisms $\Phi : M \to M$ and $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "\Phi"] \arrow[d, "\tilde{f}"'] & M \arrow[d, "\tilde{g}"] \\ \mathbb{R}^2 \arrow[r, "\varphi"'] & \mathbb{R}^2 \end{tikzcd}$$

is commutative.

For a smooth map $\tilde{f} : M \to \mathbb{R}^2$, we denote by $S(\tilde{f})$ the set of the points in M where the rank of the differential of $\tilde{f}$ is strictly less than two. We say that $S(\tilde{f}) \subset M$ is a singular set of $\tilde{f}$ and $\tilde{f}(S(\tilde{f})) \subset \mathbb{R}^2$ is an apparent contour of $\tilde{f}$.

The following characterizations of stable maps are well-known (see [2, 8], for example).

Proposition 2.1. *A smooth map $\tilde{f} : M \to \mathbb{R}^2$ of a closed surface M is a stable map if and only if the following conditions are satisfied.*

(i) *For every $q \in M$, there exist local coordinates (x, y) and (X, Y) around $q \in M$ and $\tilde{f}(q) \in \mathbb{R}^2$ respectively such that one of the following holds:*
   (a) $(X \circ \tilde{f}, Y \circ \tilde{f}) = (x, y)$ *(q : regular point),*
   (b) $(X \circ \tilde{f}, Y \circ \tilde{f}) = (x, y^2)$ *(q : fold point),*
   (c) $(X \circ \tilde{f}, Y \circ \tilde{f}) = (x, y^3 - xy)$ *(q : cusp point).*

(ii) *If $q \in M$ is a cusp point, then $\tilde{f}^{-1}(\tilde{f}(q)) \cap S(\tilde{f}) = \{q\}$.*

(iii) *The map $\tilde{f}|(S(\tilde{f}) \setminus \{\text{cusp points}\})$ is an immersion with normal crossings.*

Note that $S(\tilde{f})$ is a compact 1-dimensional submanifold of M and the number of cusp points is finite. Let $U \subset M$ be a tubular neighborhood of $S(\tilde{f})$. Then the restriction of $\tilde{f}$ on the closure $\mathrm{cl}(M \setminus U)$ is an immersion. By the apparent contour of $\tilde{f}$, $\mathbb{R}^2$ is naturally stratified into 2-, 1- and 0-dimensional strata. The union of 1- and 0-dimensional strata forms $\tilde{f}(S(\tilde{f}))$. On each 1-dimensional stratum, we can define an orientation as follows. We fix the canonical orientation on $\mathbb{R}^2$. Let Ω be a connected component of $\mathbb{R}^2 \setminus \tilde{f}(S(\tilde{f}))$. We associate to Ω a non-negative integer $n_{\tilde{f}}(\Omega)$, which is the number of points in the fiber of $\tilde{f}$ over any point of Ω. Every 1-dimensional stratum is adjacent to exactly two connected components of $\mathbb{R}^2 \setminus \tilde{f}(S(\tilde{f}))$. Since these two components have distinct $n_{\tilde{f}}(\Omega)$-values, we can orient each 1-dimensional stratum in $\tilde{f}(S(\tilde{f}))$ so that the region with the larger $n_{\tilde{f}}(\Omega)$-value is on its left. Since $\tilde{f}|(S(\tilde{f}) \setminus \{\text{cusp points}\})$ is an immersion, $S(\tilde{f}) \setminus \{\text{cusp points}\}$ is also oriented.

Suppose that M is an oriented closed surface and $\mathbb{R}^2$ is oriented plane. Let q be a cusp point of a stable map $\tilde{f} : M \to \mathbb{R}^2$. For a sufficiently small neighborhood U of $f(q)$, the map $\tilde{f}|V : V \to U$ has degree ± 1, where V is the component of $\tilde{f}^{-1}(U)$ containing q. We call q is a positive (resp. negative) cusp if the local degree of $\tilde{f}$ at q equals $+1$ (resp. -1).

3. Immersion lift

In the following, we assume that M, $\mathbb{R}^3$ and $\mathbb{R}^2$ are oriented. In this case, Haefliger's theorem is restated as follows.

