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SPECTRAL MEANS OF CENTRAL VALUES OF AUTOMORPHIC L-FUNCTIONS FOR GL(2)

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Introduction : In this short note, we announce new asymptotic mean formulas for the central values of automorphic L-functions of GL(2) over a totally real number field. We consider average of such L-values from two different aspects: (i) level aspect and (ii) Laplace eigenvalue aspect. We omit all proofs of theorems, which we refer to [11].

1. Level aspect

1.1. **Existing works.** The projective unimodular group $\mathrm{PSL}_2(\mathbb{Z})$ acts on the Poincaré upper half plane $\mathfrak{H} := \{\tau \in \mathbb{C} \,|\, \mathrm{Im}(\tau) > 0\}$ by the fractional linear transformations properly discontinuously. For $N \in \mathbb{N}$, consider the Hecke congruence subgroup

$$\Gamma_0(N) = \{[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in \mathrm{SL}_2(\mathbb{Z}) |\, c \equiv 0 \pmod{N}\}$$

of $\mathrm{PSL}_2(\mathbb{Z})$. Let $k \geqslant 2$ be an even integer. Given a positive integer N, let $S_k(N)$ denote the space of holomorphic cusp forms of weight k with level N, i.e., $S_k(N)$ is the space of holomorphic functions $\phi : \mathfrak{H} \to \mathbb{C}$ with the modular transformation property

$$\phi\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k \phi(\tau)$$

for all $[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in \Gamma_0(N)$ such that the constant term of the Fourier expansion of $\phi(\tau)$ at each cusp $c \in \mathbb{Q} \cup \{\infty\}$ vanishes. Thus the Fourier expansion of $\phi \in S_k(N)$ at the cusp ∞ takes the form

$$\phi(\tau) = \sum_{n=1}^{\infty} a_\phi(n)\, q^n, \qquad (q = e^{2\pi i \tau}),$$

where $a_\phi(n)$ is the n-th Fourier coefficient of ϕ.

It is well known that $S_k(N)$ is a finite dimensional Hilbert space with the Petersson inner-product

$$\langle \phi_1, \phi_2 \rangle := [\mathrm{PSL}_2(\mathbb{Z}) : \Gamma_0(N)]^{-1} \int_{\Gamma_0(N)\backslash\mathfrak{H}} \phi_1(\tau)\, \overline{\phi_2(\tau)} y^{k-2}\, \mathrm{d}x \mathrm{d}y.$$

The Hecke operators $T(n)$ ($n \in \mathbb{N}$, $(n, N) = 1$) act on $S_k(N)$ self-adjointly by the formula

$$[T(n)\phi](\tau) = n^{k-1} \sum_{\substack{ad=n, \\ a \geqslant 1, 0 \leqslant b < d}} \phi\left(\frac{a\tau + b}{d}\right) d^{-k}.$$

Let $S_k(N)^{\mathrm{new}}$ be the subspace of newforms in $S_k(N)$ in the usual sense. Then, as a consequence of the theory of newforms, it follows that there exists a $\mathbb{C}$-basis $\mathcal{F}_k(N) = \{\phi_j\}$ of $S_k(N)^{\mathrm{new}}$ such that

(i) $a_\phi(1) = 1$ for any $\phi \in \mathcal{F}_k(N)$.

(ii) $T(n)\phi = a_\phi(n)\,\phi$ for all $n \in \mathbb{N}$ and for all $\phi \in \mathcal{F}_k(N)$.

For any $\phi \in \mathcal{F}_k(N)$, we have Deligne's estimate

$$|a_\phi(p)| \leqslant 2p^{(k-1)/2} \quad \text{for primes } p \text{ relatively prime to the level } N.$$

Thus if we introduce the normalized Fourier coefficients of $\phi \in \mathcal{F}_k(N)$ by

$$c_\phi(p) \stackrel{\text{def}}{=} p^{(1-k)/2}\,a_\phi(p)$$

for each prime p not dividing N, then they belong to the interval $[-2,2]$. Viewing $c_\phi(p)$ as a function in p and ϕ, we may ask the following "statistical" questions from two different aspects:

(1) Fix $\phi \in \mathcal{F}_k(N)$.
With p varied, how do the numbers $c_\phi(p)$ distribute on the interval $[-2,2]$?

(2) Fix p.
With (k,N) varied, how do the numbers $\{c_\phi(p) \mid \phi \in \mathcal{F}_k(N)\}$ distribute on the interval $[-2,2]$?

