数理解析研究所講究録
第 1767 巻 2011 年 96-102

θ-correspondence for PGSp(4) and PGU(2,2)

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Introduction

Let $\mathfrak{H}_2 = \{Z = {}^tZ \in \mathrm{M}_2(\mathbb{C}) \mid \Im(Z) > 0\}$ be the Siegel upper half space of degree 2. Let

$$\theta_m(Z) = \sum_{x\in\mathbb{Z}^2} \exp\left(2\pi \mathrm{i}\left(\tfrac{1}{2}(x+\tfrac{m'}{2})Z^t(x+\tfrac{m'}{2}) + (x+\tfrac{m'}{2})^t(\tfrac{m''}{2})\right)\right)$$

be the Igusa theta constant with $m = (m', m'') \in \mathbb{Q}^2 \times \mathbb{Q}^2$. For a congruence subgroup Γ of $\mathrm{Sp}_2(\mathbb{Z})(\subset \mathrm{SL}_4(\mathbb{Z}))$, let $S_3(\Gamma)$ denote the space of Siegel modular cusp forms of weight 3 with respect to Γ, and let S_Γ the Siegel modular 3-fold associated to Γ. van Geemen and van Straten showed that $S_3(\Gamma_2(4,8))$ is spanned by certain 6-tuple products $\prod_{j=1}^6 \theta_{m_j}(Z)$ with $m_j \in \{0,1\}^4$ using the theta embedding of $S_{\Gamma(4,8)}$ into $\mathbb{P}^{13}$ (cf. [3]), where

$$\Gamma(4,8) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma(4) \mid \mathrm{diag}(B) \equiv \mathrm{diag}(C) \equiv 0 \pmod 8 \right\}. \qquad (0.1)$$

Through Igusa's transformation formula, $\mathrm{Sp}_4(\mathbb{Z})$ acts on these 6-tuple products. They showed that $S_3(\Gamma(4,8))$ is decomposed into seven irreducible $\mathrm{Sp}_4(\mathbb{Z})$-modules, and each module is generated by acting $\mathrm{Sp}_4(\mathbb{Z})$ a 6-tuple product of Igusa theta constants:

$$S_3(\Gamma(4,8)) = \sum_{i=1}^{7} \mathrm{Sp}_2(\mathbb{Z}) \cdot f_i \qquad (0.2)$$

where $\cdot$ indicates the standard action of the elements of $\mathrm{Sp}_2(\mathbb{R})$ to the Siegel modular forms of weight 3. Further, they showed that each 6-tuple products f_i lie in irreducible cuspidal automorphic representations π_{f_i} of $\mathrm{PGSp}_4(\mathbb{A})$. Calculating some eigenvalues for Hecke operators on

$$f_7(Z) := \theta_{(0,0,0,0)}(Z)^2\theta_{(1,0,0,0)}(Z)\theta_{(0,1,0,0)}(Z)\theta_{(0,0,1,1)}(Z)\theta_{(0,0,0,1)}(Z),$$

they gave the following conjecture:

Conjecture (van Geemen and van Straten [2]). *Let ρ be the unique elliptic cusp form of weight 3 of level 32 with central character χ_{-4}. Let μ be the größencharacter of $\mathbb{Q}(\mathrm{i})$ associated to the CM-elliptic curve $y^2 = x^3 - x$, and $\pi(\mu)$ be the CM-elliptic cusp form of weight 2 of level 32. Then, the irreducible cuspidal automorphic representation π_{f_7} has the partial spinor L-function (of degree 4) $L(s, \rho \otimes \pi(\mu))$ outside of 2.*

Here χ_{-4} indicates the quadratic character related to the extension $\mathbb{Q}(\mathrm{i})/\mathbb{Q}$. We will give a sketch of a proof of this conjecture.

