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QUANDLE COCYCLE INVARIANTS OF ROLL-SPUN KNOTS

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ABSTRACT. We generalize the class of roll-spun knots in 2-knot theory and study the quandle colorings for such a 2-knot. We also explain how to calculate the quandle cocycle invariant and prove that the invariant of any roll-spun knot is trivial if the second homology group of the quandle vanishes.

1. INTRODUCTION

For an oriented surface-knot F and a third cohomology class $\theta \in H^3(X;A)$ of a quandle X, the calculation of the quandle cocycle invariant of F with respect to θ is given as follows:

$$C \in \mathrm{Col}_X(F) \rightsquigarrow \gamma(C) \in H_3(X) \rightsquigarrow \Phi_X(F) \rightsquigarrow \Phi_\theta(F).$$

More precisely, each X-coloring C for F defines a third homology class $\gamma(C) \in H_3(X)$ by taking the sum of weights on triple points of a diagram, and such classes form the multi-set

$$\Phi_X(F) = \{\gamma(C) \in H_3(X) \mid C \in \mathrm{Col}_X(F)\}.$$

Under the Kronecker product $\langle\ ,\ \rangle : H_3(X) \otimes H^3(X;A) \to A$, the cocycle invariant $\Phi_\theta(F)$ is the evaluation of $\Phi_X(F)$ by $[\theta]$;

$$\Phi_\theta(F) = \{\langle \gamma(C), \theta\rangle \in A \mid C \in \mathrm{Col}_X(F)\}.$$

The deform-spun knot [8] is a 2-knot obtained from a tangle of a 1-knot with its motion. The spinning process is originally introduced by Artin [1], and generalized to twist-spinning by Fox [5] and Zeeman [11]. The quandle cocycle invariant of a twist-spun knot is calculated in some cases (cf. [2, 3, 6, 7]).

In this note, we introduce a 2-knot $F(K, K')$ associated with a tangle diagram K and a 1-knot diagram K'. In particular, $F(K, K')$ is a roll-spun knot in the special case. Under some condition for a quandle X, we prove that there is a one-to-one correspondence between $\mathrm{Col}_X(F(K, K'))$ and $\mathrm{Col}_X(K)$; that is, each X-coloring C for K can be extended to that for $F(K, K')$ naturally which is denoted by $\overline{C}$.

On the other hand, we define a normal subgroup $G_0(X)$ of the adjoint group $G(X)$ and a shifting map $S_2^w : H_2(X) \to H_3(X)$ for every element $w \in G_0(X)$ so that we have

$$\gamma(\overline{C}) = S_2^{w(C)}(\gamma(C))$$

for any $C \in \mathrm{Col}_X(K)$, where $w(C) \in G_0(X)$ and $\gamma(C) \in H_2(X)$ are the element of $G_0(X)$ and the second homology class associated with C. This implies that $\Phi_X(F(K, K'))$ is calculated in terms of $\mathrm{Col}_X(K)$, and so is $\Phi_\theta(F(K, K'))$. As an application, we give a sufficient condition for $\Phi_X(F(K, K'))$ to be trivial.

2. Definition of $F(K, K')$

Let K be an oriented tangle diagram and K' an oriented knot diagram. We assume that K' is located on a 2-sphere S^2 embedded in $\mathbb{R}^3$. We replace a tubular neighborhood of K' in S^2 with a product $K \times S^1$, where the modification near a crossing of K' is illustrated in Fiugre 1. This modification is realized by the connected sum of two copies of K as cross-sections such that one of K's passes through the other K.