In the first aspect, the following conjecture has long been believed to hold.

Sato-Tate conjecture :
For $\lambda > 0$, set $\mathcal{P}_\lambda = \{p : \text{primes} \mid p \leqslant \lambda\}$. Let $\phi \in S_k(N)^{\text{new}}$ (fixed) be a "non CM form". Then, as $\lambda \to \infty$, the numbers $\{c_\phi(p) \mid p \in \mathcal{P}_\lambda\}$ become equidistributed in $[-2,2]$ with respect to the Sato-Tate measure

$$\mathrm{d}\mu^{\mathrm{ST}}(x) = \frac{1}{2\pi}\sqrt{4-x^2}\,\mathrm{d}x,$$

i.e.,

$$\lim_{\lambda\to\infty} \frac{1}{\sharp\mathcal{P}_\lambda} \sum_{p\in\mathcal{P}_\lambda} f(c_\phi(p)) = \int_{[-2,2]} f(x)\,\mathrm{d}\mu^{\mathrm{ST}}(x)$$

for any $f \in C([-2,2])$. □

Due to a great progress brought recently by Clozel, Harris, Taylor and Shepherd-Barron together with their collaborators, this deep and difficult conjecture has been solved positively in most cases.

The question (2) is far more accessible than (1) but is still intriguing. Actually, in this aspect, the following theorem has been known for some time.

Theorem 1. (Serre[10]) : Let p be a prime (fixed) and (k_λ, N_λ) $(\lambda \in \Lambda)$ be a filtered family of pairs of natural numbers such that

(1) $k_\lambda \geqslant 2$ is even,
(2) $k_\lambda + N_\lambda \to +\infty$ as $\lambda \to +\infty$,
(3) $p \nmid N_\lambda$.

As $\lambda \to +\infty$, the numbers $\{c_\phi(p) \mid \phi \in \mathcal{F}_{k_\lambda}(N_\lambda)\}$ $(\lambda \in \Lambda)$ becomes equidistributed in $[-2,2]$ with respect to the measure

$$\mathrm{d}\mu_p^-(x) = \frac{p+1}{(p^{1/2}+p^{-1/2})^2 - x^2}\,\mathrm{d}\mu^{\mathrm{ST}}(x)$$

$$\text{i.e.,}\quad \lim_{\lambda\to\infty} \frac{1}{\sharp\mathcal{F}_{k_\lambda}(N_\lambda)} \sum_{\phi\in\mathcal{F}_{k_\lambda}(N_\lambda)} f(c_\phi(p)) = \int_{-2}^{2} f(x)\,\mathrm{d}\mu_p^-(x)$$

for any text function $f \in C([-2,2])$. □

This is proved by analyzing the trace formula of the Hecke operators in detail.

Remark: $\mathrm{d}\mu_p^-(x)$ is the spherical Plancherel measure of $\mathrm{PGL}(2,\mathbb{Q}_p)$ in an appropriate normalization.

1.2. ***L*-series.** Serre's result recalled above has a "relative analogue". To explain this, let us introduce the L-series. Suppose $\phi \in S_k(N)^{\mathrm{new}}$ satisfies the Atkin-Lehner condition

$$\phi|_{k,\gamma_N}(\tau) = w_\phi\, i^{-k}\, \phi(\tau) \qquad (w_\phi = \pm 1)$$

with $\gamma_N = \left[\begin{smallmatrix} & -N \\ 1 & \end{smallmatrix}\right]$. For any primitive Dirichlet character $\eta : (\mathbb{Z}/D\mathbb{Z})^\times \longrightarrow \{\pm 1\}$ of conductor D, define the L-series of ϕ twisted by η as

$$L(s,\phi\otimes\eta) \stackrel{\mathrm{def}}{=} \sum_{n=1}^{\infty} \frac{a_\phi(n)\,\eta(n)}{n^s} \quad (\mathrm{Re}(s) \gg 0).$$

Then, it is known that the completed L-function

$$\Lambda(s,\phi\otimes\eta) \stackrel{\mathrm{def}}{=} L(s,\phi\otimes\eta)\,(D^2N)^{s/2}\,\Gamma_{\mathbb{C}}(s)$$

has a holomorphic extension to $\mathbb{C}$ satisfying the functional equation

$$\Lambda(s,\phi\otimes\eta) = w_\phi\eta(-N)\,\Lambda(k-s,\phi\otimes\eta).$$

From this, we infer that the value at the center of symmetry $\Lambda(k/2,\phi\otimes\eta) = 0$ unless $w_\phi\eta(-N) = +1$. It is known that $\Lambda(k/2,\phi\otimes\eta)$ is always non-negative ([3]).

