Differential equations satisfied by principal series Whittaker functions on $SU(2, 2)$

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Abstract

In this talk, we discuss about a holonomic system of differential equations for Whittaker functions associated with the principal series representation of $SU(2; 2)$ with higher dimensional minimal $K$-type.

1 Introduction

Throughout, let $G$ be the simple real Lie group $SU(2, 2)$ of rank two, and let

$$K = S(U(2) \times U(2)) : \text{the maximal compact subgroup of } G$$

$$\pi : \text{an irreducible representation of } G \text{ which is } K\text{-admissible.}$$

For the representation $\pi$, there are two types of Whittaker model with respect to a character $\eta$ of $N$ (a spherical subgroup of $G$). One is the smooth model, and the other is the algebraic models induced by the space of algebraic Whittaker vectors:

$$W(\pi, \eta) := \text{Hom}_{(\mathfrak{g}, K)}(\pi |_{K}, C^\infty\text{-Ind}_{N}^{G}(\eta)),$$

Here, $\mathfrak{g}$ is the Lie algebra of $G$, $\pi |_{K}$ is the subspace of $K$-finite vectors in $\pi$ and $C^\infty\text{-Ind}_{N}^{G}(\eta)$ is the right $G$-module smoothly induced from $\eta$.

Our aim is a characterization of the space $W(\pi, \eta)$ for the principal series representation $\pi$ of $G$ associated with a minimal parabolic subgroup, which leads to a description of the following challenging question associated to $\pi$.

Question. For each intertwiner $\Phi$ in $W(\pi, \eta)$, what is the image of $\Phi$? Equivalently, for each $K$-type $\tau$ occurring in $\pi$, one can ask the image of the $\tau$-isotypic component in $\pi$. The latter image is called the space of Whittaker functions of $\pi$ with respect to $\tau$.

The natural and classical approach. Let $\tau$ be a $K$-type belonging to $\pi$, and $f_1, \ldots, f_n$ be its a basis in $\pi$. Denote by $\phi_j(g)$ the image of $f_j$ under a fixed intertwiner $\Phi$. Then, for each $j$ and $k$ in $K$, the function $(k\phi_j)(g) = \phi_j(gk)$ is a linear combination of the functions $\phi_1(g), \ldots, \phi_n(g)$. Thus, it is enough to consider the functions $\phi_j$ on $A$ for our purpose.
Assume that $C$ is a square matrix of size $\dim(\tau)$, with entries in the universal enveloping algebra of $\mathfrak{g}$, so that

$$
\pi(C) \circ \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = \gamma \cdot \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix},
$$

for some constant $\gamma \in \mathbb{C}$.

By applying $\Phi$ to the identity (1) we get the following system of differential equations (the $A$-radial part)

$$
\mathcal{R}(C) \circ \begin{pmatrix} \phi_1(a) \\ \phi_2(a) \\ \vdots \\ \phi_n(a) \end{pmatrix} = \gamma \cdot \begin{pmatrix} \phi_1(a) \\ \phi_2(a) \\ \vdots \\ \phi_n(a) \end{pmatrix},
$$

where $\mathcal{R}$ denotes the infinitesimal action of $G$ on $\mathcal{C}^\infty$-Ind$_N^G(\eta)$.

**Remark.** Recall that Whittaker functions satisfy differential equations with regular singularities at "0". The most important requirements for choosing a basis for $\tau$ are the simplicity and symmetricity of the series expansion of these functions $\phi_j(a)(a \in A)$ around 0 and of the system of differential equations arising from (1).

**Principal series $\pi_{s,\chi}$.** Let

$$
a = \{a(t_1, t_2) = \begin{pmatrix} 0 & 0 & t_1 & 0 \\ 0 & 0 & 0 & t_2 \\ t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \end{pmatrix} \mid t_1, t_2 \in \mathbb{R} \} \subset \mathfrak{g},
$$

$$
M = \{\text{diag}(e^{i\theta}, e^{-i\theta}, e^{i\theta}, e^{-i\theta})\} \oplus \{1_4, \text{diag}(1, -1, 1, -1)\}.
$$

Define linear functions $\lambda_1$ and $\lambda_2$ on $a$ by $\lambda_1(a(t_1, t_2)) = t_1$ and $\lambda_2(a(t_1, t_2)) = t_2$.