1 θ-lifts

Let $K = \mathbb{Q}(\mathrm{i})$. Let $\mathrm{Gal}(K/\mathbb{Q}) = \{1, c\}$. Let

$$J = \begin{bmatrix} 0 & -1_2 \\ 1_2 & 0 \end{bmatrix}$$

and

$$\mathrm{GU}_{2,2}(K) = \left\{ g \in \mathrm{GL}_4(K) \mid {}^t g^c J g = \nu(g) J, \nu(g) \in \mathbb{Q}^\times \right\}.$$

We define the 6-dimensional quadratic space over $\mathbb{Q}$

$$\mathrm{X}(\mathbb{Q}) = \left\{ x = \begin{bmatrix} 0 & u & a & d \\ -u & 0 & b & -a^c \\ -a & -b & 0 & -v \\ -d & a^c & v & 0 \end{bmatrix} \mid b, d, u, v \in \mathbb{Q},\ a \in K \right\}$$

with norm form $(x, x) = (bd + uv + a\bar{a})$. Let

$$\begin{aligned} e_1 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, e_{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\ e_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, e_{-2} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, e = \mathrm{i}J. \end{aligned}$$

We define a right action ϱ of $\mathrm{GU}_{2,2}(K)$ on $\mathrm{X}(\mathbb{Q})$ by

$$\varrho(h)x = h^{-1} \cdot x \cdot {}^t h^{-1}.$$

Via ϱ, we have the isomorphism

$$\mathrm{PGU}_{2,2}(K) \simeq \mathrm{PGSO}_{\mathrm{X}}(\mathbb{Q}).$$

We will denote by F a v-adic completion of $\mathbb{Q}$. Let $\mathrm{Y}(\mathrm{F})$ be the 4-dimensional symplectic space with symplectic form $\langle , \rangle$. Let $\{\varepsilon_{+1}, \varepsilon_{-1}, \varepsilon_{+2}, \varepsilon_{-2}\}$ be the standard basis of $\mathrm{Y}(\mathrm{F})$ ($\langle \varepsilon_{+i}, \varepsilon_{-j} \rangle = \delta_{ij}, \langle \varepsilon_{+i}, \varepsilon_{+j} \rangle = 0$). Let $\mathrm{Sp}_2(\mathrm{F})$ act from the right on $\mathrm{Y}(\mathrm{F})$. We will use the two polarizations

$$\begin{aligned} \mathrm{Z} = \mathrm{Y} \otimes \mathrm{X} &= \mathrm{Z}^+ + \mathrm{Z}^- \\ &= \mathrm{Z}'^+ + \mathrm{Z}'^- \end{aligned}$$

with

$$\begin{aligned} \mathrm{Z}^\pm &= \mathrm{Y} \otimes (\mathrm{F}e_{\pm 1} + \mathrm{F}e_{\pm 2}) + (\mathrm{F}\varepsilon_{\pm 1} + \mathrm{F}\varepsilon_{\pm 2}) \otimes (\mathrm{F}e + \mathrm{F}e'), \\ \mathrm{Z}'^\pm &= (\mathrm{F}\varepsilon_{\pm 1} + \mathrm{F}\varepsilon_{\pm 2}) \otimes \mathrm{X}(\mathrm{F}). \end{aligned}$$

We realize the Weil representation r_ψ, r'_ψ of $\mathrm{Sp}(\mathrm{Z})$ in $\mathcal{S}(\mathrm{Z}^+(\mathrm{F})), \mathcal{S}(\mathrm{Z}'^+(\mathrm{F}))$, respectively. Put

$$\mathcal{R}(\mathrm{F}) = \{(g, h) \in \mathrm{GSp}_2(\mathrm{F}) \times \mathrm{GU}_{2,2}(K_v) \mid \nu(g) = \nu(h)\}$$

where ν indicates the similitude norm. We embed $\mathcal{R}(\mathrm{F})$ into $\mathrm{Sp}(\mathrm{Z}(\mathrm{F}))$ through the action $z \to \varrho(h^{-1})zg$, and denote

$$\begin{aligned} r_{\psi_v}(g,h)\phi(z) &= r_{\psi_v}(g,\varrho(h^{-1}))\phi(z), \\ r'_{\psi_v}(g,h)\phi(z) &= r'_{\psi_v}(g,\varrho(h^{-1}))\phi(z). \end{aligned}$$