Theorem 2. (Ramakrishnan-Rogawski [9]) Let $\eta : (\mathbb{Z}/D\mathbb{Z})^\times \longrightarrow \{\pm 1\}$ be a primitive Dirichlet character such that $\eta(-1) = -1$. Let p be a prime such that $(p,D) = 1$. Let N be a prime such that $\eta(-N) = 1$ and $(pD,N) = 1$. Fix an integer $k \geqslant 4$. Then, for the characteristic function f of any subinterval of $[-2,2]$ and for any $\epsilon > 0$, as $N \to \infty$, we have

$$\sum_{\phi\in\mathcal{F}_k(N)} \frac{\Lambda(k/2,\phi)\,\Lambda(k/2,\phi\otimes\eta)}{\|\phi\|^2} f(c_\phi(p)) = 2\Lambda(1,\eta)\, c_k^{(\infty)} \int_{-2}^{2} f\mathrm{d}\mu_p{}^{\pm} + O_\epsilon(N^{-k/2+\epsilon})$$

where $c_k^{(\infty)}$ is an explicit constant depending only on k. When the sign $\pm = \eta(p)$ is $-$, the measure $\mathrm{d}\mu_p^-(x)$ is the same one as in Theorem 1, otherwise

$$\mathrm{d}\mu_p{}^{+}(x) = \frac{p-1}{(p^{1/2}+p^{-1/2}-x)^2}\,\mathrm{d}\mu^{\mathrm{ST}}(x).$$

□

Theorem 2 is proved by computing the relative trace formula of Jacquet in detail; this is why Theorem 2 may be regarded as a relative analogue of Serre's theorem.

Remark :

- The constant $c_k^{(\infty)}$ coincides with the Plancherel density (= formal degree) of the discrete series representation of $\mathrm{PGL}(2,\mathbb{R})$ corresponding to ϕ.

- The case when $f = 1$ was proved by W. Duke (1995) by Kuznetsov's trace formula.
- Theorem 2 is extended to the Hilbert modular setting by Feigon-Whitehause ([2]) by the relative trace formula of Jacquet.

In the next section, we announce that a similar kind of asymptotic formula for Maass cusp forms is true in a setting of automorphic representations of GL(2) over a totally real algebraic number field.

1.3. **Results.** Let F be a totally real number field, $d_F = [F : \mathbb{Q}]$ its degree and $\mathfrak{o}_F$ its maximal order. Let Σ_{fin} and Σ_∞ be the set of finite places of F and the set of infinite places of F, respectively. The completion of F at a place $v \in \Sigma_{\text{fin}} \cup \Sigma_\infty$ is denoted by F_v. If $v \in \Sigma_{\text{fin}}$, F_v is a non-archimedean local field, whose maximal order is denoted by $\mathfrak{o}_v$; we fix a uniformizer ϖ_v of $\mathfrak{o}_v$ once and for all, and denote by q_v the order of the residue field $\mathfrak{o}_v/\mathfrak{p}_v$, where $\mathfrak{p}_v = \varpi_v\mathfrak{o}_v$ is the maximal ideal of $\mathfrak{o}_v$. Let $\mathbb{A}$ and $\mathbb{A}_{\text{fin}}$ be the adele ring of F and the finite adele ring of F, respectively. For an $\mathfrak{o}_F$-ideal $\mathfrak{a}$, let us denote by $S(\mathfrak{a})$ the set of all $v \in \Sigma_{\text{fin}}$ such that $\mathfrak{a}\mathfrak{o}_v \subset \mathfrak{p}_v$.

Let S be a finite set of places containing Σ_∞, and $\eta = \prod_{v \in \Sigma_\infty \cup \Sigma_{\text{fin}}} \eta_v$ an idele-class character of order 2 with conductor $\mathfrak{f}$ such that η is unramified over $S_{\text{fin}} = S \cap \Sigma_{\text{fin}}$ and such that η_v is trivial for any $v \in \Sigma_\infty$. Let $\mathfrak{I}^*_{S,\eta}$ be the set of all the square free $\mathfrak{o}_F$-ideals $\mathfrak{n}$ with the following properties.