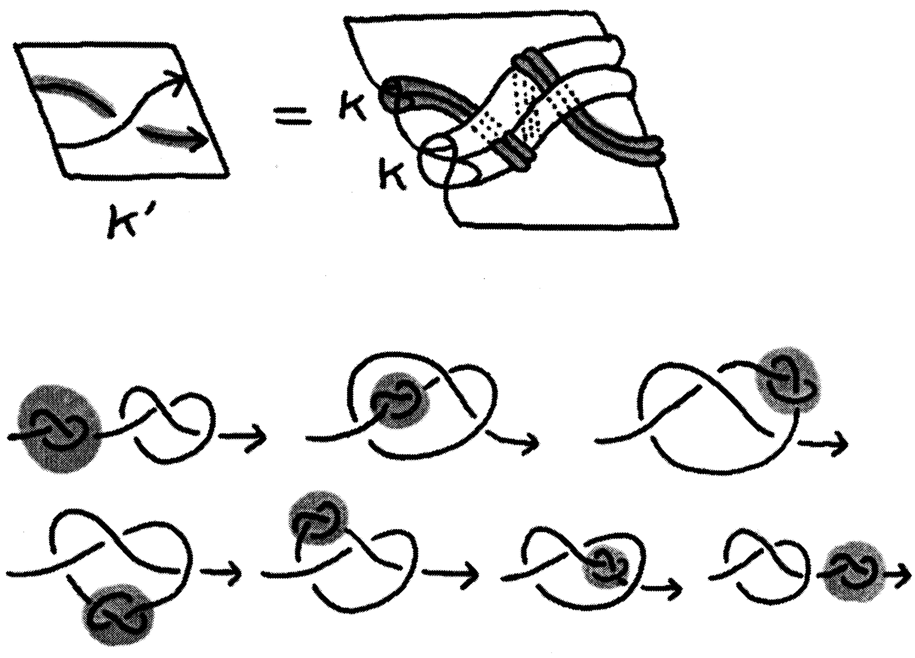

Figure 1

We denote by $F(K, K')$ the 2-knot presented by this diagram. Let $\omega(K)$ and $\omega(K')$ denote the writhes of K and K', respectively. Then we have the following.

Lemma 2.1. *If $\omega(K_0) = \omega(K_1)$ and $\omega(K'_0) = \omega(K'_1)$, then $F(K_0, K'_0) \cong F(K_1, K'_1)$.*

Proposition 2.2. *$F(K,K')$ is a deform-spun knot.*

We denote by $\tau^r\rho^s K$ the r-twist-s-roll-spinning of K (cf. [8]).

Theorem 2.3. *If K' is a diagram of the trivial knot, then $F(K,K') \cong \tau^{-rs}\rho^{-s}K$, where $r = \omega(K)$ and $s = \omega(K')$. In particular, if K is a tangle diagram with $\omega(K) = 0$, then $F(K,K') \cong \rho^{-s}K$.*

3. Definition of $G_0(X)$

For a quandle X and an element $x \in X$, we denote by $\varphi_x : X \to X$ the right action by x; that is, $\varphi_x(a) = a * x$. The axiom of the right distribution induces the equality

$$\varphi_y \circ \varphi_x = \varphi_{x*y} \circ \varphi_y$$

for any $x, y \in X$. Let $W(X)$ denote the set of words on X. For a word $w = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n} \in W(X)$, we define a quandle isomorphism $\varphi_w : X \to X$ to be

$$\varphi_w(a) = \varphi_{x_n}^{\varepsilon_n} \circ \cdots \circ \varphi_{x_1}^{\varepsilon_1}(a).$$

We also use the notation $\varphi_w(a) = a * w$. We remark that w in the definition of φ_w can be regarded as an element of the adjoint group

$$G(X) = \langle x \in X \mid a * b = b^{-1}ab \ (a, b \in X) \rangle.$$

The index of an element $w = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n} \in G(X)$ is defined by $\mathrm{ind}(w) = \varepsilon_1 + \cdots + \varepsilon_n$.

Definition 3.1. $G_0(X) = \{w \in G(X) \mid \varphi_w = \mathrm{id}_X \text{ and } \mathrm{ind}(w) = 0\}$.

We remark that $G_0(X)$ is a normal subgroup of $G(X)$.

Lemma 3.2. (i) *$G_0(R_p) = \{0\}$, where $R_p = \mathbb{Z}[t, t^{-1}]/(p, t+1)$ for odd prime p.*
(ii) *$G_0(S_4) = \mathbb{Z}_2$, where $S_4 = \mathbb{Z}[t, t^{-1}]/(2, t^2 + t + 1)$.*

Example 3.3. We consider the case $X = S_4 = \mathbb{Z}[t, t^{-1}]/(2, t^2 + t + 1)$. The element $w = 1 \cdot t^{-1} \cdot 0 \cdot (t+1)^{-1}$ satisfies $\varphi_w = \mathrm{id}_{S_4}$; in fact, we have

$$\begin{array}{lllll}
0 & \xmapsto{\varphi_1} t+1 & \xmapsto{\varphi_t^{-1}} 1 & \xmapsto{\varphi_0} t & \xmapsto{\varphi_{t+1}^{-1}} 0 \\
1 & \longmapsto 1 & \longmapsto 0 & \longmapsto 0 & \longmapsto 1 \\
t & \longmapsto 0 & \longmapsto t+1 & \longmapsto 1 & \longmapsto t \\
t+1 & \longmapsto t & \longmapsto t & \longmapsto t+1 & \longmapsto t+1.
\end{array}$$

Since $\mathrm{ind}(w) = 0$, it holds that $w \in G_0(S_4)$. Figure 2 shows that $w^2 = 1$ in $G_0(S_4)$. Moreover, we see that w is the generator of $G_0(S_4) \cong \mathbb{Z}_2$.