Let ψ be a nontrivial additive character of $\mathbb{Q}\backslash\mathbb{A}$ and $r_\psi = \otimes_v r_{\psi_v}, r'_\psi = \otimes_v r_{\psi'_v}$. Let τ be an automorphic form on $\mathrm{PGSp}_2(\mathbb{A})$. For $\phi \in \mathcal{S}(\mathrm{Z}^+(\mathbb{A}))$ and $h \in \mathrm{GU}_{2,2}(K_\mathbb{A})$, we define

$$\theta_\psi(\phi,\tau)(h) = \int_{\mathrm{Sp}_2(\mathbb{Q})\backslash\mathrm{Sp}_2(\mathbb{A})} \sum_{z\in\mathrm{Z}^+(\mathbb{Q})} r_{\psi^{-1}}(g_1 g,h)\phi(z)\tau(g_1 g)dg_1,$$

where $g \in \mathrm{GSp}_2(\mathbb{A})$ is taken so that $(g,h) \in \mathcal{R}(\mathbb{A})$. Then, the value $\theta_\psi(\phi,\tau)(h)$ does not depend on the choice of g, and $\theta_\psi(\phi,\tau)$ is an automorphic form on $\mathrm{GU}_{2,2}(K_\mathbb{A})$. If τ has the trivial central characacter, then so does $\theta_\psi(\phi,\tau)$. For an irreducible cuspidal automorphic representation π of $\mathrm{PGSp}_2(\mathbb{A})$, we define $\Theta_\psi(\pi)$ the space of these automorphic forms on $\mathrm{PGU}_{2,2}(K_\mathbb{A})$ obtained from all $\tau \in \pi$ and all $\phi \in \mathcal{S}(\mathrm{Z}^+(\mathbb{A}))$. For an irreducible cuspidal automorphic representation σ of $\mathrm{PGU}_{2,2}(K_\mathbb{A})$, we define $\Theta'_\psi(\sigma)$ the space of these automorphic forms on $\mathrm{PGSp}_2(\mathbb{A})$ using $\mathcal{S}(\mathrm{Z}'^+(\mathbb{A}))$ and r'_ψ, similarly.

For $s,x,y,z \in \mathbb{Q}$, let

$$n(s,x,y,z) = \begin{bmatrix} 1 & s & & \\ & 1 & & \\ & & 1 & \\ & & -s & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{bmatrix} \in \mathrm{GSp}_2(\mathbb{Q}).$$

Let $N(\mathbb{Q}) = \{n(s,x,y,z) \mid s,x,y,z \in \mathbb{Q}\}$. On $N(\mathbb{Q})$, for an nontrivial additive character ψ, we define $\psi_N(n(s,x,y,z)) = \psi(s+z)$, and the Whittaker function $W_\psi(f;g)$ of an automorphic form f with respect to ψ by

$$W_\psi(f;g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \psi_N(n)f(ng)dn.$$

We say π is globally generic, if there is an $f \in \pi$ having a nontrivial Whittaker function with respect to some nontrivial ψ. For $x,z \in \mathbb{Q}, s,y \in K$, let

$$n_K(s,x,y,z) = \begin{bmatrix} 1 & s & & \\ & 1 & & \\ & & 1 & \\ & & -s^c & 1 \end{bmatrix} \begin{bmatrix} 1 & & x & y \\ & 1 & y^c & z \\ & & 1 & \\ & & & 1 \end{bmatrix} \in \mathrm{GU}_{2,2}(K).$$