- The set $S(\mathfrak{n})$ is disjoint from $S(\mathfrak{f}) \cup S_{\text{fin}}$ and has the even cardinality.
- $\eta_v(\varpi_v) = -1$ for any $v \in S(\mathfrak{n})$.

For any ideal $\mathfrak{n}$, let $\Pi_{\text{cus}}(\mathfrak{n})$ be the set of all the irreducible cuspidal representations π of $\mathrm{GL}(2, \mathbb{A})$ with trivial central characters having non zero $\mathbf{K}_\infty\mathbf{K}_0(\mathfrak{n})$-fixed vectors, where $\mathbf{K}_\infty$ is a standard maximal compact subgroup of $\mathrm{GL}(2, F \otimes_\mathbb{Q} \mathbb{R})$ and

$$\mathbf{K}_0(\mathfrak{n}) = \prod_{v \in \Sigma_{\text{fin}}} \{ \left[\begin{smallmatrix} a_v & b_v \\ c_v & d_v \end{smallmatrix}\right] \in \mathrm{GL}(2, \mathfrak{o}_v) \,|\, c_v \in \mathfrak{n}\mathfrak{o}_v \}.$$

Let $\pi \in \Pi_{\text{cus}}(\mathfrak{n})$ with $\mathfrak{n} \in \mathfrak{I}^*_{S,\eta}$. Then, there exists a family of irreducible smooth representations π_v of $\mathrm{GL}(2, F_v)$ for all places v such that π_v is unramified for almost all $v \in \Sigma_{\text{fin}}$ and such that $\pi \cong \bigotimes_v \pi_v$. When $v \in \Sigma_\infty$ or $v \in \Sigma_{\text{fin}}$ prime to $\mathfrak{n}$, then π_v is isomorphic to an unramified principal series representation $I_v(|\,|_v^{\nu_v})$ with the parameter ν_v belonging to the space $\mathfrak{X}_v^{0+}$, where $\mathfrak{X}_v^{0+}$ denotes $i\mathbb{R}_+ \cup (0,1)$ or $i(0, 2\pi(\log q_v)^{-1}) \cup (0,1)$ according to $v \in \Sigma_\infty$ or $v \in \Sigma_{\text{fin}}$, respectively. Note that the set of $\mathrm{PGL}(2, \mathfrak{o}_v)$-spherical unitary dual of $\mathrm{PGL}(2, F_v)$ coincides with the union of $\{I_v(|\,|_v)^{\nu/2} \,|\, \nu \in \mathfrak{X}_v^{0+}\}$ and the set of unitary characters of $\mathrm{PGL}(2, F_v)$ trivial on $\mathrm{PGL}(2, \mathfrak{o}_v)$. The *spectral parameters* at S of π is defined to be the point $\nu_{\pi,S} = (\nu_v)_{v \in S}$ lying in the product space

$$\mathfrak{X}_S^{0+} = \prod_{v \in S} \mathfrak{X}_v^{0+}.$$

We endow the set $\mathfrak{X}_S^{0+}$ with the induced topology from $\mathbb{C}^S$. On the one hand, the set $\Pi_{\text{cus}}(\mathfrak{n})$ defines a Borel measure $\lambda_S^\eta(\mathfrak{n})$ on the space $\mathfrak{X}_S^{0+}$ by

$$\langle \lambda_S^\eta(\mathfrak{n}), f \rangle = \sum_{\pi \in \Pi^*_{\text{cus}}(\mathfrak{n})} \frac{L(1/2, \pi)\, L(1/2, \pi \otimes \eta)}{\mathrm{N}(\mathfrak{n})\, L(1, \pi; \mathrm{Ad})} f(\nu_{\pi,S}), \qquad f \in C_c^0(\mathfrak{X}_S^{0+}),$$

where $\Pi_{\text{cus}}^*(\mathfrak{n})$ is the set of all $\pi \in \Pi_{\text{cus}}(\mathfrak{n})$ with conductor $\mathfrak{n}$, $L(s,\pi)$ is the standard L-function of π ([5]) and $L(s,\pi;\mathrm{Ad})$ the adjoint square L-function of π. We remark that, by the Rankin-Selberg method, the special value $L(1,\pi;\mathrm{Ad})$ is identified with the squared Petersson norm of the newform φ_π^{new} of π up to an elementary positive constant. Thus, by a result of Guo ([3]), the measure $\lambda_S^\eta(\mathfrak{n})$ is non-negative. On the other hand, the space $\mathfrak{X}_S^{0+}$ carries a measure