Theorem 3.1 (Haefliger [3]). *A stable map $\tilde{f} : M \to \mathbb{R}^2$ has an immersion lift if and only if each connected component of $S(\tilde{f})$ has even number of cusp points.*

Let $\tilde{f} : M \to \mathbb{R}^2$ be a stable map which has an immersion lift. On each connected component of fold points $S(\tilde{f}) \setminus \{\text{cusp points}\}$, we can put a sign $+1$ or -1 which satisfies the following rule.

- Let C and C' be two connected components which adjacent to the same cusp point. Then C and C' have the opposite signs.

If a sign of C is $+1$ (resp. -1), we call C a positive (resp. negative) fold and a sign can be put on each image of fold component. Such a stable map $\tilde{f}$ is called a signed stable map.

Let $\tilde{f} : M \to \mathbb{R}^2$ be a signed stable map and U a tubular neighborhood of $S(\tilde{f})$. Since M is oriented, $U \setminus S(\tilde{f})$ is divided into two regions U_+ and U_- where $\tilde{f}|U_+$ (resp. $\tilde{f}|U_-$) is an orientation preserving (resp. reversing) immersion. We construct an immersion lift $f_U : U \to \mathbb{R}^3$ over $\tilde{f}|U$ which satisfies the following.

(1) If C is a positive fold, f is defined as Figure 1(a).
(2) If C is a negative fold, f is defined as Figure 1(b).
(3) If q is a positive cusp and negative fold comes in q for the orientation of $S(\tilde{f})$, f is defined as Figure 2(a).
(4) If q is a positive cusp and positive fold comes in q for the orientation of $S(\tilde{f})$, f is defined as Figure 2(b).
(5) If q is a negative cusp and negative fold comes in q for the orientation of $S(\tilde{f})$, f is defined as Figure 2(c).
(6) If q is a negative cusp and positive fold comes in q for the orientation of $S(\tilde{f})$, f is defined as Figure 2(d).

Definition 3.2. Let $\tilde{f} : M \to \mathbb{R}^2$ be a signed stable map and U a tubular neighborhood of $S(\tilde{f})$. If an immersion $f : M \to \mathbb{R}^3$ satisfies the above rules (1)–(6) on $f|U$, we call f an immersion lift over the signed stable map $\tilde{f}$.

4. $\tilde{f}$-regular homotopy

In this section, we state that $\tilde{f}$-regular homotopy classes can be determined.

Theorem 4.1. *If f and $g : M \to \mathbb{R}^3$ are immersion lifts over the signed stable map $\tilde{f} : M \to \mathbb{R}^2$, then f and g are $\tilde{f}$-regularly homotopic.*

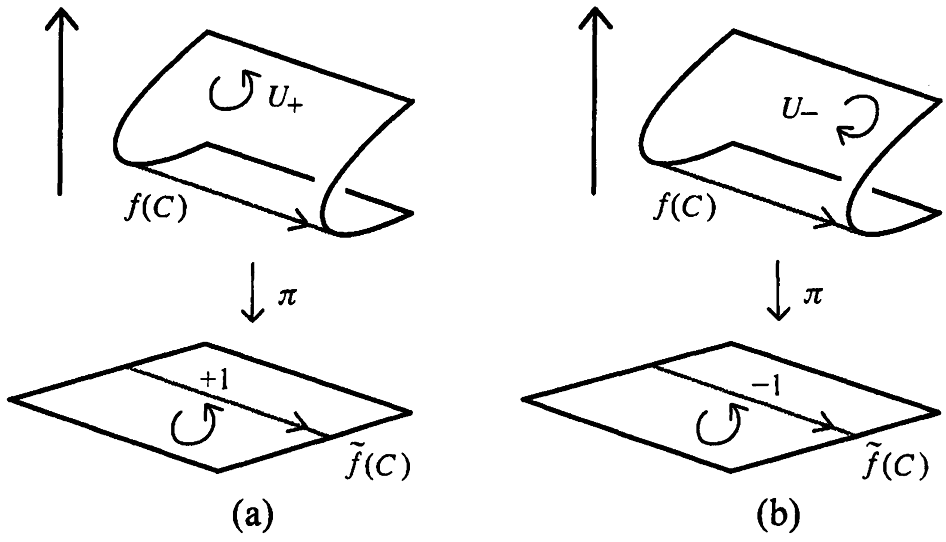

FIGURE 1. Immersion lifts if (a) C is a positive fold, (b) C is a negative fold.