Then the set $\{\pm \lambda_1 \pm \lambda_2, \pm 2\lambda_1, \pm 2\lambda_2\}$ forms the restricted root system of type $C_2$ for the pair $(\mathfrak{g}, a)$. Define $\lambda_1 \pm 2\lambda_2$ and $2\lambda_1 \pm 2\lambda_2$ to be positive. Let $P_{\text{min}}$ be the minimal parabolic subgroup of $G$ with Langlands decomposition $P_{\text{min}} = MAN$, where $N$ is the unipotent subgroup defined by the root spaces corresponding to positive roots. For the character $s \otimes \chi$ of $M$, $s \in \mathbb{Z}$, and the $\mathbb{C}$-valued real linear form $\mu = \mu_1 \lambda + \mu_2 \lambda_2$, one has the principal series representation

$$
\pi_{s,\chi} := \text{Ind}_N^G((s \otimes \chi)_M \otimes e^{\mu+\rho} \otimes 1_N),
$$

where $1_N$ is the trivial character of $N$.

The main object in the paper is the 8-dimensional space $W(\pi_{s,\chi}, \eta)$ of algebraic Whittaker vectors (see Kostant [2]) for non-degenerate character $\eta$ (unitary) of $N$. Note that it is sufficient for our purpose to assume that $s \geq 0$. 
1.1 Some previous results

Let us recall some known identities as in (1) and previous results for the space \( W(\pi, \eta) \). The first example is the classical Casimir equation: let \( \Omega \) be the Casimir operator of \( G \). Then we have the following identity

\[
\pi_{s,\chi}(\Omega) v = \chi_{\pi_{s,\chi}}(\Omega) v,
\]

where \( \chi_{\pi_{s,\chi}} \) is the infinitesimal character and \( v \) is a differential vector. This identity gives us an injection of \( W(\pi_{s,\chi}, \eta) \) into the solution space \( \text{Sol}(\mathcal{R}(\Omega)) \) of the above equation. Note that the space \( \text{Sol}(\mathcal{R}(\Omega)) \) is of infinite dimension.

Let \( \pi \) be a discrete series representation of \( G \) and \( \tau \) be its minimal \( K \)-type. Then Yamashita [10] defined an operator \( D_{\pi,\tau} \) on \( \tau \) under \( \pi \):

\[
\pi(D_{\pi,\tau}) \tau = 0.
\]

This gives us an injection of \( W(\pi, \eta) \) into the solution space \( \text{Sol}(\mathcal{R}(D_{\pi,\tau})) \) of the operator \( \mathcal{R}(D_{\pi,\tau}) \). Moreover, under certain conditions, he showed that

\[
W(\pi, \eta) \cong \text{Sol}(\mathcal{R}(D_{\pi,\tau}))
\]

as vector spaces. This result is not just for the group \( G \) (see [10] and [11]).

Let \( \pi \) be the principal series representation of \( G = Sp(2, \mathbb{R}) \) as in [6], and \( \tau \) be the minimal \( K \)-type of \( \pi \). In [6], the authors obtained a matrix, of size \( \dim(\tau) \), formula of the form \( \pi(D)v = \gamma v \) which implies

\[
W(\pi, \eta) \cong \text{Sol}(\mathcal{R}(\Omega), \mathcal{R}(D))
\]

where \( \Omega \) stands for the Casimir operator of \( Sp(2, \mathbb{R}) \). Note that the possible value of \( \dim(\tau) \) is 1 or 2. The degree of \( D \) is 4 if \( \dim(\tau) = 1 \), and 2 for the case of dimension 2.

**Remark.** In the case \( s = 0 \) and \( s = 1 \), the corresponding spaces \( W(\pi_{s,\chi}, \eta) \) behave quite similar to the above mentioned cases for \( G = Sp(2, \mathbb{R}) \), and are studied in [4], .

2 Differential equations

We begin by providing some formulas for the multiplicity one \( K \)-types \( \tau_{[0,s;l]} \) in the principal series \( \pi_{s,\chi} \). These formulas come from the explicit \((\mathfrak{g}, K)\)-module structure of \( \pi_{s,\chi} \) which originally discussed by Oda [7].

Note that the space of the adjoint \( K \)-representation \((Ad, p_\mathbb{C})\) is generated by the matrix units \( E_{ij+2} \) and \( E_{i+2j} \) \((1 \leq i, j \leq 2)\) and denote by \( E_{ij+2} \) and \( E_{i+2j} \) their infinitesimal actions with respect to \( \pi_s \). Let denote \( F_{[s;l]} \) the transpose of the vector \((f_0, f_1, \ldots, f_s)\), where \( \{f_j : 0 \leq j \leq s\} \) is the "nice" basis of \( \tau_{[0,s;l]} \) introduced in [1] and \( c_q := q/s \) for \( 0 \leq q \leq s \).