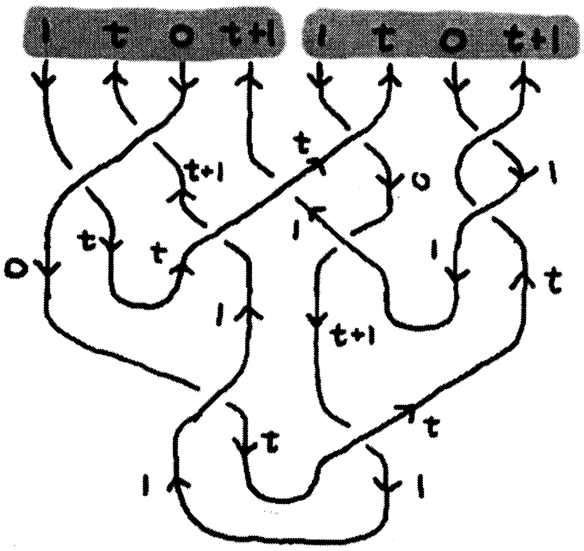

FIGURE 2

4. DEFINITION OF $S_n^w : H_n(X) \to H_{n+1}(X)$

Let $C_n(X)$ denote the quandle n-chain group which is the free abelian group generated by the n-tuples $(a_1, \ldots, a_n) \in X^n$ with $a_i \neq a_{i+1}$ for any $1 \leq i \leq n-1$.

Definition 4.1. For an n-chain $\gamma = \sum \pm(a_1, \ldots, a_n) \in C_n(X)$, a word $w \in W(X)$, and an element $x \in X$, we denote by

$$(\gamma * w, x) = \sum \pm(a_1 * w, \ldots, a_n * w, x).$$

Definition 4.2. For a word $w = x_1^{\varepsilon_1} \ldots x_k^{\varepsilon_k} \in W(X)$, we define words $w(i)$ $(1 \leq i \leq k)$ by

$$w(i) = \begin{cases} x_1^{\varepsilon_1} \ldots x_{i-1}^{\varepsilon_{i-1}} & (\varepsilon_i = +1) \\ x_1^{\varepsilon_1} \ldots x_{i-1}^{\varepsilon_{i-1}} x_i^{\varepsilon_i} & (\varepsilon_i = -1) \end{cases}$$

Definition 4.3. For a word $w = x_1^{\varepsilon_1} \ldots x_k^{\varepsilon_k} \in W(X)$, the shifting map $S_n^w : C_n(X) \to C_{n+1}(X)$ is defined by

$$S_n^w(\gamma) = \sum_{i=1}^{k} \varepsilon_i(\gamma * w(i), x_i).$$

Lemma 4.4. *If $w \in G_0(X)$, then S_n^w induces a shifting map $H_n(X) \to H_{n+1}(X)$.*

Example 4.5. For the element $w = 1 \cdot t^{-1} \cdot 0 \cdot (t+1)^{-1} \in G_0(S_4)$, the shifting map $S_n^w : H_n(X) \to H_{n+1}(X)$ is given by

$$S_n^w(\gamma) = +(\gamma, 1) - (\gamma * (1 \cdot t^{-1}), t) + (\gamma * (1 \cdot t^{-1}), 0) - (\gamma, 1 + t).$$

See Figure 3.

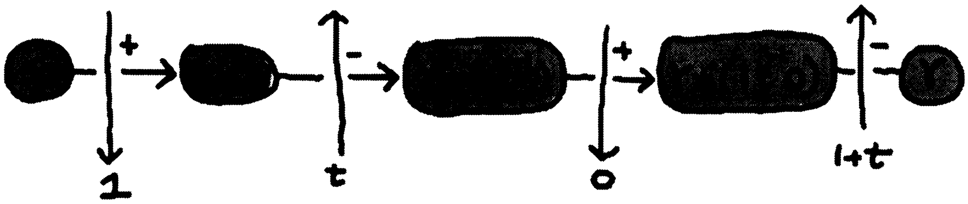

FIGURE 3

5. $\mathrm{Col}_X(F(K,K'))$ AND $\mathrm{Col}_X(K)$

Let K be a tangle diagram with k crossings and $\omega(K) = 0$, and $C \in \mathrm{Col}_X(K)$ an X-coloring for K. Let ε_i and x_i $(1 \le i \le k)$ be the sign and the color of the upper arc at ith lower crossing along K, respectively. The element of $G(X)$ associated with C is given by

$$w(C) = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_k^{\varepsilon_k}.$$

Example 5.1. We consider the S_4-coloring C for the tangle diagram of the figure-eight knot as shown in Figure 4. Then the element associated with C is given by $w(C) = 1 \cdot t^{-1} \cdot 0 \cdot (t+1)^{-1}$.