Let $N_K(K) = \{n_K(s,x,y,z) \mid x,z \in \mathbb{Q}, s,y \in K\}$ and, for a nontrivial ψ on $\mathbb{Q}$, we define $\psi_{N_K}(n(s,x,y,z)) = \psi(\mathrm{Re}(s)+z)$. We define Whittaker functions of automorphic forms on $\mathrm{GU}_{2,2}(K_\mathbb{A})$, and globally generic representation, similarly. Further, we define $\psi'_{N_K}(n_K(0,x,y,z)) = \psi(\mathrm{Im}(y))$ on the subgroup composed of $n_K(0,x,y,z)$. For an automorphic form f on $\mathrm{GU}_{2,2}(K_\mathbb{A})$, the Shalike model of f with respect to ψ is defined by

$$\int_{N'_K(K)\backslash N'_K(K_\mathbb{A})} \psi'_{N_K}(n)f(ng)dn.$$

First, we recall Ranakrishnan, Shahidi's result in [21].

Proposition 1.1 (Ranakrishnan-Shahidi). *Let ρ, μ be as in the conjecture. Then, there is an irreducible, globally, generic, cuspidal, automorphic representation π^{gn} such that*

$$L_S(s, \pi^{gn}; \mathrm{spin}) (:= \prod_{v \notin S} L(s, \pi_v^{gn}; \mathrm{spin})) = L_S(s, \rho \otimes \pi(\mu)).$$

We start an argument from this π^{gn}.

Proposition 1.2. *Let ψ_0 be the standard additive character on $\mathbb{Q}\backslash\mathbb{A}$. If π is an irreducible, globally generic, cuspidal representation of $\mathrm{PGSp}_2(\mathbb{A})$, then $\Theta_{\psi_0}(\pi)$ is nontrivial and a globally generic representation.*

Proof. Through a computation similar to used in the proof of Proposition 2.2 of Piatetski-Shapiro, Soudry [16], we get

$$\begin{aligned} &\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \psi_0(s)\theta_{\psi_0}(\phi, f)(n(s)h)\mathrm{d}s \\ = &\int_{N(\mathbb{A})\backslash \mathrm{Sp}_2(\mathbb{A})} r_{\psi_0}(g,h)\phi\left(\varepsilon_{-1}\otimes e_1 + \varepsilon_{-2}\otimes e_2 - \varepsilon_+ \otimes e.\right) W_{\psi_0}(f;g)\mathrm{d}g. \end{aligned}$$

It is possible to choose ϕ so that the right hand side of (1.1) is not zero at $h = 1$ if $W_\psi(f;1) \neq 0$. Thus the assertion. □

Let σ be an irreducible constituent of the above nontrivial $\Theta_{\psi_0}(\pi)$. Thanks to the next result due to Furusawa and Morimoto announced in the last year,

Theorem 1.3. *An irreducible, globally generic, cuspidal automorphic representation Π of $\mathrm{PGU}_{2,2}(K_{\mathbb{A}})$ has a Shalike model, if and only if $L_S(s, \Pi; \Lambda_t^2) = 1$ (a partial L-function of Π with respect to outer exterior representation Λ_t^2 (c.f. [9])) has a simple pole at $s = 1$.*

and the observation that, if an irreducible cuspidal automorphic representation π of $\mathrm{PGSp}_2(\mathbb{A})$ has a partial spinor L-function $L_S(s, \rho \otimes \pi(\mu))$ for some finite set S of places, then

$$\begin{aligned} L_S(s, \sigma; \Lambda_t^2) &= \zeta_S(s) L_S(s, \pi, \chi_{-4}; r_5) \\ &= \zeta_S(s) L_S(s, \rho, \chi_{-4}; \mathrm{sym}_2) L_S(s, \mu^2) \end{aligned}$$

has a simple pole at $s = 1$, we find that σ has a Shalike model, where $L_S(s, \pi, \chi_{-4}; r_5)$ indicates the L-function of π of degree 5 twisted by the quadratic character χ_{-4}. Further,

Proposition 1.4. *An irreducible, globally generic, cuspidal automorphic representation Π of $\mathrm{PGU}_{2,2}(K_{\mathbb{A}})$ has a nontrivial θ-lift $\Theta'_{\psi_0}(\Pi)$ to $\mathrm{PGSp}_2(\mathbb{A})$, if and only if Π has a Shalike model with respect to ψ_0.*