**Formula 1.** *(Casimir equation)* Let \( \Omega \) be the Casimir operator. Then we have

\[
\pi_{s,\chi}(\Omega) \cdot F_{[s;l]} = (\mu_1^2 + \mu_2^2 + \frac{1}{2}s^2 - 10)F_{[s;l]}.
\]
Formula 2. (Shift equations) Set $\nu_1 = \frac{1}{2}(s + l)$ and $\nu_2 = \frac{1}{2}(s - l)$. Then we have

$$\pi_{s,\chi}(\overline{Q}) \cdot F_{[s;l]} = \frac{1}{4} (\mu_1^2 - (\nu_1 + 1)^2) F_{[s;l]},$$

and

$$\pi_{s,\chi}(Q) \cdot F_{[s;l]} = \frac{1}{4} (\mu_2^2 - (\nu_2 - 1)^2) F_{[s;l]},$$

where $\overline{Q} = \{\overline{Q}_{ij}\}_{0 \leq i, j \leq s}$ and $Q = \{Q_{ij}\}_{0 \leq i, j \leq s}$ are square matrices given by

$$\overline{Q}_{qq-1} = -c_q (E_{24}E_{32} + E_{14}E_{31})$$
$$\overline{Q}_{qq+1} = -(1 - c_q) (E_{23}E_{42} + E_{13}E_{41})$$
$$\overline{Q}_{qq} = (1 - c_q) (E_{23}E_{32} + E_{13}E_{31}) + c_q (E_{14}E_{41} + E_{24}E_{42})$$

and

$$Q_{qq-1} = c_q (E_{32}E_{24} + E_{31}E_{14})$$
$$Q_{qq+1} = (1 - c_q) (E_{42}E_{23} + E_{41}E_{13})$$
$$Q_{qq} = c_q (E_{32}E_{23} + E_{31}E_{13}) + (1 - c_q) (E_{41}E_{14} + E_{42}E_{24})$$

for $0 \leq q \leq s$, but all other entries are 0.

Formula 3. (Annihilation equations) We have

$$\pi_{s,\chi}(A) \cdot F_{[s;l]} = 0,$$

and

$$\pi_{s,\chi}(\overline{A}) \cdot F_{[s;l]} = 0,$$

where $A = \{A_{ij}\}$ and $\overline{A} = \{\overline{A}_{ij}\}$ are square matrices whose non-zero entries are given by

$$A_{jj-1} = -E_{31}E_{14} - E_{32}E_{24},$$
$$A_{jj} = E_{41}E_{14} + E_{42}E_{24} - E_{31}E_{13} - E_{32}E_{23},$$
$$A_{jj+1} = E_{41}E_{13} + E_{42}E_{23},$$

and

$$\overline{A}_{jj-1} = -E_{14}E_{31} - E_{24}E_{32},$$
$$\overline{A}_{jj} = E_{14}E_{41} + E_{24}E_{42} - E_{13}E_{31} - E_{23}E_{32},$$
$$\overline{A}_{jj+1} = E_{13}E_{41} + E_{23}E_{42},$$

for $1 \leq j \leq s - 1$.

Proposition 2.1. On the $K$-type $\tau_{[0,s;l]}$ with respect to the action $\pi_{s,\chi}$ we have

$$Q + \overline{Q} = \Omega/4.$$
2.1 A holonomic system of rank 8

**Coordinate system.** Since the $\mathbb{R}$-split torus $A$ for our case is two dimensional, one may choose the coordinate system $(y_1, y_2)$. Denote the Euler operators $y_1 \frac{\partial}{\partial y_1}$ and $y_2 \frac{\partial}{\partial y_2}$ with respect to this system by $\partial_1$ and $\partial_2$, respectively.

We now define the matrix differential operator $\overline{D}$ by

\[
\begin{pmatrix}
\bar{d}_{00} & \bar{d}_{01} & 0 & \ldots & 0 \\
0 & \bar{d}_{11} & \bar{d}_{12} & \ldots & 0 \\
0 & 0 & \bar{d}_{22} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \bar{d}_{s-2s-2} & \bar{d}_{s-2s-1} & 0 \\
0 & 0 & 0 & \ldots & 0 & \bar{d}_{s-1s-1} & \bar{d}_{s-1s} \\
0 & 0 & 0 & \ldots & 0 & \bar{d}_{ss-1} & \bar{d}_{ss}
\end{pmatrix}
\]

where

\[
d_{qq} = \frac{1}{4}((\partial_1 - q)^2 - \mu_1^2) - \xi \bar{\xi} y_1^2, \quad d_{q+1} = \bar{\xi} y_1(\partial_2 + \frac{1}{2}s - q) + \bar{\xi} y_1 y_2
\]

for $q = 0, \ldots, s - 1$ and

\[
d_{ss} = \frac{1}{4}((\partial_1 - 2\partial_2)^2 - \mu_2^2) - \xi \bar{\xi} y_1^2 - y_2^2 - \nu_1 y_2
\]

\[
d_{ss-1} = -\xi y_1(\partial_2 + \frac{1}{2}s) + \xi y_1 y_2.
\]

