Discrete series for symmetric spaces over $p$-adic fields

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0 Introduction

Let $G$ be a connected reductive group over a non-archimedean local field $F$ equipped with an $F$-involution $\sigma : G \to G$, and $H$ the subgroup of all $\sigma$-fixed points of $G$. The quotient space $G/H$ of $F$-points $G = G(F)$ by $H = H(F)$ is called a symmetric space over $F$. We are interested in representations of $G$ which can be realized in the space of functions on $G/H$. Such representations are said to be $H$-distinguished. We are concerned especially with representations which can be realized in the space $L^2(G/Z_GH)$ (in rough notation) of square integrable functions on $G/Z_GH$, where $Z_G$ denotes the center of $G$. An irreducible representation having such a realization is said to be in the discrete series for $G/H$.

We give a criterion for such realizability, i.e., square integrability on $G/H$, in terms of exponents of Jacquet modules. The result has already appeared in [KT2]. In this report we give a brief survey of the main result of [KT2] (and also a part of our preceding work [KT1]). It is a symmetric space analogue of Casselman's well-known criterion for the group case ([C], which we recall below in Section 1). In our symmetric space analogue, we only consider Jacquet modules along $\sigma$-split parabolics, and consider exponents on $(\sigma, F)$-split components (see Section 3). The set of relative exponents used in our criterion is given in 5.1, and the main theorem is stated in 5.2. Several examples are included in the final section.
1 Casselmann's criterion for the usual square integrability

Let $F$ be a non-archimedean local field with the valuation ring $\mathcal{O}$ and the absolute value $|\cdot|_F$. Let $G$ be a connected reductive group defined over $F$ and $Z$ the $F$-split component of $G$, that is, the largest $F$-split torus in the center of $G$. Let us write $G = G(F), Z = Z(F)$.

Let $(\pi, V)$ be a smooth representation of $G$. Suppose that $\pi|Z$ is a unitary character of $Z$. Let $(\pi, \overline{V})$ denote the contragredient of $(\pi, V)$. The usual matrix coefficients of $\pi$ are functions on $G$ of the form

$$c_{v, \overline{v}}(g) = \overline{\langle v, \pi(g^{-1})v \rangle} \quad (g \in G)$$

for $v \in V$ and $\overline{v} \in \overline{V}$. The representation $\pi$ is said to be square integrable (in the usual sense) if

$$\int_{G/Z} |c_{v, \overline{v}}(g)|^2 dg < \infty$$

for all $v \in V$ and $\overline{v} \in \overline{V}$.

We briefly say that $P$ is a parabolic subgroup of $G$ if $P$ is the group of $F$-points of a parabolic $F$-subgroup $P = P$ of $G$ etc, by abuse of terminology.

For a smooth representation $(\pi, V)$ of $G$ and a parabolic subgroup $P = MU$, let $(\pi_P, V_P)$ denote the normalized Jacquet module of $(\pi, V)$ along $P$: The space $V_P$ is the quotient of $V$ by the $M$-stable subspace

$$V(U) = \langle \{\pi(u)v - v \mid u \in U, v \in V\} \rangle_{\mathbb{C}}.$$

Let $j_P : V \to V/V(U) = V_P$ be the canonical projection. Then $M$ acts on $V_P$ by

$$\pi_P(m)j_P(v) = \delta_P^{-1/2}(m)j_P(\pi(m)v)$$

where $\delta_P$ is the modulus of $P$. It is known ([C, §3]) that if $\pi$ is finitely generated (resp. admissible), then so is the $M$-module $\pi_P$.

Let $\mathcal{X}(A)$ be the set of all quasi-characters of the $F$-split component $A$ of the Levi subgroup $M$. For a $\chi \in \mathcal{X}(A)$, consider the generalized $\chi$-eigenspace

$$(V_P)_{\chi, \infty} = \left\{ \overline{v} \in V_P \mid \text{There exists a } d \in \mathbb{N} \text{ such that } (\pi_P(a) - \chi(a))^d \cdot \overline{v} = 0 \text{ for all } a \in A \right\}.$$
A quasi-character $\chi \in \mathcal{X}(A)$ is called an exponent of $\pi_P$ if $(V_P)_{\chi,\infty} \neq \{0\}$. The set of all exponents of $\pi_P$ is denoted by $E_A(\pi_P)$.

