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COMPUTING SIEGEL MODULAR FORMS AND PARAMODULARITY

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ABSTRACT. We give a survey of techniques for computing Siegel paramodular forms in degree 2, with the main application being to give evidence for the Paramodular Conjecture (see [2]).

This is joint work with Cris Poor, Fordham University.

1. Modularity and Paramodularity

We begin with definitions and motivation from genus 1. An elliptic modular form with respect to a subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ of weight k is a holomorphic function

$$f : \mathcal{H}_1 = \{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0\} \to \mathbb{C}$$

that is also holomorphic at the cusps such that

$$\forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma, f((a\tau+b)(c\tau+d)^{-1}) = (c\tau+d)^k f(\tau).$$

We write $f \in M_1^k(\Gamma)$. We say f is a cusp form if it vanishes at the cusps. Define

$$\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N\}.$$

The famous Modularity Theorem (Taniyama-Shimura-Weil Conjecture, proven by Wiles, Breuil, Conrad, Diamond, Taylor) is

Theorem 1 (Modularity Theorem). *For an elliptic curve defined over the rationals with conductor N, there exists a modular form which is a rational Hecke eigenform in $S_1^2(\Gamma_0(N))^{new}$ which has the same L-function.*

We can ask if there is a higher genus version of this. We define a Siegel modular form of degree (genus) $n > 1$ with respect to a subgroup $\Gamma \subseteq \mathrm{Sp}_n(\mathbb{Q})$ of weight k to be a holomorphic function

$$f : \mathcal{H}_n = \{\Omega \in \mathbb{C}_{n\times n}^{\mathrm{sym}} : \operatorname{Im}\Omega > 0\} \to \mathbb{C}$$

such that

$$\forall \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma, f((A\Omega+B)(C\Omega+D)^{-1}) = \det(C\Omega+D)^k f(\Omega).$$

We write $f \in M_n^k(\Gamma)$. If $\Gamma \supseteq \{\left(\begin{smallmatrix} I & S \\ 0 & I \end{smallmatrix}\right) : S \in \ell\mathbb{Z}_{n\times n}\}$, some $\ell \in \mathbb{N}$, then f has a Fourier expansion

$$f(\Omega) = \sum_{T \in \mathcal{P}_n^{\mathrm{semi}}(\mathbb{Q})} a(T; f) e(\operatorname{tr}(T\Omega)),$$

where $e(x) = e^{2\pi i x}$. And it is a cusp form if the expansion is

$$f(\Omega) = \sum_{T \in \mathcal{P}_n(\mathbb{Q})} a(T; f) e(\operatorname{tr}(T\Omega)),$$

with a similar expansion at each cusp. We write $f \in S_n^k(\Gamma)$ for cusp forms. Define

$$\Gamma_0^{(n)}(N) = \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z}) : C \equiv 0 \bmod N\}.$$

In 1980 [13], Yoshida predicted that every abelian surface defined over $\mathbb{Q}$ should have the same L-function as some Siegel modular form of some suitable level in degree two. But what is the suitable level? Yoshida's examples were the following: For $p \in \{23, 29, 31\}$, let $f, \bar{f}$ span $S_1^2(\Gamma_0(p))$. There exists a Yoshida lift $\mathrm{Yosh}(f, \bar{f}) \in S_1^2(\Gamma_0^{(2)}(p))$ such that

$$L(\mathrm{Yosh}(f,\bar{f}), s, \mathrm{spin}) = L(f,s)L(\bar{f},s)$$
$$\overset{\text{(by Shimura)}}{=} L(\mathrm{Jac}(X_0(p)), s, \text{Hasse-Weil}).$$

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Note that $\mathrm{Jac}(X_0(p))$ is a abelian surface defined over $\mathbb{Q}$ of conductor p^2. So somehow $\Gamma_0^{(2)}(p)$ is not the "right" group in $g = 2$. Later on, Brumer and Kramer realized that the paramodular group is the "right" group. The paramodular group is defined as

$$K(N) = \{\begin{pmatrix} * & N* & * & * \\ * & * & * & */N \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix} : * \in \mathbb{Z}\} \cap Sp_2(\mathbb{Q}).$$