Corollary 4.2. *Let $\tilde{f} : M \to \mathbb{R}^2$ be a stable map which has an immersion lift. (Note that $\tilde{f}$ is not signed.) The number of $\tilde{f}$-regular homotopy classes is $2^{\sharp S(\tilde{f})}$, where $\sharp S(\tilde{f})$ is the number of connected components of $S(\tilde{f})$.*

We have a following example which is related to Theorem 4.1.

Example 4.3. Let T^2 be an oriented torus and l and m longitude and meridian of T^2, respectively. Let $\tilde{f}$ and $\tilde{g} : T^2 \to \mathbb{R}^2$ be signed stable maps which satisfy the following properties. They do not have cusp points, $\tilde{f}(S(\tilde{f})) = \tilde{g}(S(\tilde{g}))$, both signs are the same, $\tilde{f}|l = \tilde{g}|l$ and $\tilde{g}|m$ are plane curves whose rotation numbers equal 2 (or -2), $\tilde{f}|m$ is a simple closed plane curve. See Figure 3. By the theorem of Pinkall [6], immersion lifts f and $g : T^2 \to \mathbb{R}^3$ over $\tilde{f}$ and $\tilde{g}$ respectively are not regularly homotopic.

Theorem 3.1 and Example 4.3 mean that if $M \neq S^2$, an apparent contour with sign does not determine a regular homotopy class. We need information of immersion $\tilde{f}|(M \setminus S(\tilde{f}))$.

5. REGULAR HOMOTOPY LIFT OVER A GENERIC HOMOTOPY

Let $\tilde{f}$ and $\tilde{g} : M \to \mathbb{R}^2$ be stable maps and $\tilde{F} : M \times [0,1] \to \mathbb{R}^2$ a homotopy between $\tilde{f}$ and $\tilde{g}$. If $\tilde{F}$ satisfies the following conditions, we call $\tilde{F}$ a generic homotopy between $\tilde{f}$ and $\tilde{g}$ (see [5]).

(1) There is a finite set of parameter values $0 < t_1 < \cdots < t_n < 1$ (possibly empty) in $(0,1)$.
(2) For any $t \in (0,1) \setminus \{t_1, \ldots, t_n\}$, $\tilde{F}|M \times \{t\} : M \times \{t\} \to \mathbb{R}^2$ is a stable map.
(3) For each t_i and a sufficiently small positive value ε, the moves of apparent contours of $\tilde{F}|M \times \{t\}$ $(t \in (t_i - \varepsilon, t_i + \varepsilon))$ are classified into lips (type L), beaks (type B), swallowtail (type S), cusp-fold (type C), self-tangency (type K) or triple point (type T).