We also define the matrix differential operator $D$ by

\[
\begin{pmatrix}
d_{00} & d_{01} & 0 & \ldots & 0 \\
d_{10} & d_{11} & 0 & \ldots & 0 \\
0 & a_{32} & d_{33} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & d_{s-2s-2} & 0 & 0 \\
0 & 0 & 0 & \ldots & d_{s-1s-2} & d_{s-1s-1} & 0 \\
0 & 0 & 0 & \ldots & 0 & d_{ss-1} & d_{ss}
\end{pmatrix}
\]

where

\[
d_{00} = \frac{1}{4}((\partial_1 - 2\partial_2)^2 - \mu_2^2) - \xi \bar{\xi} y_1^2 - y_2^2 - \nu_2 y_2
\]

\[
d_{01} = -\bar{\xi} y_1(\partial_2 - \frac{1}{2}s) - \bar{\xi} y_1 y_2
\]

and

\[
d_{qq} = \frac{1}{4}((\partial_1 - s + q)^2 - \mu_2^2) - \xi \bar{\xi} y_1^2, \quad d_{q+1} = \xi y_1(\partial_2 + q - \frac{1}{2}s) - \xi y_1 y_2
\]

for $q = 1, \ldots, s$. Here, the parameters $\xi$ and $\bar{\xi}$ are associated to the character $\eta$. 
By using Formulas 2 and 3, one can see that the Whittaker functions of \( \pi_{s,\chi} \) with respect to \( \tau_{\{0, s;l\}} \) satisfy the system of differential equations \( D = 0 \) and \( \overline{D} = 0 \). Moreover, we have the following result which characterizes the Whittaker functions of \( \pi_{s,\chi} \) with respect to \( \tau_{\{0, s;l\}} \).

**Theorem 2.2.** For \( s \geq 2 \), the natural map from \( W(\pi_{s,\chi}, \eta) \) into \( \text{Ker}(\overline{D}, D) \) is bijection if \( \pi_{s,\chi} \) is irreducible and \( \eta \) is a nondegenerate unitary character of \( N \).

Here, we also have the following formula in the case \( s = 0 \), which is analogue to the class one case for \( Sp(2, \mathbb{R}) \) in [5]. Write \( W \) for the little Weyl group for \((g, a)\), and \((\rho_1, \rho_2)\) for the pair \((3, 2)\) related to the half sum.

**Theorem 2.3.** Let \( \pi_{0,\chi} \) be an irreducible principal series with parameter \( \mu = (\mu_1, \mu_2) \in a_\mathbb{C}^* \), and set \( \varepsilon = \frac{1-\chi(-1)}{2} \). Then the function \( \phi_\mu \) on \( A \) defined by

\[
\phi_\mu(y_1, y_2) = y_1^{\rho_1} y_2^{\rho_2} \sum_{m,n \geq 0} \frac{U_{m,n}^0}{2^{2n}(\frac{\mu_1-\varepsilon}{2}+1)m(\frac{\mu_2-\varepsilon}{2}+1)n} \times y_1^{\mu_1+2m} y_2^{\mu_2+2n} \\
+ \varepsilon \frac{U_{m,n}^1}{2^{2n+1}(\frac{\mu_1-\varepsilon}{2}+1)m(\frac{\mu_2-\varepsilon}{2}+1)n+1} \times y_1^{\mu_1+2m} y_2^{\mu_2+2n+1},
\]

is a Whittaker function, on \( A \), of \( \pi_{0,\chi} \) with the \( K \)-type \( \tau_{\{0,0;2\varepsilon\}} \). Moreover, the intertwiners \( \Phi_{\omega(\mu)} \) attached to the function \( \phi_{\omega(\mu)}(y_1, y_2) \) form a basis of the 8-dimensional space \( W(\pi_{0,\chi}, \eta) \). Here,

\[
U_{m,n}^t := \sum_{j=0}^{\min(m,n)} \frac{(\frac{\mu_1-\varepsilon}{2}+n+1+t)m-j}{(m-j)!(n-j)!j!(\frac{\mu_1+\mu_2}{2}+1)j(\frac{\mu_1-\mu_2}{2}+1)m-j}
\]

for \( t = 0, 1 \).

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**References**

[1] Bayarmagnai, G. The \((g, K)\)-module structure of principal series of \( SU(2, 2) \), J. Math. Soc, Vol. 61, No. 3 (2009), 661-686