Perhaps a more natural definition is

$$K(N) = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&N \end{pmatrix} \{M \in \mathrm{GL}_4(\mathbb{Z}) : M' E_N M = E_N\} \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&N \end{pmatrix}^{-1}.$$

where $E_N = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&N \\ -1&0&0&0 \\ 0&-N&0&0 \end{pmatrix}$. This paramodular group arises naturally because $K(N)\backslash\mathcal{H}_2$ is isomorphic to the moduli space of abelian surfaces with $(1, N)$ polarization. Brumer and Kramer make the following conjecture for abelian surfaces defined over $\mathbb{Q}$ of prime conductor (the general version will be stated later in this article) [2].

Conjecture 2 (Paramodular Conjecture for prime conductors (Brumer-Kramer) [2]). *Let N be prime. There is a bijection between lines of Hecke eigenforms $f \in S_2^2(K(N))$ with rational eigenvalues that are not Gritsenko lifts and isogeny classes of abelian surfaces $\mathcal{A}$ defined over $\mathbb{Q}$ of conductor N. In this correspondence,*

$$L(\mathcal{A}, s, \textit{Hasse-Weil}) = L(f, s, \textit{spin}).$$

Currently there are no proven examples. Our goal is to find examples or give strong evidence of examples to support this conjecture. The prime 277 is the smallest prime conductor of an abelian surface defined over $\mathbb{Q}$. The abelian surface is the Jacobian of the curve

$$y^2 = x^6 - 2x^5 - x^4 + 4x^3 + 3x^2 + 2x + 1.$$

Let us define Jacobi forms and Gritsenko lifts. Define

$$\Gamma_\infty(\mathbb{Z}) = \mathrm{Sp}_2(\mathbb{Z}) \cap \left\{ \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

A (level one) Jacobi Form of weight k, index m, is a holomorphic $\phi : \mathcal{H}_1 \times \mathbb{C} \to \mathbb{C}$ such that the function

$$\tilde{\phi}(\left(\begin{smallmatrix} \tau & z \\ z & w \end{smallmatrix}\right)) \stackrel{\text{def}}{=} \phi(\tau, z) e(mw)$$

transforms so that $\tilde{\phi} \in M_2^k(\Gamma_\infty(\mathbb{Z}))$ and ϕ has Fourier expansion

$$\phi(\tau, z) = \sum_{n,r : 4mn - r^2 \geq 0} c(n, r) e(n\tau) e(rz).$$

We write $\phi \in J_{k,m}$. We say it is a cusp form if the above sums only over $4mn - r^2 > 0$; in that case we write $\phi \in J_{k,m}^{\mathrm{cusp}}$. The idea is that a level one Siegel modular cusp form f has Fourier expansion that collects into

$$f(\left(\begin{smallmatrix} \tau & z \\ z & w \end{smallmatrix}\right)) = \sum_m \phi_m(\tau, z) e(mw)$$

where each $\phi_m \in J_{k,m}^{\mathrm{cusp}}$. The Gritsenko lift [4] is a tool that is not available in genus 1.

Theorem 3 (Gritsenko). *Let $\phi \in J_{k,N}^{cusp}$, with*

$$\phi(\tau, z) = \sum_{n,r : 4Nn - r^2 > 0} c(n, r) e(n\tau) e(rz).$$

There exists $Grit(\phi) \in S_2^k(K(N))$ given by

$$Grit(\phi)(\left(\begin{smallmatrix} \tau & z \\ z & w \end{smallmatrix}\right)) = \sum_{n,r,m} \left(\sum_{\delta | (n,r,m)} \delta^{k-1} c(\tfrac{mn}{\delta^2}, \tfrac{r}{\delta}) \right) e(n\tau + rz + mNw).$$

This gives an injective linear map of Jacobi cusp forms $J_{k,N}^{\text{cusp}} \hookrightarrow S_2^k(K(N))$. Note that Gritsenko lifts have L-functions that come from lower degree L-functions, and so it makes sense that they would be excluded in the Paramodular Conjecture. It is important to note that there is a formula for the dimensions of $J_{k,m}^{\text{cusp}}$.