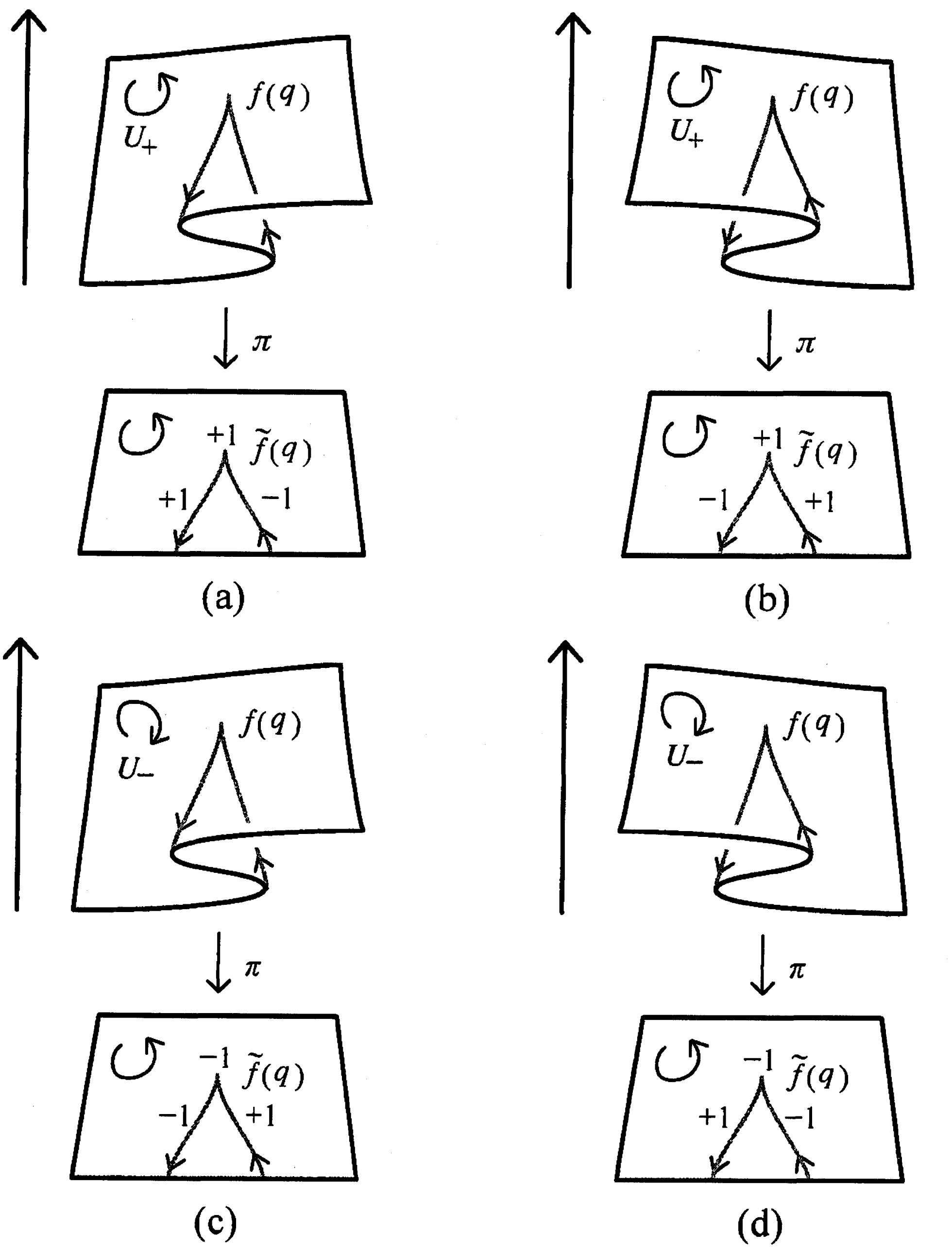

FIGURE 2. Immersion lifts if (a) q is a positive cusp and negative fold comes in, (b) q is a positive cusp and positive fold comes in, (c) q is a negative cusp and negative fold comes in, (b) q is a negative cusp and positive fold comes in.

We call each t_i is a bifurcation point on a generic homotopy $\tilde{F}$.

Let $\tilde{F} : M \times [0,1] \to \mathbb{R}^2$ be a generic homotopy between signed stable maps $\tilde{f}$ and $\tilde{g}$ and let f and g immersion lifts over $\tilde{f}$ and $\tilde{g}$, respectively. If there exists a regular homotopy $F : M \times [0,1] \to \mathbb{R}^3$ between f and g such that $\pi \circ F = \tilde{F}$, we call F a regular homotopy lift over $\tilde{F}$.