$$\mathrm{d}\lambda_S^\eta = 4D_F^{3/2}\, L(1,\eta) \bigotimes_{v\in S} \mathrm{d}\lambda_v^{\eta_v},$$

where D_F is the absolute discriminant of F, $L(1,\eta)$ is the completed Hecke L-series of η and $\mathrm{d}\lambda_v^{\eta_v}$ is a measure on $\mathfrak{X}_v^{0+}$ supported on the purely imaginary points, on which it is given by

$$\mathrm{d}\lambda_v^{\eta_v}(iy) = \frac{L(1/2, I_v(|\,|_v^{iy/2}))\, L(1/2, I_v(|\,|_v^{iy/2}) \otimes \eta_v)}{L(1,\pi_v,\mathrm{Ad})} \frac{\zeta_{F,v}(2)}{L_v(1,\eta_v)} \mathrm{dP}_v(y) \tag{1.1}$$

with

$$\mathrm{dP}_v(y) = \begin{cases} \dfrac{(1+q_v^{-1})\log q_v}{4\pi} \left|\dfrac{1-q_v^{-iy}}{1-q_v^{-(1+iy)}}\right|^2 \mathrm{d}y, & v \in S_{\text{fin}}, \\[2ex] \dfrac{1}{4\pi}\left|\dfrac{\Gamma((1+iy)/2)}{\Gamma(iy/2)}\right|^2 \mathrm{d}y, & v \in \Sigma_\infty \end{cases} \tag{1.2}$$

the $\mathrm{PGL}(2,\mathfrak{o}_v)$-spherical Plancherel measure of $\mathrm{PGL}(2,F_v)$. Here, $L(s,\pi_v)$ denotes the local L-factor of an irreducible smooth representation π_v of $\mathrm{GL}(2,F_v)$ and $L(1,\eta_v)$ denotes Tates' local L-factor of η_v.

Then, our first main result is stated as follows.

Theorem 3. *As* $\mathrm{N}(\mathfrak{n}) \to +\infty$ *with* $\mathfrak{n} \in \mathfrak{J}_{S,\eta}^*$, *the measure* $\lambda_S^\eta(\mathfrak{n})$ *converges weakly to the measure* λ_S^η, *i.e.,*

$$\langle \lambda_S^\eta(\mathfrak{n}), f\rangle \to \langle \lambda_S^\eta, f\rangle \qquad \text{for any} \quad f \in C_c^0(\mathfrak{X}_S^{0+}).$$

As a corollary to this theorem, we have

Corollary 4. *Let* $\{J_v\}_{v\in S}$ *be a family of intervals such that* $J_v \subset [1/4,+\infty)$ *if* $v \in \Sigma_\infty$ *and* $J_v \subset [-2,2]$ *if* $v \in S_{\text{fin}}$. *Then, for any* $\delta > 0$, *there exists an irreducible cuspidal automorphic representation* π *with trivial central character having the following properties.*

(1) *The conductor* $\mathfrak{f}_\pi$ *of* π *belongs to* $\mathfrak{J}_{S,\eta}^*$ *and* $\mathrm{N}(\mathfrak{f}_\pi) > \delta$.
(2) $L(1/2,\pi) \neq 0$ *and* $L(1/2,\pi\otimes\eta) \neq 0$.
(3) *The spectral parameters* $\nu_{\pi,S} = (\nu_v)_{v\in S}$ *at* S *of* π *satisfies* $(1-\nu_v^2)/4 \in J_v$ *for any* $v \in \Sigma_\infty$ *and* $q_v^{-\nu_v/2} + q_v^{\nu_v/2} \in J_v$ *for any* $v \in S_{\text{fin}}$.

If we take a function ϕ on $\mathfrak{H}^{d_F}$ corresponding to the new vector φ_π^{new} on $\mathrm{GL}(2,\mathbb{A})$, then $(1-\nu_v^2)/4$ coincides with the v-th Laplace eigenvalue for ϕ and $q_v^{\nu_v/2} + q_v^{-\nu_v/2}$ coincides with the v-th Hecke eigenvalue for ϕ. Thus, Corollary 4 is regarded as an analogue of [9, Corollary B].