First, a summary of some of what is known about the dimensions of paramodular cusp forms. Ibukiyama [5] [6] has proven formulas for $\dim S_2^k(K(p))$ for p prime and weights $k \geq 3$. And for weight 2, the weight that is relevant for paramodularity, Ibukiyama had proven that $\dim S_2^2(K(p)) = 0$ for primes $p \leq 23$. Of course, if the Paramodular Conjecture is true, then the first prime p for which $\dim S_2^2(K(p)) \neq \dim J_{k,m}^{\text{cusp}}$ is $p = 277$. We will survey three methods of computing paramodular forms:

- Restriction to elliptic forms
- Integral Closure
- Restriction to Jacobi Forms

2. Restriction to Elliptic Modular Forms

The technique of "restriction to elliptic modular forms" had been successfully used (see [7] [9] [10]) to compute higher degree Siegel modular forms for the spaces of degree 4 cusp forms S_4^k for weights $k \leq 16$, and even higher degree spaces of cusp forms S_5^8, S_5^{10}, S_6^8. And it was also successful in degree 2 for subgroups, $S_2^2(\Gamma_0^{(2)}(p))$ for primes $p \leq 43$, and some others. This method in the paramodular case, in brief, amounts to taking $s = \begin{pmatrix} a & b \\ b & c/N \end{pmatrix} > 0$ with $a, b, c \in \mathbb{Z}$ and looking at the map $\phi_s : \mathcal{H}_1 \to \mathcal{H}_2$ by $\phi_s(\tau) = \tau s$. The pullback of this map gives a homomorphism

$$\phi_s^* : S_2^2(K(N)) \to S_1^4(\Gamma_0(N\ell))$$

where $\ell = \det(s)$ and $\phi_s^* f = f \circ \phi_s$. Using known dimension formulas for $S_1^4(\Gamma_0(N\ell))$ and finding a basis of $S_1^4(\Gamma_0(N\ell))$, and also computing $S_1^4(\Gamma_0(N\ell))|\gamma$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ (this is manageable if $N\ell$ is square-free) and having algorithms to compute $(\phi_s^* f)|\gamma$ in terms of the original f (this is manageable for $K(N)$ where N is prime), we get relations on the Fourier coefficients of f.

To make a conclusion about how these relations give an upper bound on the dimension of $S_2^2(K(N))$, we need to know a determining set of Fourier coefficients. In the elliptic case, we have that a level one elliptic modular form $f \in M_1^k$ is determined by its Fourier coefficients $a(j; f)$ with $j \leq \frac{k}{12}$. For degree 2 Siegel modular forms, we have the following theorems from [8] [11].

Theorem 4 (P-Y). *Let $f \in M_2^k$. Denote $\mathcal{X}_2 = \{\begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \geq 0 : a, b, c \in \mathbb{Z}\}$. Then*

$$\forall T \in \mathcal{X}_2,\ a(T; f) \in \mathbb{Z}\langle a(s; f) : w(s) \leq \tfrac{k}{6}\rangle$$

where w is the dyadic trace function

$$w(\begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}) = a + c - \tfrac{1}{2}|b| \text{ if } |b| \leq a \leq c.$$

As a corollary, a vanishing and congruence theorem for paramodular forms [11] is

Theorem 5 (P-Y). *Let $f \in S_2^k(K(p))$, where p is prime. The vanishing of f and congruences of f are determined by $a(T; f)$ for the finite number of classes of T satisfying*

$$w(T) \leq \frac{k}{6} \frac{p^2 + 1}{p + 1}.$$

Then using this set of determining coefficients and the restriction technique, we were able to prove that

$$\dim S_2^2(K(p)) = \dim J_{2,p}^{\text{cusp}}, \text{ for } p \leq 83.$$

This is still far from 277.