Theorem 5.1. *Let $\tilde{f}$ and $\tilde{g} : M \to \mathbb{R}^2$ be signed stable maps. If there exists a generic homotopy $\tilde{F} : M \times [0,1] \to \mathbb{R}^2$ between $\tilde{f}$ and $\tilde{g}$ which preserves sign*

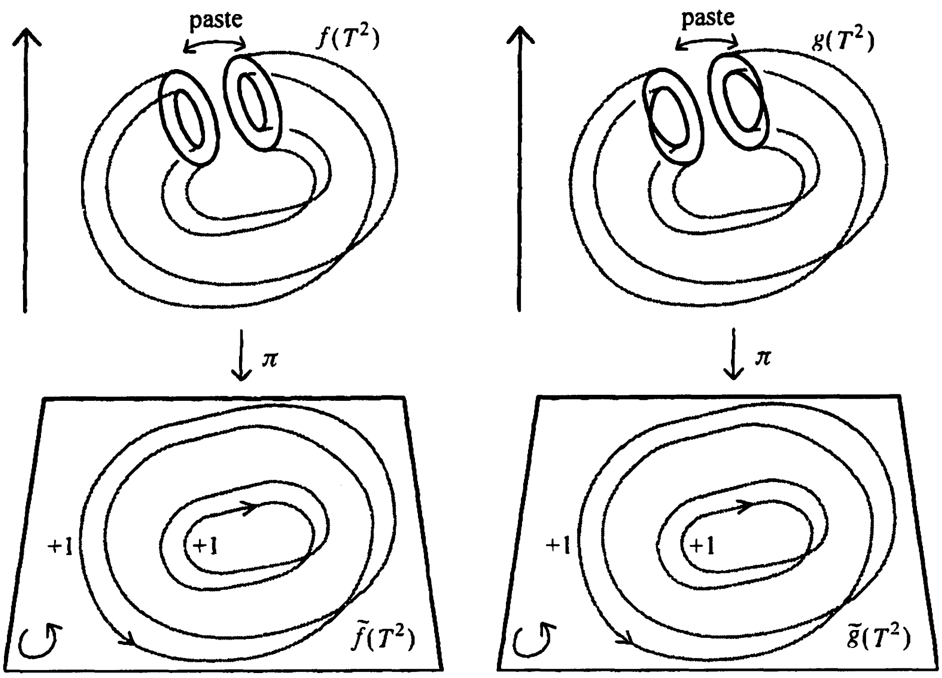

FIGURE 3. Two stable maps $\tilde{f}$ and $\tilde{g} : T^2 \to \mathbb{R}^2$ which satisfy that $\tilde{f}(S(\tilde{f})) = \tilde{g}(S(\tilde{g}))$ and both apparent contours have positive signs. But their immersion lifts f and $g : T^2 \to \mathbb{R}^3$ are not regularly homotopic.

convention as depicted in Figures 4 *and* 5*, then* $\tilde{F}$ *has a regular homotopy lift* $F : M \times [0,1] \to \mathbb{R}^3$.

As an application of Theorem 5.1, we have the following example.

Example 5.2. If $\tilde{f}$ and $\tilde{g} : S^2 \to \mathbb{R}^2$ are signed stable maps such that $\tilde{f}(S^2) = \tilde{g}(S^2) = D^2$, $\tilde{f}(S(\tilde{f})) = \tilde{g}(S(\tilde{g}))$ is a simple closed curve and the sign of $S(\tilde{f})$ (resp. $S(\tilde{g})$) is $+1$ (resp. -1). Then there is a generic homotopy $\tilde{F} : S^2 \times [0,1] \to \mathbb{R}^2$ between $\tilde{f}$ and $\tilde{g}$ which has a regular homotopy lift $F : S^2 \times [0,1] \to \mathbb{R}^3$. See Figure 6. By the definitions of $\tilde{f}$, $\tilde{g}$, the regular homotopy lift F over $\tilde{F}$ corresponds to an eversion of the embedded sphere.

Our eversion in Example 5.2 is almost same as the eversion given by Francis [1]. But in his picture, self intersections of immersed spheres were not drawn. Professor Mikami Hirasawa and the author draw a regular homotopy over the generic homotopy of Figure 6, precisely. So, we can follow how self intersections move during our sphere eversion. Our eversion will appear in their preparing paper.

REFERENCES

[1] G. K. Francis, A topological picturebook, Springer, New York, 2007.