2. Laplace eigenvalue aspect (Weyl's law)

2.1. **Motivation.** We start with the classical situation. Let

$$\Delta = -y^2(\partial^2/\partial x^2 + \partial^2/\partial x^2)$$

be the hyperbolic Laplacian acting on $L^2\,(\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}; y^{-2}\,dxdy)$, and

$$\{\lambda_j = 1/4 + \kappa_j^2 | \, j \geqslant 1\}$$

the non-decreasing sequence of the cuspidal eigenvalues of Δ, counted with multiplicity. Fix an orthonormal system of Maass cusp forms $\{u_j\}_{j=1}^{\infty}$ such that $\Delta\, u_j = \lambda_j\, u_j$ and $u_j(x+iy) = \epsilon_j\, u_j(-x+iy)$ with some $\epsilon_j \in \{\pm 1\}$ for each j. Then, the L-series of u_j is defined to be the absolutely convergent series

$$L_j(s) = \sum_{n=1}^{\infty} \frac{c_j(n)}{n^s}, \quad \mathrm{Re}(s) \gg 0,$$

where

$$u_j(\tau) = \sum_{n\in\mathbb{Z}-\{0\}} c_j(n)\, y^{1/2}\, K_{i\kappa_j}(2\pi|n|y)\, e^{2\pi i n x}$$

is the Fourier expansion of u_j at the cusp ∞. As is well-known, the completed L-series

$$\Lambda_j(s) = \Gamma_{\mathbb{R}}(s - i\kappa_j)\, \Gamma_{\mathbb{R}}(s + i\kappa_j)\, L_j(s)$$

is continued meromorphically to $\mathbb{C}$ with the functional equation

$$\Lambda_j(1-s) = \epsilon_j\, \Lambda_j(s).$$

The asymptotic behaviors of various spectral means of the central values $L_j(1/2)$ are extensively studied by Motohashi [8] by means of Kuznetsov's formula. Among other things, the asymptotic formula of the square mean values

$$(2.1)\quad \sum_{\kappa_j \leqslant t} \frac{|L_j(1/2)|^2}{\cosh(\pi\kappa_j)} = \frac{2}{\pi^2}\, t^2\, (\log t + C_{\mathrm{Euler}} - 1/2 - \log(2\pi)) + \mathcal{O}(t(\log t)^6), \quad t \to +\infty$$

is obtained ([8, Theorem 2]), where C_{Euler} is the Euler constant. In this article, we announce an analogous asymptotic formula (not for the mean of $|L_j(1/2)|^2/\cosh(\pi\kappa_j)$'s but) for the mean of $|\Lambda_j(1/2)|^2$'s in the context of automorphic representations of GL(2) over a totally real number field.

2.2. **Results.**

2.2.1. *Weyl's law.* We keep the notation introduced in the paragraph 1.3. Let $\mathfrak{X}_{\Sigma_\infty}^0$ be the tempered locus of $\mathfrak{X}_{\Sigma_\infty}^{0+}$, i.e.,

$$\mathfrak{X}_{\Sigma_\infty}^0 = \prod_{\iota\in\Sigma_\infty} i[0, +\infty),$$

which is viewd as a subset of $i\mathbb{R}^{d_F}$.

Theorem 5. *Let $\mathfrak{n}$ be any square free ideal of $\mathfrak{o}_F$. Let $J \subset i^{-1}\mathfrak{X}^0_{\Sigma_\infty}$ be a compact subset with smooth boundary, which is "positive", i.e., $\nu_\iota > 0$ for any $\nu \in J$ and for any $\iota \in \Sigma_\infty$. Then, for any $\epsilon > 0$,*

$$\sum_{\substack{\pi\in\Pi^*_{\mathrm{cus}}(\mathfrak{n})\\ \nu_{\pi,\Sigma_\infty}\in itJ}} \frac{|L(1/2,\pi)|^2}{\mathrm{N}(\mathfrak{n})\,L(1,\pi;\mathrm{Ad})} = \frac{4(1+\delta_{\mathfrak{n},\mathfrak{o}_F})\,D_F^{3/2}\,\mathrm{vol}(J)}{(4\pi)^{d_F}}\,t^{d_F}\,(d_F\,\log t + \mathbf{C}(F,\mathfrak{n}))$$