Proof. Let τ be an automorphic form of Π, and $B_{\psi_0}(\tau; *)$ the Shalike model of τ. Then, the Whittaker function of $F = \theta_{\psi_0}(\varphi, \tau)$ with respect to ψ_0 is

$$\int_{N'_K(K_{\mathbb{A}})\backslash \mathrm{SU}_{2,2}(K_{\mathbb{A}})} r'_{\psi_0}(g,h)\varphi(\varepsilon_1 \otimes e_{+1} + \varepsilon_2 \otimes e_{+2}) B_{\psi_0}(h)\mathrm{d}h. \qquad (1.1)$$

It is possible to choose φ so that this function of g is nontrivial. Hence the assertion. □

Therefore, we conclude that an irreducible, globally generic, cuspidal, automorphic representation π^{gn} of $\mathrm{PGSp}_2(\mathbb{A})$ with $L_S(\pi;\mathrm{spin}) = L_S(\rho \otimes \pi(\mu))$ can come back through these θ-lifts $\Theta_{\psi_0}, \Theta'_{\psi_0}$.

Now then, we will observe the levels of these automorphic representations. First, π^{gn} has the spinor L-function, from the functional equation of the L-function and the result of Roberts-Schmidt [22], one can estimate the paramodular level of π^{gn} divides 2^{10}. More precisely, π^{gn} has a right $K^{para}(2^{10})$ (paramodular group) invariant Whittaker function W_{ψ_0} such that $W_{\psi_0}(1) \neq 0$. Let $\mathfrak{O}_K$ be the ring of integers of K and

$$\Gamma_0(2^5)^K = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{GU}_{2,2}(\mathfrak{O}_K) \mid C \equiv 0 \pmod{2^5\mathfrak{O}_K} \right\}.$$

Setting a ϕ suitably in (1.1), we can construct a right $\Gamma_0(2^5)^K_2$-invariant Shalike model B_{ψ_0} of $\Theta_{\psi_0}(\pi^{gn})$ such that $B_{\psi_0}(1) \neq 0$. Setting a φ suitably in (1.1), we can construct a right $\Gamma(4,8)_2$-invariant Whittaker model of $\Theta'_{\psi_0}(\Theta_{\psi_0}(\pi^{gn}))$. Thus, by the strong multiplicity one theorem for globally generic representation of $\mathrm{GSp}_2(\mathbb{A})$, due to Soudry [23], π^{gn} has a right $\Gamma(4,8)_2$-invariant vector. One can deduce the following from Weissauer's result

Proposition 1.5 (Proposition 1.5 of [25]). *If an irreducible globally generic cuspidal automorphic representation π of $\mathrm{GSp}_2(\mathbb{A})$ has a cohomological weight, then there is an irreducible cuspidal automorphic representation π^{hol} such that*

- $\pi_v^{hol} \simeq \pi_v$ *for all nonarchimedean places v.*
- π_∞^{hol} *is a holomorphic discrete series with a cohomological weight.*

Remark 1. Ramakrishnan, Shahidi [21] showed the existence of some holomorphic Siegel modular cusp forms of degree 2 with interesting spinor L-functions, using this result.

Applying this, and looking the Γ-factor of $L(s, \rho\otimes\pi(\mu)) = L(s, \mathrm{BC}_K(\rho)\otimes\mu)$ ($\mathrm{BC}_K(\rho)$ indicates the base change lift of ρ to $\mathrm{GL}_2(K_\mathbb{A})$), one finds that there is an eigenform $F \in S_3(\Gamma(4,8))$ such that $L(s, F; \mathrm{spin}) = L(s, \rho \otimes \pi(\mu))$. In [15], as conjectured by van Geemen, van Straten [2], we showed that all irreducible cuspidal automorphic representation π_{f_i} $(1 \leq i \leq 6)$ have different spinor L-functions from $L(s, \rho \otimes \pi(\mu))$. Hence, the conjecture is true.