[2] M. Golubitsky and V. Guillemin, Stable mappings and their singularities, Graduate Texts in Mathematics, Vol. 14, Springer-Verlag, New York, Heidelberg, 1973.

[3] A. Haefliger, *Quelques remarques sur les applications differentiables d'une surface dans le plan*, Ann. Inst. Fourier Grenoble **10** (1960), 47–60.

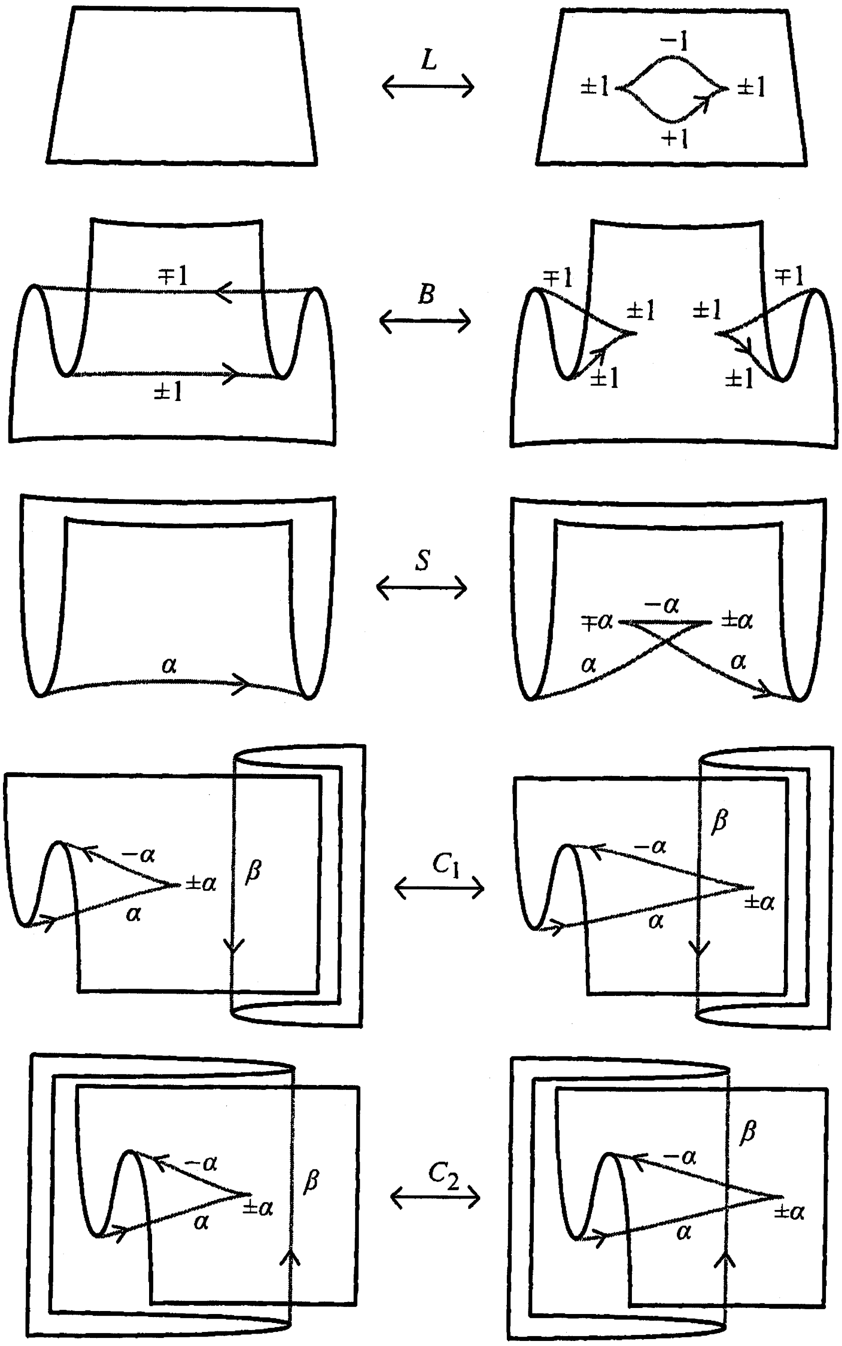

FIGURE 4. Bifurcations of type L, B, S and C which have regular homotopy lifts. Here, $\alpha = \pm 1$ and $\beta = \pm 1$ and α and β vary independently.