$$+\,\mathcal{O}\left(t^{d_F-1}\,(\log t)^3\right) + \mathcal{O}\left(t^{d_F+4\theta+2\epsilon}\,(\log t)^{1-d_F}\right), \quad t \to +\infty,$$

where $tJ = \{t\nu|\,\nu \in J\}$,

$$\mathbf{C}(F,\mathfrak{n}) = \mathrm{CT}_{s=1}\zeta_F(s) + \left\{\frac{d_F}{2}\,(C_{\mathrm{Euler}} + 2\log 2 - \log\pi) + \log(D_F\,\mathrm{N}(\mathfrak{n}))\right\}\,\mathrm{Res}_{s=1}\zeta_F(s)$$

with

$$\zeta_F(s) = \Gamma_{\mathbb{R}}(s)^{d_F}\prod_{v\in\Sigma_{\mathrm{fin}}}(1-q_v^{-s})^{-1}, \qquad \mathrm{Re}(s) > 1$$

the completed Dedekind zeta function of F, and $\theta \in \mathbb{R}$ is any constant satisfying

$$|L_{\mathrm{fin}}(1/2+it,\chi)| = \mathcal{O}((1+|t|)^{d_F/4+\theta}), \quad t \in \mathbb{R} \tag{2.2}$$

for all the unramified idele-class characters χ of $F^\times$. Here, $L_{\mathrm{fin}}(s,\chi) = \prod_{v\in\Sigma_{\mathrm{fin}}}(1 - \chi_v(\varpi_v)\,q_v^{-s})^{-1}$ is the L-series of χ. □

This formula also can be viewed as an analogue of the multidimensional Weyl's law for tempered cuspidal multiplicities of automorphic representations ([1], [6]).

2.2.2. *Subconvexity bounds.* For $\pi \in \Pi^*_{\mathrm{cus}}(\mathfrak{n})$, let $L_{\mathrm{fin}}(s,\pi)$ denotes the finite part of the standard L-function of π. Then, we are interested in the size of the L-value $L_{\mathrm{fin}}(1/2,\pi)$ measured by means of the analytic conductor $C(\pi)$ of π. When $\pi \in \Pi^*_{\mathrm{cus}}(\mathfrak{n})$, it is confirmed that $C(\pi) \asymp \prod_{\iota\in\Sigma_\infty}(1+|\nu_{\pi,\iota}|^2)$. The bound

$$|L_{\mathrm{fin}}(1/2,\pi)| \ll C(\pi)^{1/4}, \quad \pi \in \Pi^*_{\mathrm{cus}}(\mathfrak{n})$$

is called the convexity bound and is known in a general context ([4]). Any bound breaking this is called a subconvexity bound (in the Laplace eigenvalue aspect). Our third result is stated as follows.

Theorem 6. *Let $\mathfrak{n}$ be a square free ideal. Let $\theta \in \mathbb{R}$ be a constant such that* (2.2) *holds for all unramified idele-class characters χ of $F^\times$. Let J be any closed cone in $i^{-1}\mathfrak{X}^0_{\Sigma_\infty}$ such that $\nu_\iota > 0$ for any $\nu \in J$ and for any $\iota \in \Sigma_\infty$. Then, for any $\epsilon > 0$,*

$$|L_{\mathrm{fin}}(1/2,\pi)| = \mathcal{O}_{\epsilon,J}\left(C(\pi)^{1/4+\sup(\theta,-1/4)/d_F+\epsilon}\right), \qquad \pi \in \Pi^*_{\mathrm{cus}}(\mathfrak{n})_J,$$

where $\Pi^*_{\mathrm{cus}}(\mathfrak{n})_J = \{\pi \in \Pi^*_{\mathrm{cus}}(\mathfrak{n})|\,i^{-1}\,\nu_{\pi,\Sigma_\infty} \in J\}$.

This breaks the convexity bound if $\theta < 0$ when π varies over the set $\Pi^*_{\mathrm{cus}}(\mathfrak{n})_J$. We remark that a recent work of Michel and Venkatesh ([7]) shows the existence of a subconvexity bound (in any aspect) for a class of automorphic L-functions of GL(1) and GL(2) over an arbitrary number field in a uniform way. In particular, not only a bound (2.2) with $\theta < 0$ but also a subconvexity bound for $L_{\mathrm{fin}}(1/2,\pi)$ in Laplace eigenvalues aspect follow from their work. Despite this, we believe that the qualitative nature of the exponent in our bound is of some interest.

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