The set $E_A(\pi_P)$ is finite if $\pi$ is finitely generated and admissible. One has a direct sum decomposition

$$V_P = \bigoplus_{\chi \in E_A(\pi_P)} (V_P)_{\chi,\infty}.$$ 

Set

$$A^- = \{a \in A \mid |a^\alpha|_F \leq 1 \text{ for all simple roots } \alpha \}.$$

Let us take up the following condition imposed on $P$:

$$(b_P) \quad |\chi(a)| < 1 \text{ for all } \chi \in E_A(\pi_P) \text{ and all } a \in A^- \setminus ZA(\mathcal{O}_F).$$

Now the well-known Casselman’s criterion is stated as follows:

**Theorem 1.1** (Casselman [C, 4.4.6]) A finitely generated admissible representation $\pi$ of $G$ is square integrable if and only if the condition $(b_P)$ is satisfied for every parabolic subgroup $P$ of $G$.

## 2 H-square integrable representations

From now on we assume that the residual characteristic of $F$ is not equal to 2. Let $\sigma : G \to G$ be an $F$-involution on $G$. We put

$$H = \{h \in G \mid \sigma(h) = h\}, \quad Z_0 = \{z \in Z \mid \sigma(z) = z^{-1}\}^0,$$

We call $Z_0$ the $(\sigma, F)$-split component of $G$. We write $G = G(F)$, $H = H(F)$ etc.

A smooth representation $(\pi, V)$ of $G$ is said to be $H$-distinguished if the space $\text{Hom}_H(\pi, \mathbb{C}) = (V^*)^H$ of $H$-invariant linear forms on $V$ is nonzero. For a while, suppose that $\pi|_{Z_0}$ is a unitary character of $Z_0$, say, $\omega_0 : Z_0 \to \mathbb{C}_1$. Let $(\pi, V)$ be $H$-distinguished and take a non-zero $H$-invariant linear forms $\lambda \in (V^*)^H$. We consider functions $\varphi_{\lambda,v}$ on $G/H$ for $v \in V$ defined by

$$\varphi_{\lambda,v}(g) = \langle \lambda, \pi(g^{-1})v \rangle \quad (g \in G).$$

Such functions are called $(H, \lambda)$-matrix coefficients of $\pi$. Note that these are not the matrix coefficients in the usual sense, but are generalized matrix coefficients, since
$H$-invariant linear forms are not smooth in general. We have an obvious equivariance for $(H, \lambda)$-matrix coefficients:

$$\varphi_{\lambda, v}(z_0 gh) = \omega_0(z_0)^{-1} \varphi_{\lambda, v}(g) \quad \forall z_0 \in Z_0, \ g \in G, \ h \in H.$$ 

We also have $\varphi_{\lambda, \pi(g)v} = L(g)\varphi_{\lambda, v}$ (where $L(\cdot)\varphi$ denotes the left translation). Thus, for a fixed $\lambda \in (V^*)^H$, the set of functions $\{\varphi_{\lambda, v} \mid v \in V\}$ gives a realization of $\pi$ in the space of functions on $G/H$.

Since $H$ is reductive, $H$ is unimodular. So the quotient space $G/Z_0H$ carries a unique (up to constant) left $G$-invariant measure, denoted by $\int_{G/Z_0H} \ldots dg$.

**Definition 2.1** We say that $\pi$ is $(H, \lambda)$-square integrable if $|\varphi_{\lambda, v}(\cdot)|$ is square integrable on $G/Z_0H$ for all $v \in V$, namely, if

$$\int_{G/Z_0H} |\varphi_{\lambda, v}(g)|^2 dg < \infty$$

for all $v \in V$.