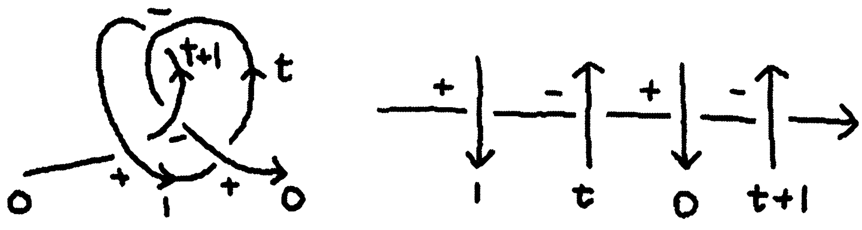

FIGURE 4

Lemma 5.2. *Let a and a' be the colors assigned to the initial and terminal arcs of K, respectively. Then it holds that $\varphi_w(a) = a'$.*

We consider the following condition (#) for a quandle X;

(#) For any tangle diagram K with $\omega(K) = 0$ and any X-coloring C for K, it holds that $w(C) \in G_0(X)$.

We remark that since $\omega(K) = 0$, we have $\mathrm{ind}(w(C)) = 0$. Therefore, the condition $(\#)$ is equivalent to $\varphi_{w(C)} = \mathrm{id}_X$.

Proposition 5.3. *Any Alexander quandle satisfies the condition* $(\#)$.

Theorem 5.4. *Suppose that a quandle X satisfies the condition $(\#)$. If K is a tangle diagram with $\omega(K) = 0$, then there is a one-to-one correspondence between* $\mathrm{Col}_X(F(K, K'))$ *and* $\mathrm{Col}_X(K)$.

6. Computation of $\Phi_X(F(K, K'))$

We consider the connected sum of two copies of a tangle diagram K colored by $C \in \mathrm{Col}_X(K)$. Let $\gamma \in H_3(X)$ be the class associated with the motion where the small tangle passes through the big one as shown in Figure 5. We divide γ into γ_+ and $\gamma_- \in H_3(X)$ corresponding to the motions where the small tangle passes over and under the transverse arc, respectively.

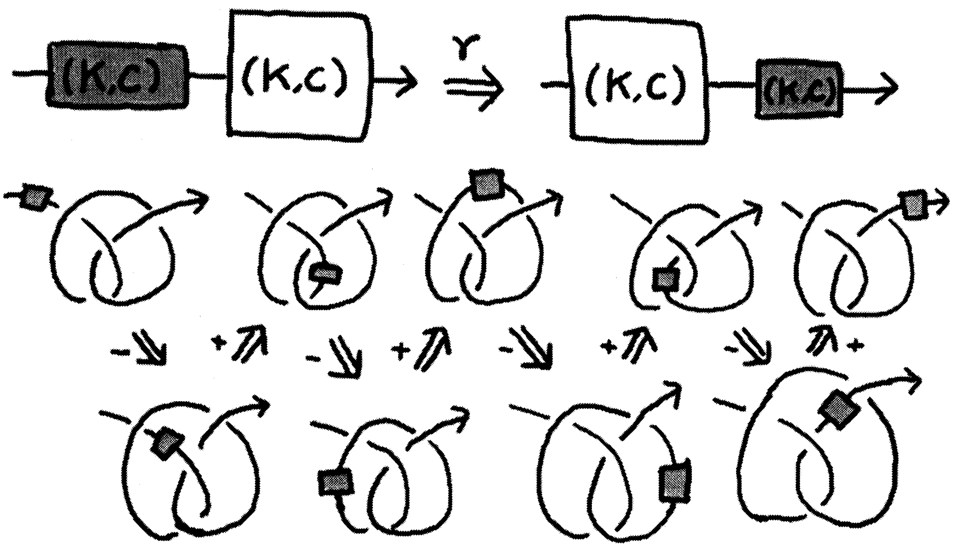

FIGURE 5

The third homology class γ_+ is the sum of weights on the triple points as shown in Figure 6 which is equivalent to the shadow cocycle invariant of K.

Lemma 6.1. $\gamma_+ = 0$.