3. Integral Closure

Next, we explain the method of "integral closure" (see [11]). Given a prime p, to find <u>non</u>-Gritsenko lifts $f \in S_2^2(K(p))$, we look for weight 4 forms $h_1, h_2 \in S_2^4(K(p))$ and weight 2 forms $g_1, g_2 \in S_2^2(K(p))$ such that

$$\frac{h_1}{g_1} = \frac{h_2}{g_2} \ (= \text{alleged } f).$$

Lemma 6. *Let $g_1, g_2 \in S_2^2(K(p))$. The map*

$$S_2^2(K(p)) \to \{(h_1, h_2) \in S_2^4(K(p)) \times S_2^4(K(p)) : h_1 g_2 = h_2 g_1\}$$

given by $f \mapsto (g_1 f, g_2 f)$ is injective.

A useful corollary is

Corollary 7. *If $\dim\{(h_1, h_2) \in S_2^4(K(p)) \times S_2^4(K(p)) : h_1 g_2 = h_2 g_1\} \leq \dim J_{2,p}^{cusp}$, then $S_2^2(K(p))$ is spanned by Gritsenko lifts.*

One major issue in using this technique is that we need to find a basis of $S_2^4(K(p))$. In practice, we try to span $S_2^4(K(p))$ by the following methods:

- Gritsenko lifts $J_{4,p}^{\mathrm{cusp}} \to S_2^4(K(p))$.
- Products of weight 2 Gritsenko lifts.
- Hecke operators on the above products.
- Tracing $\mathrm{Tr} : S_2^4(\Gamma_0(p)) \to S_2^4(K(p))$, by tracing theta series of lattices.

We note that tracing is the only one of the above methods that can hit the "minus" forms in $S_2^4(K(p))$ and is also a very expensive calculation. It is worth remarking that tracing theta series in weight 2 will not produce non-Gritsenko lifts because of

Theorem 8 (P-Y). *The trace map $Tr : S_2^2(\Gamma_0(p)) \to S_2^2(K(p))$ in weight 2 is zero on theta series.*

Another issue after we find $\frac{h_1}{g_1} = \frac{h_2}{g_2}$ is trying to prove that $\frac{h_1}{g_1}$ is holomorphic. One approach is to find the candidate weight 4 form $F \in S_2^4(K(p))$ such that allegedly $(\frac{h_1}{g_1})^2 = F$. We then prove the weight 8 identity

$$h_1^2 = g_1^2 F,$$

and thus infer that $\frac{h_1}{g_1}$ is holomorphic. The major hurdle of this technique is that to get a useful set of determining Fourier coefficients for the weight 8 space, we need to find a basis for $S_2^8(K(p))$. We have used this technique to successfully prove

Theorem 9 (P-Y). $\dim S_2^2(K(277)) = 1 + \dim J_{2,277}^{cusp}$. *That is, there is exactly one nonlift eigenform. Furthermore, its L-function has Euler factors at 2, 3, 5 that match those of the L-function of the known abelian surface of conductor 277.*

We like to emphasize that this is the first example of a truly new L-function of a higher degree Siegel modular eigenform in the sense that the L-function is not constructed from those of GL_2-type.

This integral closure technique was applied to primes $p < 600$ and now we discuss the evidence for the prime version of the Paramodular Conjecture.