**Remark 2.2** In our preceding work [KT1], we have defined that $(\pi, V)$ is $(H, \lambda)$-relatively cuspidal if the support of $\varphi_{\lambda, v}$ is compact modulo $Z_0H$ for all $v \in V$. So, by definition, $(H, \lambda)$-relatively cuspidal representations are $(H, \lambda)$-square integrable provided that $\omega_0$ is unitary.

### 3 Tori and parabolics associated to $\sigma$

We recollect some notation and terminology for tori and parabolic subgroups associated to the involution $\sigma$. Basic reference is [HH].

**Definition 3.1** (i) A parabolic $F$-subgroup $P$ of $G$ is said to be $\sigma$-split if $P$ and $\sigma(P)$ are opposite, i.e., if $P \cap \sigma(P)$ is a ($\sigma$-stable) Levi subgroup of $P$.

(ii) An $F$-split torus $S$ is said to be $(\sigma, F)$-split if $\sigma(s) = s^{-1}$ hold for all $s \in S$.

Let us fix a maximal $(\sigma, F)$-split torus $S_0$ of $G$ and take a maximal $F$-split torus $A_\emptyset$ containing $S_0$. Then $A_\emptyset$ is necessarily $\sigma$-stable, so $\sigma$ acts naturally on $X^*(A_\emptyset)$. Let $\Phi \subset X^*(A_\emptyset)$ be the root system of $(G, A_\emptyset)$. It is $\sigma$-stable. We can choose a
\(\sigma\)-basis \(\Delta\) of \(\Phi\) that has the property
\[
\alpha > 0, \sigma(\alpha) \neq \alpha \Rightarrow \sigma(\alpha) < 0
\]
in the corresponding order. The subset of all \(\sigma\)-fixed roots in \(\Delta\) is denoted by \(\Delta_{\sigma}\).

Let \(P_{0}\) be the minimal parabolic subgroup corresponding to the choice of \(\Delta\). Standard parabolic subgroups \(P_{I} = M_{I} U_{I}\) (i.e., those containing \(P_{0}\)) correspond to subsets \(I\) of \(\Delta\) as usual. We can decide exactly when \(P_{I}\) is \(\sigma\)-split.

**Lemma 3.2** ([HH, 2.6], [KT1, 2.5]) (i) \(P_{I}\) is \(\sigma\)-split if and only if \(I \supset \Delta_{\sigma}\) and the subsystem \(\Phi_{I}\) of \(\Phi\) generated by \(I\) is \(\sigma\)-stable. (In such a case we call \(I\) a \(\sigma\)-split subset.)

(ii) Any \(\sigma\)-split parabolic subgroup of \(G\) is written in the form \(\gamma^{-1} P_{I} \gamma\) for some \(\sigma\)-split subset \(I \subset \Delta\) and \(\gamma \in (M_{0} H)(F)\), where \(M_{0} = Z_{G}(S_{0})\) denotes the centralizer of \(S_{0}\) in \(G\).

Therefore, a minimal \(\sigma\)-split parabolic subgroup \(P_{0}\) of \(G\) can be given as the one corresponding to the minimal \(\sigma\)-split subset \(I = \Delta_{\sigma}\). Alternatively it is given by \(P_{0} = P_{\emptyset} M_{0}\). Note also that \(M_{0}\) is the \(\sigma\)-stable Levi subgroup of \(P_{0}\).