References

[1] M. Asgari, F. Shahidi, Generic transfer from GSp(4) to GL(4), Composito Math., **142** (2006), 541-550.

[2] B. van Geemen, D. van Straten, The cuspform of weight 3 on $\Gamma_2(2,4,8)$, Math. Comp., **61** (1993), 849-872.

[3] B. van Geemen, N. O. Nygaard, On the Geometry and Arithmetic of Some Siegel Modular Threefolds, J. Number Theory, **53** (1995), 45-87.

[4] Y. Gon, Generalized Whittaker functions on SU(2,2) with respect to the Siegel parabolic subgroups, Mem. A.M.S., **155** (2002).

[5] M. Harris, S. Kudla, Arithmetic automorphic forms for the nonholomorphic discrete series of $GSp(2)$, Duke Math., **66** (1992), 59-121.

[6] R. Howe, I.I. Piatetski-Shapiro, Some examples of automorphic forms on Sp_4, Duke. Math., **50** (1983), 55-105.

[7] H. Jacquet, R.P. Langlands, *Automorphic forms on GL(2)*, L.N.M., **114** (1970), Springer.

[8] M. Harris, D. Soudry, and R. Taylor, l-adic representations associated to modular forms over imaginary quadratic fields, Invent. math., **112** (1993), 377-411.

[9] H. Kim, M. Krishnamurthy, Twisted exterior square lift from $GU(2,2)$E/F to GL_6/F, J. Ramanujan. Math. Soc., **23** no.4, (2008), 381-412.

[10] S. Kudla, On the local theta-correspondence, Invent. Math., **83** (1986), 229-255.

[11] S. Kudla, S. Rallis, A regularized Siegel-Weil formula: The first term identity, Annals of Math., **140** (1994), 1-80.

[12] Max. A. Knus, *Quadratic forms, Clifford algebras and Spinors*, Lecture notes, Seminars in Mathematics 1, Campinas, (1988).

[13] G. Laumon, Sur la cohomologie a supports compacts des varietes de Shimura pour GSp(4)$\mathbb{Q}$. Composito Math., 105 (1997), no. 3, 267–359.

[14] T. Oda, J. Schwermer, Mixed Hodge structures and automorphic forms for Siegel modular varieties of degree two. Math. Ann., 286 (1990), no. 1-3, 481–509.

[15] T. Okazaki, L-functions of $S_3(\Gamma(4,8))$, J. Number Theory, **125** (2007), 117-132.

[16] Ilya I. Piatetski-Shapiro, D. Soudry, On a correspondence of automorphic forms on orthogonal groups of order five, J. math. pures et appl., **66** (1987), 407-436.

[17] D. Prasad, Theta correspondence for unitary groups, Pacific. J. Math., **194**, No. 2, (2000).

[18] T. Przebinda, The oscillator duality correspondence for the pair $O(2,2)$, $Sp(2,\mathbb{R})$, Memoirs of A.M.S., **79** Number 403 (1989).

[19] B. Roberts, The nonarchimedean theta correspondence for GSp(2) and GO(4), Trans. A.M.S., **351**, Num 2, (1999) 781-811.

[20] —, Global L-packets for GSp(2) and theta lifts, Docu. Math., **6** (2001) 247-314.

[21] D. Ramakrishnan, F. Shahidi, Siegel modular forms of genus 2 attached to elliptic curves, Math. Res. Lett., **13** (2006), 10001-10018.

[22] B. Roberts, R. Schmidt, *Local newforms for GSp(4)*, L.N.M., **1918** (2007), Springer.

[23] D. Soudry, A uniqueness theorem for representations of GSO(6) and the strong multiplicity one theorem for generic representations of GSp(4), Israel. J., **38**, No 3, (1987), 257-287.

[24] T. Watanabe, Theta lifting of cusp forms on the unitary group $U(d, d)$, Duke. Math., **67**, no.1 (1992), 159-187.

[25] R. Weissauer, Four dimensional Galois representations, Astérisque, **302**, (2005), 67-150.

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