[4] I. James and E. Thomas, *Note on the classification of cross-sections*, Topology **4** (1966) 351–359.

[5] T. Ohmoto and F. Aicardi, *First order local invariants of apparent contours*, Topology **45** (2006), 27–45.

[6] U. Pinkall, *Regular homotopy classes of immersed surfaces*, Topology **24** (1985), 421–434.

[7] S. Smale, *A classification of immersions of the two-sphere*, Trans. Amer, Math. Soc. **90** (1958), 281–290.

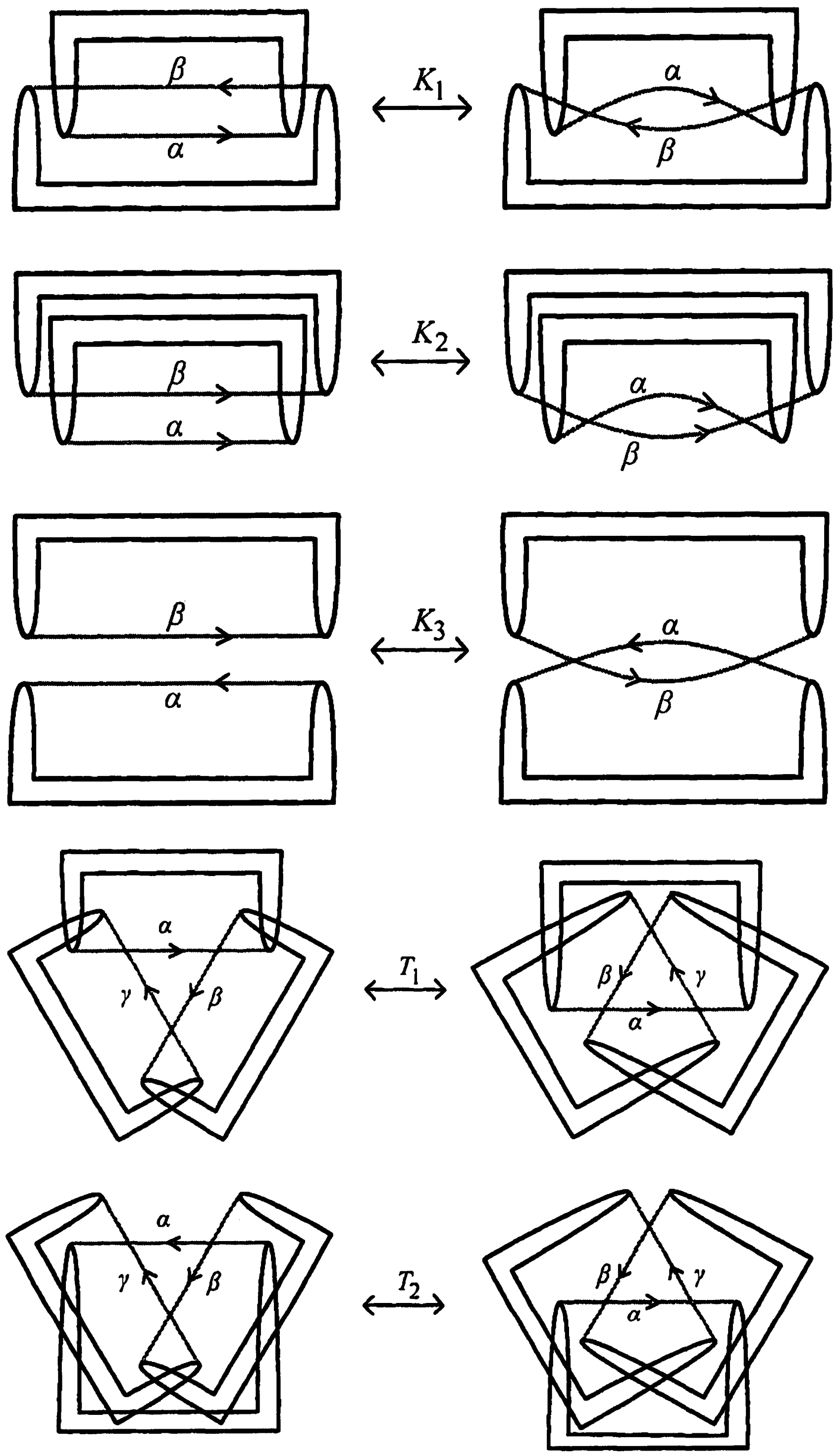