For a subset \(I \subset \Delta\), let \(A_{I}\) be the \(F\)-split component of \(M_{I}\). If \(I\) is a \(\sigma\)-split subset, let \(S_{I}\) denote the \(\sigma\)-split part of \(A_{I}\), i.e., the identity component of \(A_{I} \cap S_{0}\). We call \(S_{I}\) the \((\sigma, F)\)-split component of \(P_{I}\). For a positive real number \(\epsilon \leq 1\), we put
\[
S_{I}^{-}(\epsilon) = \{ s \in S_{I} = S_{I}(F) \mid |s^{\alpha}|_{F} \leq \epsilon (\alpha \in \Delta \setminus I) \}
\]
and
\[
i S_{0}^{-}(\epsilon) = \{ s \in S_{0} = S_{0}(F) \mid |s^{\alpha}|_{F} \leq \epsilon (\alpha \in \Delta \setminus I), \epsilon < |s^{\alpha}|_{F} \leq 1 (\alpha \in I) \}.\]

We abbreviate \(S_{I}^{-} = S_{I}^{-}(1)\) and \(S_{0}^{-} = S_{\Delta_{\sigma}}^{-}(1)\). We note that if \(\alpha \in \Delta_{\sigma}\) and \(s \in S_{0}\), then \(s^{\alpha} = s^{\sigma(\alpha)} = (s^{-1})^{\alpha}\), so that \(|s^{\alpha}|_{F} = 1\).

**Lemma 3.3** ([KT2, Lemma 1.6]) For any \(\epsilon < 1\), one has
\[
S_{0}^{-} = \bigcup_{I \subset \Delta \setminus \sigma\text{-split}} i S_{0}^{-}(\epsilon) \ (\text{disjoint}).
\]

It will turn out that the behaviors of \(H\)-matrix coefficients are determined essentially on \(S_{0}^{-}\), and furthermore, on the subset \(i S_{0}^{-}(\epsilon)\), they are connected to \(M_{I} \cap H\)-matrix coefficients of the Jacquet module along \(P_{I}\).
4 Asymptotic behaviors of $H$-matrix coefficients

Let $(\pi, V)$ be an admissible representation of $G$. Only when $P = MU$ is a $\sigma$-split parabolic subgroup, we have defined in [KT1] a linear mapping

$$r_P : (V^*)^H \to (V_P^*)^{M \cap H}$$

between the spaces of invariant linear forms. If $v \in V$ is a canonical lifting ([C, §4]) of $\overline{v} \in V_P$ with respect to a suitable $\sigma$-stable open compact subgroup, then $r_P(\lambda)$ for $\lambda \in (V^*)^H$ is well-defined by the relation

$$\langle r_P(\lambda), \overline{v} \rangle = \{\lambda, v\}$$

(see [KT1, 5.3(2)]). The same mapping was constructed independently by N. Lagier [L] in a different manner. P. Delorme extended the construction of such mappings to any smooth representations by using Bernstein’s second adjointness theorem in [D].

Now, through the mapping $r_P : (V^*)^H \to (V_P^*)^{M \cap H}$, the $H$-matrix coefficients of $\pi$ are related to the $M \cap H$-matrix coefficients of the Jacquet module $\pi_P$ as follows:

**Proposition 4.1** ([KT2, 3.3]) Let $I$ be a $\sigma$-split subset of $\Delta$ and $P = P_I$ the corresponding $\sigma$-split parabolic subgroup with the $(\sigma, F)$-split component $S = S_I$. Let $(\pi, V)$ be an $H$-distinguished admissible representation of $G$ and $v \in V$, $\lambda \in (V^*)^H$. There exists a positive real number $\epsilon \leq 1$ such that

$$\langle \lambda, \pi(s)v \rangle = \delta_P^{1/2}(s)\langle r_P(\lambda), \pi_P(s)j_P(v) \rangle$$

for all $s \in IS_0^-(\epsilon)$.

**Remark 4.2** In [KT1, 6.2], we have shown the following criterion for $(H, \lambda)$-relative cuspidality in terms of $r_P$: The representation $\pi$ is $(H, \lambda)$-relatively cuspidal if and only if $r_P(\lambda) = 0$ for every proper $\sigma$-split parabolic subgroup $P$.