- For primes $p < 600$ and $p \notin \{277, 349, 353, 389, 461, 523, 587\}$, we have proven that $\dim S_2^2(K(p))$ $= \dim J_{2,p}^{\mathrm{cusp}}$ (that is, no nonlifts) and Brumer and Kramer have proven that there are no abelian surfaces defined over $\mathbb{Q}$ with conductor p.
- For $p = 277$, we have proven $\dim S_2^2(K(277)) = 1 + \dim J_{2,277}^{\mathrm{cusp}}$ (there is one nonlift) and Brumer and Kramer have found only one (they cannot rule out there are no others) isogeny class of abelian surface defined over $\mathbb{Q}$ of conductor 277. Also, the 2, 3, 5 -Euler factors match.
- For $p \in \{349, 353, 389, 461, 523\}$, we have proven $\dim S_2^2(K(p)) \leq 1 + \dim J_{2,p}^{\mathrm{cusp}}$ (there is at most one nonlift) and Brumer and Kramer have found one isogeny class of abelian surfaces defined over $\mathbb{Q}$ of conductor p. Also, the 2, 3 -Euler factors of the alleged nonlift match those of the corresponding surface. Only holomorphicity of the alleged nonlifts remains to be proven.
- For $p = 587$, we have proven $\dim S_2^2(K(587)) \leq 2 + \dim J_{2,587}^{\mathrm{cusp}}$ (there are at most two nonlifts) and Brumer and Kramer have found two isogeny classes of abelian surfaces defined over $\mathbb{Q}$ of conductor 587. Also, the 2, 3 -Euler factors match.
- Also, in the above $p \in \{277, 349, 353, 389, 461, 523, 587\}$ whether the alleged nonlifts are plus or minus forms matches whether the abelian surface has even or odd rank.
- Also, in the above $p \in \{277, 349, 353, 389, 461, 523, 587\}$ any congruences of the alleged nonlift to a lift matches any torsion defined over $\mathbb{Q}$ of the abelian surface.

Details of these calculations (including exactly how the weight 4 space $S_2^4(K(p))$ was spanned using what Hecke operators and the tracing of what theta series) can be found at

http://math.lfc.edu/~yuen/paramodular

Note that the integral closure technique was only applied to prime N because we needed to span $S_2^4(K(N))$ and currently the dimension of such spaces are only known for prime N.

4. Restriction to Jacobi Forms

Now, we discuss the technique of "restriction to Jacobi forms". This technique is at present heuristic but may be applied to composite N. We start with an abstract $f \in S_2^k(K(N))$ (where the coefficients $a(n,r,m)$ are abstract variables), and collect

$$f\left(\left(\begin{smallmatrix}\tau & z\\ z & w\end{smallmatrix}\right)\right) = \sum_{4mn-r^2>0,\ N|m} a(n,r,m)\, e(n\tau + rz + mw)$$
$$\overset{\text{(collect)}}{=} \sum_{N|m} (\psi_m f)(\tau, z)\, e(mw),$$

where each $\psi_m f$ is required to be a Jacobi form. That is, for each $m = N, 2N, 3N, \ldots$, we have a map

$$\psi_m : S_2^k(K(N)) \to J_{k,m}^{\text{cusp}}.$$

If we can find a basis of $J_{k,m}^{\text{cusp}}$, then these maps would pull back to linear relations on the Fourier coefficients $a(n,r,m)$. Now, there is a formula for $\dim J_{k,m}^{\text{cusp}}$ from Eichler, Zagier, Skoruppa [3], and we can try to find a basis using "theta blocks" (recent work of Gritsenko, Skoruppa, Zagier) and Hecke operators. Finding a basis by this method appears to always succeed in weight 2, at least for the m investigated so far, although the search may become expensive for large m. And for $k > 2$, this method still appears to succeed for m attempted so far. We now discuss the calculations only for weight $k = 2$. The algorithm, in brief, is as follows. For each N, pick the following parameters for the calculation:

- A range of m.
- A finite set of Fourier coefficients $a(n,r,m)$ of f to track. In practice, we pick some determinant bound D and use the set $\{a(n,r,m) : 4mn - r^2 \leq D\}$.