On the other hand, the third homology class γ_- is the sum of triple points as shown in Figure 7. Let $\gamma(C) \in H_2(X)$ denote the class associated with the X-coloring C for K. Then we have the following.

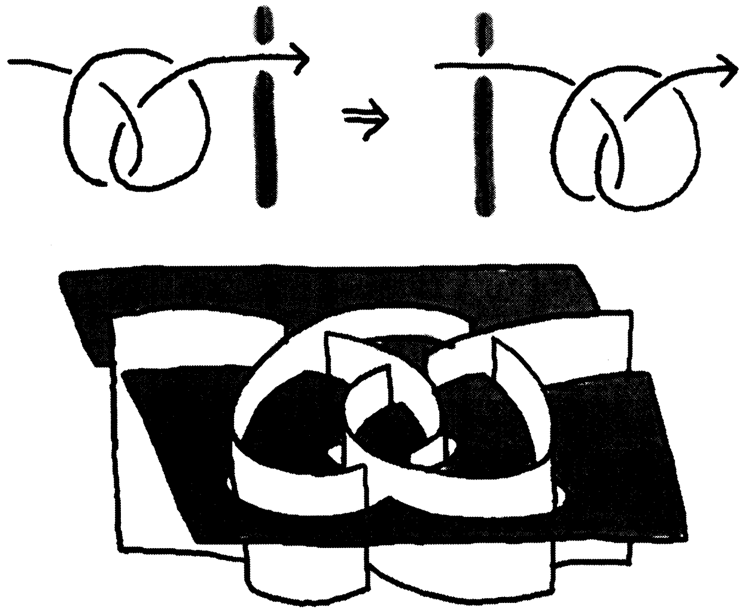

FIGURE 6

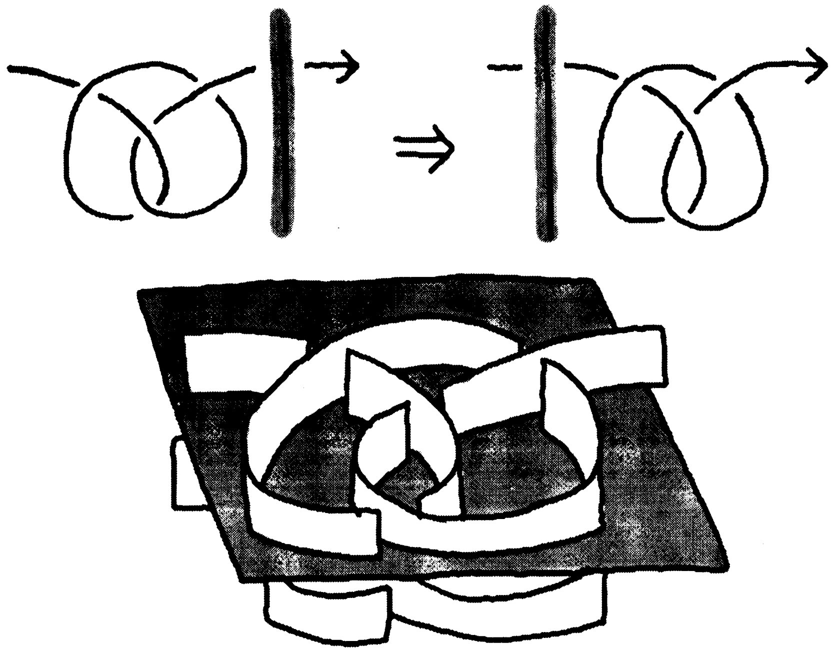

FIGURE 7

Lemma 6.2. $\gamma_- = S_2^{w(C)}(\gamma(C))$.

Theorem 6.3. *Suppose that a quandle X satisfies the condition (#). If K is a tangle diagram with $\omega(K) = 0$, then it holds that*

$$\Phi_X(F(K, K')) = \{-\omega(K') \cdot S_2^{w(C)}(\gamma(C)) \mid C \in \mathrm{Col}_X(K)\}$$

Corollary 6.4. *Suppose that a quandle X satisfies the condition (#) with $G_0(X) = 0$ or $H_2(X) = 0$. If K is a tangle diagram with $\omega(K) = 0$, then $\Phi_X(F(K, K'))$ is trivial.*

After the conference, Nosaka pointed out that $G_0(X)$ and $H_2(X)$ are isomorphic for any Alexander quandle (cf. [4]). Therefore, the conditions $G_0(X) = 0$ and $H_2(X) = 0$ are equivalent if X is an Alexander quandle.

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