5 Main theorem

Let $(\pi, V)$ be a finitely generated $H$-distinguished admissible representation of $G$, with a non-zero $H$-invariant linear form $\lambda \in (V^*)^H$. Let $P = MU$ be a $\sigma$-split
parabolic subgroup of $G$ with the $(\sigma, F)$-split component $S$. We let $\mathcal{X}(S)$ denote the set of all quasi-characters of $S$ and for a $\chi \in \mathcal{X}(S)$, we consider the generalized $\chi$-eigenspace $(V_P)_{\chi, \infty}$ as in section 1.

**Definition 5.1** A quasi-character $\chi \in \mathcal{X}(S)$ is called an exponent of $\pi_P$ relative to $r_P(\lambda)$ if the induced linear form $r_P(\lambda)$ on $V_P$ is non-zero on the generalized $\chi$-eigenspace $(V_P)_{\chi, \infty}$. The set of all such exponents is denoted by $\mathcal{E}_S(\pi_P; r_P(\lambda))$:

$$\mathcal{E}_S(\pi_P; r_P(\lambda)) = \{\chi \in \mathcal{X}(S) | r_P(\lambda)|_{(V_P)_{\chi, \infty}} \neq 0\}.$$

Now we consider the following condition imposed on $P$ and $\lambda$:

$$(\#_{P, \lambda}) \quad |\chi(s)| < 1 \text{ for all } \chi \in \mathcal{E}_S(\pi_P; r_P(\lambda)) \text{ and all } s \in S^{-} \setminus Z_0S(\mathcal{O}_F).$$

The main theorem of [KT2] is the following:

**Theorem 5.2** ([KT2, 4.7]) Let $(\pi, V)$ be a finitely generated $H$-distinguished admissible representation of $G$, with a non-zero $H$-invariant linear form $\lambda \in (V^*)^H$. Then, the representation $(\pi, V)$ is $(H, \lambda)$-square integrable if and only if the condition $(\#_{P, \lambda})$ is satisfied for every $\sigma$-split parabolic subgroup $P$ of $G$.

**Remark 5.3** By combining our criterion and Casselman's criterion, we have the following (possibly non-trivial) corollary: If $(\pi, V)$ is $H$-distinguished and is square integrable in the usual sense, then it is $(H, \lambda)$-square integrable for any $\lambda \in (V^*)^H$.

### 6 Ingredients of the proof

To evaluate the $L^2$-norm of the $(H, \lambda)$-matrix coefficients, we first decompose $G/Z_0H$ according to the analogue of Cartan decomposition. We fix a $\sigma$-stable open compact subgroup $K_0$ of $G$ which has Iwahori factorization with respect to each $\sigma$-split parabolic subgroup. This is not a maximal compact subgroup. [BO] and [DS] gave the following: There is a finite set $\Xi$ of $G$ and a finite set $\Gamma$ of $(M_0H)(F)$ such that

$$G = \bigcup_{\xi \in \Xi} \bigcup_{\gamma \in \Gamma} \xi K_0 s^{-1} \gamma H.$$
Choose $\epsilon$ so that Proposition 4.1 is valid for all $I$ and put

$$G_{I,\gamma} = \bigcup_{s \in I S_{0}^{-}(\epsilon)/Z_{0}S_{0}(O)} K_{0}s^{-1}\gamma H$$

for each $\sigma$-split subset $I \subset \Delta$ and $\gamma \in \Gamma$. Then we have

$$G/Z_{0}H = \bigcup_{\xi,\gamma,I} \xi G_{I,\gamma}/Z_{0}H$$

by Lemma 3.3. Now, the evaluation of the $L^{2}$-norm of $\varphi = \varphi_{\lambda,v}$ starts from

$$\int_{G/Z_{0}H} |\varphi(g)|^{2}dg \leq \sum_{\xi,\gamma,I} \left( \int_{\xi G_{I,\gamma}/Z_{0}H} |\varphi(g)|^{2}dg \right).$$

We may drop $\xi$ by changing vector suitably. To prove the if part of the main theorem, it is enough to show the following:

**Claim.** If the condition $(\#_{P,\lambda})$ is satisfied for $P = \gamma^{-1}P_{I}\gamma$ (see Lemma 3.2 (ii)), then

$$\int_{G_{I,\gamma}/Z_{0}H} |\varphi(g)|^{2}dg < \infty.$$  

The integral is bounded by the series

$$\sum_{s \in I S_{0}^{-}(\epsilon)/Z_{0}S_{0}(O)} \int_{K_{0}s^{-1}\gamma Z_{0}H/Z_{0}H} |\varphi(g)|^{2}dg$$

over the lattice $I S_{0}^{-}(\epsilon)/Z_{0}S_{0}(O)$. Each term can be evaluated as

$$\int_{K_{0}s^{-1}\gamma Z_{0}H/Z_{0}H} |\varphi(g)|^{2}dg \leq C \cdot \text{vol}(K_{0}s^{-1}\gamma Z_{0}H/Z_{0}H) \cdot |\varphi(s^{-1}\gamma)|^{2}$$

by a constant $C$. Look at the term where $\gamma = 1$ for simplicity. In the right hand side,

$$\varphi(s^{-1}) = \langle \lambda, \pi(s)v \rangle = \delta_{P}(s)^{1/2} \langle r_{P}(\lambda), \pi_{P}(s)j_{P}(v) \rangle$$

by Proposition 4.1. To proceed further, we need the following volume computation.

**Lemma 6.1 ([KT2, 2.6])** For each $\gamma \in \Gamma$, there exist positive real constants $c_{1}$ and $c_{2}$ such that

$$c_{1} \cdot \delta_{P_{0}}(s^{-1}) \leq \text{vol}(K_{0}s^{-1}\gamma Z_{0}H/Z_{0}H) \leq c_{2} \cdot \delta_{P_{0}}(s^{-1})$$

for all $s \in S_{0}^{-}$. 
From this lemma, the volume factor can be replaced by (a constant times) $\delta_{P_0}(s^{-1})$, so the problem reduces to the series

$$\sum_{s \in iS_0^{-}(e)/Z_0S_0(\mathcal{O})} |(r_{P}(\lambda), \pi_{P}(s)j_{P}(v))|^2,$$

whose convergence will follow from the condition $(\sharp_{P,\lambda})$.

7 Examples

7.1 The symmetric space $GL_2(E)/GL_2(F)$ where $E/F$ is quadratic

Let $E/F$ be a quadratic extension of non-archimedean local fields and $\omega_{E/F}$ be the unique non-trivial character of $F^\times$ trivial on the norm image $N_{E/F}(E^\times)$.

Let $G$ be the group $GL_2(E)$ and consider the involution $\sigma$ on $G$ defined by

$$\sigma(g) = (_{10}^{01})\overline{g}(_{10}^{01}),$$

where $(_{\cdot}^{\cdot})$ denotes the conjugation over $F$. Then the subgroup $H$ of $\sigma$-fixed points in $G$ is isomorphic to $GL_2(F)$. The standard parabolic subgroup $P = \{(_{0*}^{**}) \in G\}$ is $\sigma$-split, with the $\sigma$-stable Levi subgroup $T = \{(_{0*}^{*0}) \in G\}$. Any proper $\sigma$-split parabolic subgroup is $H$-conjugate to $P$.

Non-cuspidal irreducible $H$-distinguished representations of $G$ are completely determined in Hakim's thesis [H]:

1. The irreducible principal series $\text{Ind}(\chi_1, \chi_2)$, with $\chi_2 = \overline{\chi_1}^{-1}$.
2. The irreducible principal series $\text{Ind}(\chi_1, \chi_2)$, with $\chi_i|_{F^\times} \equiv 1$ and $\chi_1 \neq \chi_2$.
3. The spacial representation $\text{sp}(\chi_1, \chi_2)$, with $\chi_1\chi_2^{-1} = |.|_E$ and $\chi_1|.|_E^{-1/2} = \chi_2|.|_E^{1/2} = \omega_{E/F}$ on $F^\times$.