We try to pick these so that the solution space (the space of Fourier coefficients that satisfy all the linear relations) appear to stabilize as m increases. If the solution space contains more than the Gritsenko lifts, we compute Hecke operators to find the alleged nonlift eigenforms and compute alleged Hecke eigenvalues. This method produced data for $N \leq 1000$, and we now discuss the data preliminarily.

- The data is consistent with the data produced by the integral closure method for primes $N < 600$.
- We have a list of N where the evidence is that there are no nonlifts.
- We have a list of "interesting" N where the evidence is that there is at least one nonlift, along with some alleged eigenvalues.
- For the most part, the surface data of Brumer and Kramer match up with this data, but not exactly at first until Brumer and Kramer finalized their Paramodular Conjecture to composite conductors. That is, this data was helpful in giving evidence that led to the following conjecture.

Conjecture 10 (Paramodular Conjecture (Brumer and Kramer) [2]). *Let $N \in \mathbb{N}$. There is a bijection between lines of rational Hecke eigenforms in $S_2^2(K(N))^{\text{new}}$ which are not Gritsenko lifts and isogeny classes of abelian surfaces $\mathcal{A}$ defined over $\mathbb{Q}$ of conductor N with $\text{End}_{\mathbb{Q}}(\mathcal{A}) = \mathbb{Z}$. In this correspondence, the L-functions match.*

Here, $S_2^2(K(N))^{\text{new}}$ is defined using the concept of old forms in the recent work of Roberts and Schmidt [12]. Note that the above condition is that the ring of endomorphisms defined over $\mathbb{Q}$ is trivial.

Continuing the discussion of the data, we see that there is very strong evidence for the Paramodular Conjecture.

- Where the data indicates no nonlifts, Brumer and Kramer have found no abelian surfaces (and have proven there are no such abelian surfaces in many cases).
- Where the paramodular data indicates a nonlift eigenform exists, of the 86 alleged rational nonlift eigenforms, Brumer and Kramer have found a corresponding abelian surface defined over $\mathbb{Q}$ in 68 of these cases. In these 68 cases, the computed eigenvalues of the paramodular form (usually only λ_2, λ_3 or λ_5, and sometimes $\lambda_4, \lambda_7, \lambda_9$ also) always match those of the corresponding abelian surface. Also, the even/odd ranks match whether the eigenform is a plus or minus form. And the torsion data appear to match as well.

- Of the 18 remaining cases, there are 9 cases where the conductor of the candidate surfaces need to be confirmed because calculating the power of 2 in the conductor is tedious ($N = 464, 472, 688, 704, 768, 832, 856, 924, 932$), and the other 9 cases are work in progress as far as either finding a surface of that conductor or proving that none exist ($N = 550, 702, 760, 816, 903, 945, 969, 976, 988$).
- It should be emphasized that in no case is there data that contradicts the Paramodular Conjecture.

5. Example of Twisting?

In $S_2^2(K(954))$, there are two alleged nonlift rational eigenforms that appear to be twists of each other. They have eigenvalues

$$\lambda_5 : 3, -3$$
$$\lambda_7 : -2, -2.$$

Now, it is easy to twist the corresponding abelian surfaces. Currently, there is no known way of twisting paramodular forms. The coefficients of these two paramodular forms can be seen at:

http://math.lfc.edu/~yuen/eigenforms/954htm/

and there do not seem to be any obvious relations among their coefficients.

6. An Application in weight 3

In [1], Ash, Gunnells, and McConnell studied $H^5(\Gamma_0(p), \mathbb{C})$ and found eigenforms and computed Euler 2, 3 factors at $p = 61, 73, 79$. They predicted the existence of corresponding Siegel modular forms. Brumer brought this to our attention and noted that $\dim S_2^3(K(p)) = 1 + \dim J_{3,p}^{\mathrm{cusp}}$ for these p. Using the rigorous integral closure technique, we computed the one nonlift in each of these cases and found that the nonlift eigenforms have matching Euler factors.

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