FIGURE 5. Bifurcations of type K and T which have regular homotopy lifts. Here, $\alpha = \pm 1, \beta = \pm 1$ and $\gamma = \pm 1$ and α, β and γ vary independently.

[8] H. Whitney, *On singularities of mappings of euclidean spaces.* I. *Mappings of the plane into the plane*, Ann. of Math. (2) **62** (1955), 374–410.

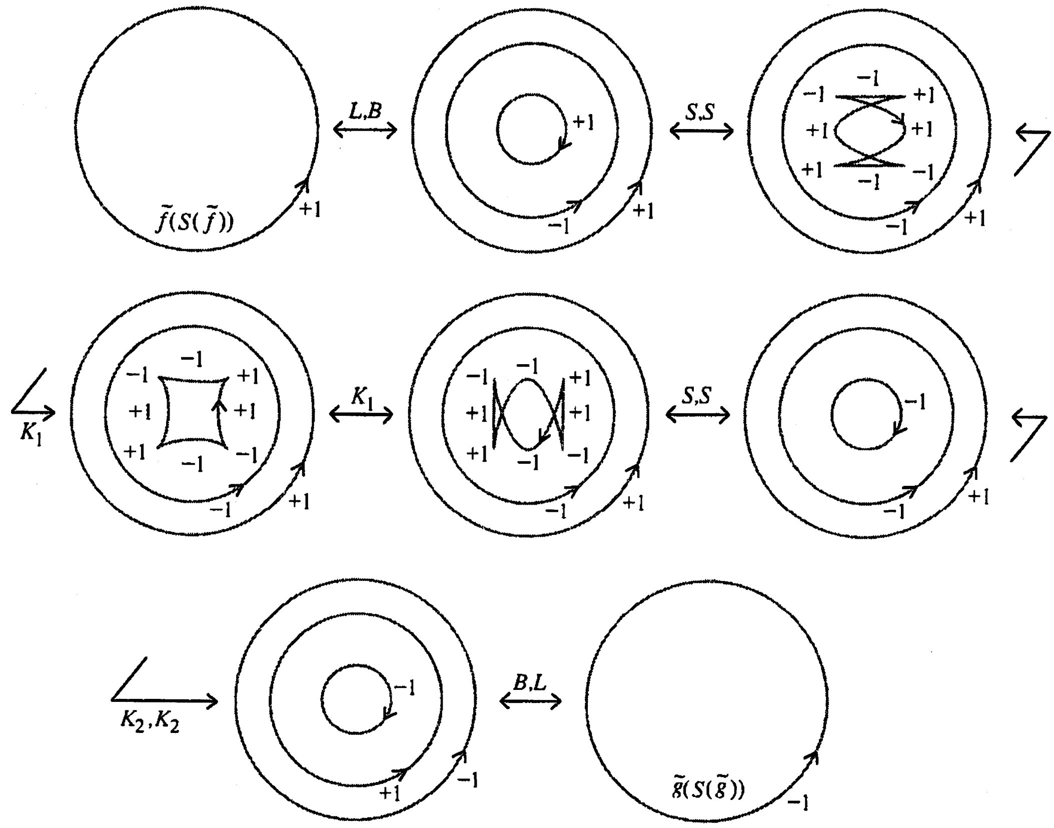

FIGURE 6. A sequence of apparent contours of a generic homotopy between $\tilde{f}$ and $\tilde{g} : S^2 \to \mathbb{R}^2$ which has a regular homotopy lift. This regular homotopy corresponds to a sphere eversion.