Here, for quasi-characters $\chi_1, \chi_2$ of $E^\times$, $\text{Ind}(\chi_1, \chi_2)$ stands for the normalized induction determined by the quasi-character $(\begin{smallmatrix} \sigma & 0 \\ 0 & d \end{smallmatrix}) \mapsto \chi_1(a)\chi_2(d)$ of $T$. In any case, it is known that the dimension of the space of $H$-invariant linear forms is one.

The representations in the second class are $(H, \lambda)$-relatively cuspidal (see Remark 2.2 for the definition, and Remark 4.2 for the criterion). Indeed, the Jacquet module along $P$ is the direct sum $(\chi_1, \chi_2) \oplus (\chi_2, \chi_1)$ of characters of $T$. If $\chi_i|_{F} \equiv 1$, then we have

$$\chi_1 = \overline{\chi_1}^{-1}, \quad \chi_2 = \overline{\chi_2}^{-1},$$
so \( \chi_2 \neq \overline{\chi_1}^{-1} \) provided that \( \chi_1 \neq \chi_2 \). Thus, neither \((\chi_1, \chi_2)\) nor \((\chi_2, \chi_1)\) cannot be trivial on \( T \cap H = \{(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}) \} \).

The representations in the third class provide examples of Remark 5.3.

### 7.2 The symmetric space \( GL_3(F)/(GL_2(F) \times GL_1(F)) \)

Let \( G \) be the group \( GL_3(F) \) and \( \sigma \) the inner involution \( \sigma = \text{Int}(\epsilon) \) defined by the anti-diagonal permutation matrix \( \epsilon = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \). Then the \( \sigma \)-fixed point subgroup \( H \) is isomorphic to \( GL_2(F) \times GL_1(F) \). For this symmetric space, all the irreducible \( H \)-distinguished representations were determined by D. Prasad [P].

Form the normalized induction

\[
\pi(\rho) = \text{Ind}^{G}_{P_{1,2}}(1_{GL_1(F)} \otimes \rho)
\]

from the standard parabolic \( P_{1,2} \) of type \((1, 2)\) and an irreducible representation \( \rho \) of \( GL_2(F) \). Then \( \pi(\rho) \) is irreducible, and is \( H \)-distinguished. The Borel subgroup \( P_0 \) consisting of upper triangular matrices is the only proper \( \sigma \)-split parabolic of \( G \) up to \( H \)-conjugacy. It is easy to determine exponents of \( \pi(\rho) \) along \( P_0 \). By using our criterion, we may conclude that \( \pi(\rho) \) belongs to the discrete series for \( G/H \) if \( \rho \) is the Steinberg representation of \( GL_2(F) \). See [KT2, 5.1] for details.

### 7.3 The symmetric space \( GL_4(F)/Sp_2(F) \)

Let \( G \) be the group \( GL_4(F) \) and \( \sigma \) the involution on \( G \) defined by

\[
\sigma(g) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^t g^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \quad (g \in G).
\]

Then the \( \sigma \)-fixed point subgroup \( H \) is the symplectic group \( Sp_2(F) \). For this symmetric space, \( H \)-distinguished representations were studied by Heumos-Rallis [HR]. Let \( \rho \) be an irreducible admissible representation of \( GL_2(F) \) and form the normalized induction

\[
\text{I}(\rho) = \text{Ind}^{G}_{P_{2,2}}(\rho \cdot |\det(\cdot)|_F^{1/2} \otimes \rho \cdot |\det(\cdot)|_F^{-1/2}).
\]

where \( P_{2,2} \) is the standard parabolic of type \((2, 2)\). Then \( \text{I}(\rho) \) has the unique irreducible quotient \( \pi(\rho) \) which is \( H \)-distinguished ([HR, 11.1(b)]). One can show that \( \pi(\rho) \) belongs to the discrete series for \( G/H \) if \( \rho \) is the Steinberg representation of \( GL_2(F) \). See [KT2, 5.2] for details.
References


